On generalized Pell numbers generated by Fibonacci and Lucas numbers

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Abstract

In this paper we introduce a new kind of distance Pell numbers which are generated using the classical Fibonacci and Lucas numbers. Generalized companion Pell numbers is closely related to distance Pell numbers which were introduced in [12]. We present some relations between distance Pell numbers, distance companion Pell numbers and their connections with the Fibonacci numbers. To study properties of these numbers we describe their graph interpretations which in the special case gives a distance generalization of the Jacobsthal numbers. We also use the concept of a lexicographic product of graphs to obtain a new interpretation of distance Jacobsthal numbers.

Keywords: Fibonacci numbers, Lucas numbers, Pell numbers, Jacobsthal numbers, d-independent set

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1 Introduction and preliminary results

Let F_n be the nth Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial terms $F_0 = F_1 = 1$. There are many numbers of the Fibonacci type defined by the linear recurrence relation,

(1.1) Lucas numbers
$L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$

(1.2) Pell numbers
$P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with $P_0 = 0$, $P_1 = 1$

(1.3) companion-Pell numbers
$Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$ with $Q_0 = Q_1 = 1$

(1.4) Jacobsthal numbers
$J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ with $J_0 = 0$, $J_1 = 1$

(1.5) Jacobsthal-Lucas numbers
$j_n = j_{n-1} + 2j_{n-2}$, for $n \geq 2$ with $j_0 = 2$, $j_1 = 1$

(1.6) Tribonacci numbers

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \text{ for } n \geq 3 \text{ with } T_0 = 0,\ T_1 = T_2 = 1$$

For other types of known sequences see The On-Line Encyclopedia of Integer Sequences, [18]. In this paper we define one parameter generalization of the Pell numbers in the distance sense.

In general we say that we generalize the numbers of the Fibonacci type in the distance sense if these numbers are defined by the kth order linear recurrence relations, for an arbitrary $k \geq 3$. Distance generalizations of Fibonacci numbers and Pell numbers are studied recently, see [2], [10], [12]-[17].

In [12] the special distance generalizations of Pell numbers was introduced. We recall it.

Let $k \geq 1$, $n \geq 0$ be integers. The nth distance Pell numbers we define recursively in the following way

$$Pd(k,n) = Pd(k,n-1) + Pd(k,n-2) + Pd(k,n-k), \text{ for } n \geq k$$

with the initial conditions
$Pd(k,0) = 0$,
$Pd(k,i) = 1$ for $k \leq i+2$, $i \geq 1$ and
$Pd(k,i) = 0$ for $k > i+2$, $i \geq 1$.

For $k = 1,2,3$ this sequence reduces to classical Pell sequence, the Jacobsthal sequence and Tribonacci sequence, respectively.

The following Table presents few initial distance Pell sequences. The numbers marked by bold type are Fibonacci numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11
$Pd(1,n)$	0	1	2	5	12	29	70	169	408	985	2378	5741
$Pd(2,n)$	0	1	**1**	3	5	11	21	43	85	171	341	683
$Pd(3,n)$	0	**1**	**1**	**2**	4	7	13	24	44	81	149	274
$Pd(4,n)$	0	0	**1**	**1**	**2**	**3**	6	10	18	31	55	96
$Pd(5,n)$	0	0	0	**1**	**1**	**2**	**3**	**5**	9	15	26	44
$Pd(6,n)$	0	0	0	0	**1**	**1**	**2**	**3**	**5**	**8**	14	23

Table 1. The distance Pell numbers $Pd(k,n)$.

Let $k \geq 2$, $n \geq 0$ be integers. Then
$Pd^*(k,n) = Pd^*(k,n-1) + Pd^*(k,n-2) + Pd^*(k,n-k)$, for $n \geq k+1$
and
$Pd^*(k,0) = 0$,
$Pd^*(k,n) = F_{n-1}$, for $1 \leq n \leq k$.
Clearly $Pd^*(k,n) = Pd(k,n+k-3)$ for $k \geq 3$.

In this paper we introduce the distance companion Pell sequence and this kind of the generalization is inspired by results given in [12].

Let $k \geq 1$, $n \geq 0$ be integers. The nth distance companion Pell number $Qd(k,n)$ we define in the following way
$Qd(k,n) = Qd(k,n-1) + Qd(k,n-2) + Qd(k,n-k)$ for $n \geq k$
$Qd(k,0) = k$,
$Qd(1,1) = 1$ and
$Qd(k,n) = L_n$ for $n = 1, ..., k-1$.

If $k = 1$ we have the classical companion Pell numbers P_n.
If $k = 2$ then we obtain the Jacobsthal-Lucas numbers j_n.
If $k = 3$ then $Qd(3,n)$ gives the Tribonacci numbers T_n with $T_0 = 3$, $T_1 = 1$, $T_2 = 3$.

The following Table presents few initial distance companion Pell sequences. The numbers marked by bold type are Lucas numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11
$Qd(1,n)$	1	1	3	7	17	41	99	239	577	1393	3363	8119
$Qd(2,n)$	**2**	**1**	5	7	17	31	65	127	257	511	1025	2047
$Qd(3,n)$	3	**1**	**3**	7	11	21	39	71	131	241	443	815
$Qd(4,n)$	4	**1**	**3**	**4**	11	16	30	50	91	157	278	485
$Qd(5,n)$	5	**1**	**3**	**4**	**7**	16	24	43	71	121	208	353
$Qd(6,n)$	6	**1**	**3**	**4**	**7**	**11**	24	36	63	103	173	287

Table 2. The distance companion Pell numbers $Qd(k,n)$.

Question about relations between generalizations of numbers of the Fibonacci type and the classical Fibonacci numbers is the natural in the context of results obtained by E. Kiliç also with D. Tasci, see [8], [9].

We will prove some relations between $Pd(k,n)$, $Qd(k,n)$ and F_n.

Theorem 1. *Let $k \geq 1$, $n \geq k$ be integers. Then for fixed $1 \leq i \leq n-k$ holds*

$$Pd(k,n) = F_{i+1}Pd(k,n-(i+1)) + F_iPd(k,n-(i+2)) + \sum_{t=0}^{i} F_tPd(k,n-k-t).$$

Proof. (by induction on i) Let $i = 1$. Then

$$Pd(k,n) = F_2Pd(k,n-2) + F_1Pd(k,n-3) + \sum_{t=0}^{1} F_tPd(k,n-k-t) =$$
$$= 2Pd(k,n-2) + Pd(k,n-3) + Pd(k,n-k) + Pd(k,n-k-1) =$$
$$= Pd(k,n-2) + Pd(k,n-3) + Pd(k,n-k-1) + Pd(k,n-2)+$$
$$+Pd(k,n-k) = Pd(k,n-1) + Pd(k,n-2) + Pd(k,n-k)$$

by definition of $Pd(k,n)$.

Assume now that the identity is true for an arbitrary $i \geq 2$. We shall show that it is true for $i+1$, i.e.

$$Pd(k,n) = F_{i+2}Pd(k,n-(i+2))+F_{i+1}Pd(k,n-(i+3))+\sum_{t=0}^{i+1} F_tPd(k,n-k-t).$$

$$(F_{i+1} + F_i)\, Pd(k,n-(i+2)) + F_{i+1}Pd(k,n-(i+3))+$$
$$+ \sum_{t=0}^{i} F_tPd(k,n-k-t) + F_{i+1}Pd(k,n-k-(i+1)) =$$
$$= F_{i+1}Pd(k,n-(i+2)) + F_iPd(k,n-(i+2)) + F_{i+1}Pd(k,n-(i+3))+$$
$$+F_{i+1}Pd(k,n-k-(i+1)) + \sum_{t=0}^{i} F_tPd(k,n-k-t) =$$
$$= F_{i+1}\left(Pd(k,n-(i+2)) + Pd(k,n-(i+3)) + Pd(k,n-k-(i+1))\right) +$$
$$+F_iPd(k,n-(i+2)) + \sum_{t=0}^{i} F_tPd(k,n-k-t) =$$
$$= F_{i+1}Pd(k,n-(i+1)) + F_iPd(k,n-(i+2)) + \sum_{t=0}^{i} F_tPd(k,n-k-t) =$$
$$= Pd(k,n+1)$$

by induction's assumption. □

Using the same method we can prove the similar identity for $Qd(k,n)$ and F_n.

Theorem 2. *Let $k \geq 1$, $n \geq k$ be integers. Then for fixed $1 \leq i \leq n-k$ holds*

$$Qd(k,n) = F_{i+1}Qd(k,n-(i+1))+F_iQd(k,n-(i+2))+\sum_{t=0}^{i} F_tQd(k,n-k-t).$$

2 Graph interpretations of $Qd(k,n)$ and relations with $Pd(k,n)$

In this section we use the graph tools for studying properties of $Pd(k,n)$ and $Qd(k,n)$. These properties are closely related to the concept of $\mathcal{H}$-matchings in graphs. Let G be a given graph. For a given collection $\mathcal{H} = \{H_1, H_2, ..., H_m\}$, $m \geq 1$ of graphs by an $\mathcal{H}$-matchings M of G we mean a family of subgraphs of G such that each connected component of M is isomorphic to some H_i, $1 \leq i \leq m$. If all $H_i \in \mathcal{H}$ are isomorphic to the same graph H then $\mathcal{H}$-matching is an $\mathcal{H}$-matching M in graph. If M cover the set $V(G)$ then M is a perfect matching. If M is also an induced subgraph of G, then the $\mathcal{H}$-matching is called induced. We can observe that if $H = K_2$, then K_2-matching is a matching in the classical sense. If $H = K_1$, then an induced K_1-matching is a well-known independent set. The number of all matchings in the graph G is known as the Hosoya index and it is denote by $Z(G)$, see [7]. The number of all induced K_1-matchings (i.e. independent sets) in the graph G is the Merrifield-Simmons index and it is denoted by $\sigma(G)$, see the last survey and its references, [5]. For graph and combinatorics concepts not defined here see [1] and [3].

In [12] it has been proved:

Theorem 3. [12] *Let $k \geq 3$, $n \geq 1$ be integers. Then the number of perfect $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_k\}$-matchings of a graph $\mathbb{P}_n$ is equal to $Pd(k, n+k-2)$.*

Using this theorem we can prove the result for the graph interpretation of $Qd(k, n)$.

Theorem 4. *Let $k \geq 3$, $n \geq k$ be integers. Then the number of all perfect $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_k\}$-matchings of $\mathbb{C}_n$ is equal to $Qd(k, n)$.*

Proof. Let k, n be as in the statement of the theorem, and let $M \subset \mathbb{C}_n$ be a perfect $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_k\}$-matchings of $\mathbb{C}_n$. Then for an arbitrary vertex $x \in V(\mathbb{C}_n)$ there exists a subgraph $\mathbb{P}_i \in M$, for $i \in \{1, 2, k\}$ such that $x \in \mathbb{P}_i$. Assume that vertices from $V(\mathbb{C}_n)$ are numbered in the natural fashion and without loss of the generality we can choose the vertex x_1. Let q_i be the number of all perfect $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_k\}$-matchings such that $x_1 \in \mathbb{P}_i$, $i \in \{1, 2, k\}$. Hence the number of all perfect matchings of a graph $\mathbb{C}_n$ is equal to $q_1 + q_2 + q_k$. We distinguish the following cases.

1. $x_1 \in \mathbb{P}_1$, where $\mathbb{P}_1 \in M$.
   Then $M = M_1 \cup \mathbb{P}_1$ where M_1 is a perfect $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_k\}$-matching of the graph $\mathbb{C}_n \setminus \{x_1\} \simeq \mathbb{P}_{n-1}$. Using Theorem 3 we have that $q_1 = Pd(k, n+k-3)$.

2. $x_1 \in \mathbb{P}_2$, where $\mathbb{P}_2 \in M$.
   Then $M = M_2 \cup \mathbb{P}_2$ where M_2 is a perfect $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_k\}$-matching either of a the graph $\mathbb{C}_n \setminus \{x_1, x_2\}$ or $\mathbb{C}_n \setminus \{x_n, x_2\}$. Both of these graphs are isomorphic to $\mathbb{P}_{n-2}$. Using Theorem 3 we have that $q_2 = 2Pd(k, n+k-4)$.

3. $x_1 \in \mathbb{P}_k$, where $\mathbb{P}_k \in M$, $k \geq 3$.
   Then $M = M_3 \cup \mathbb{P}_k$ where M_3 is a perfect $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_k\}$-matching of

the graph $\mathbb{C}_n \setminus \mathbb{P}_k$, which is isomorphic to $\mathbb{P}_{n-k}$. Since there exists k subgraphs $\mathbb{P}_k$ containing the vertex $x_k \in \mathbb{P}_k$ so $q_3 = k \cdot Pd(k, n-2)$.

From the above cases
$q_1 + q_2 + q_k = Pd(k, n+k-3) + 2Pd(k, n+k-4) + kPd(k, n-2)$.
Claim. $Qd(k,n) = Pd(k, n+k-3) + 2Pd(k, n+k-4) + kPd(k, n-2)$.

Proof. (Proof of Claim by induction on n).
If $n = 2$, then $Pd(k, k-1) + 2Pd(k, k-2) + kPd(k, 0) = 3 = Qd(k, 2)$.
Assume that the Claim is true for $t < n$. From the definition $Qd(k,n) = Qd(k, n-1) + Qd(k, n-2) + Qd(k, n-k)$. Transforming right side of the above equation using our induction's assumption we have

$$\begin{aligned}&Pd(k,n+k-4) + 2Pd(k,n+k-5) + kPd(k,n-3)+\\&+Pd(k,n+k-5) + 2Pd(k,n+k-6) + kPd(k,n-4)+\\&+Pd(k,n-3) + 2Pd(k,n-4) + kPd(k,n-k-2) =\end{aligned}$$

$$\begin{aligned}&= Pd(k,n+k-4) + Pd(k,n+k-5) + Pd(k,n-3)+\\&+2\left(Pd(k,n+k-5) + Pd(k,n+k-6) + Pd(k,n-4)\right)+\\&+k\left(Pd(k,n-3) + Pd(k,n-4) + Pd(k,n-k-2)\right)\end{aligned}$$

The definitions of the number $Pd(k,n)$ implies that
$Qd(k,n) = Pd(k, n+k-3) + 2Pd(k, n+k-4) + kPd(k, n-2)$,
which ends the proof. □

If $k = 1$, then it is well-known that $Pd(1,n)$ is the classical Pell numbers P_n and $P_n = Z(\mathbb{P}_n \circ K_1)$, where $\mathbb{P}_n \circ K_1$ is the corona of graph $\mathbb{P}_n$ and K_1, see [5]. For $k = 1$ the number $Qd(1,n)$ is the classical companion Pell number Q_n and $2Q_n = Z(\mathbb{C}_n \circ K_1)$. Now we study the graph interpretation of $Pd(k,n)$ and $Qd(k,n)$ for $k \geq 2$.

If $k = 2$, then $Pd(2,n)$ gives the Jacobsthal number J_n and $Qd(2,n)$ gives the Jacobsthal-Lucas numbers j_n. In [12] it was observed that for $n \geq 1$, $t \geq 1$ we have $\sigma\left(P_n[K_t]\right) = J_{t,n+2}$.

For the Jacobsthal-Lucas numbers we can observe

Observation. Let $n \geq 3$ be integer. Then $\sigma(\mathbb{C}_n \circ K_1) = j_n$.

We give the two-parameter generalizations of the Jacobsthal numbers J_n and j_n which are closely related to distance d-independent sets in graphs.

Let $d \geq 2$ be integer. A subset $I \subseteq V(G)$ is a d-independent set of G if for each $u, v \in I$, $d_G(u,v) \geq d$. For $d = 2$ we obtain the definition of independent set in the classical sense. Let $\sigma_d(G)$ be the number of all d-independent sets in G. The definition of d-independent sets immediately implies that for an arbitrary graph G holds $\sigma_d(G) \geq |V(\sigma)| + 1$, for $d \geq 2$.

The theory of independence in graphs is intensively studied in the literature. It is worth to mention that special distance independent sets in digraphs are studied by H. Galeana-Sánchez also with C. Hernández-Cruz, see their last interesting papers [4], [6].

Let $n \geq 0$, $t \geq 1$, $d \geq 2$ be integers. The nth distance Jacobsthal number $J(d,t,n)$ we define recursively in the following way:

$$J(d,t,n) = J(d,t,n-1) + t \cdot J(d,t,n-d) \text{ for } n \geq d$$

with initial conditions
$J(d,t,0) = 0$,
$J(d,t,n) = 1$, for $n = 1,...,d$.

The below Table presents initial words of some distance Jacobsthal sequences for $d = 3$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$J(3,1,n)$	0	1	1	1	2	3	4	6	9	13	19	28	41
$J(3,2,n)$	0	1	1	1	3	5	7	13	23	37	63	109	183
$J(3,3,n)$	0	1	1	1	4	7	10	22	43	73	139	268	487
$J(3,4,n)$	0	1	1	1	5	9	13	33	69	121	253	529	1013
$J(3,5,n)$	0	1	1	1	6	11	16	46	101	181	411	916	1821
$J(3,6,n)$	0	1	1	1	7	13	19	61	139	253	619	1453	2971

Table 3. The distance Jacobsthal numbers $J(3,t,n)$.

If $t = 1$ and $d = 2$, then this definition reduces to the definition classical Fibonacci numbers.

For $t = 1$ and arbitrary $d \geq 2$ we obtain generalized Fibonacci number $F(d,n)$ introduced in [10].

We shall show some applications of numbers $J(d,t,n)$ for determining of $\sigma_d(G)$, where G is a special graph product. Let G be a graph on $V(G) = \{u_1,...,u_n\}$, $n \geq 2$ and $h_n = (H_i)_{i \in \{1,...,n\}}$ be a sequence of vertex disjoint graphs on $V(H_i) = \{(u_i,y_1),...,(u_i,y_x)\}$, $x \geq 1$. By the generalized lexicographic product of G and $h_n = (H_i)_{i \in \{1,...,n\}}$ we mean the graph $G[h_n]$ such that $V(G[h_n]) = \bigcup_{i=1}^{n} V(H_i)$ and $E(G[h_n]) = \{(u_i,y_p)(u_j,y_q); (u_i = u_j \text{ and } (u_i,y_p)(u_j,y_q) \in E(H_i)) \text{ or } u_iu_j \in E(G)\}$.

By H_i^c, $i = 1,...,n$ we mean the copy of the graph H_i in $G[h_n]$. If $H_i = H$ for $i = 1,...,n$, then $G[h_n]$ gives the classical lexicographic product of two graphs $G[H]$. For d-independent sets in $G[h_n]$, see [11].

If $d = 2$, then the Jacobsthal number $J(2,t,n)$ denoted as $J_{t,n}$ has been studied in [12] with respect to the number of independent sets in $\mathbb{P}_n[K_t]$.

Theorem 5. [12] *Let $n \geq 1, t \geq 1$ be integers. Then* $\sigma(\mathbb{P}_n[K_t]) = J_{t,n+2} = J(2,t,n+2)$.

Theorem 6. *Let $n \geq 2$, $t \geq 1$, $d \geq 3$ be integers. Then for an arbitrary sequence $h_n = (H_i)_{i\in\{1,\dots,n\}}$ of vertex disjoint graphs on t vertices* $\sigma_d(\mathbb{P}_n[h_n]) = J(d,t,n+d)$.

Proof. (by induction on n) Let n, t, d be as in the statement of the theorem. If $n = 2,\dots,d$, then $d \geq 3$ implies that every nonempty d-independent set of $\mathbb{P}_n[h_n]$ is a singleton, so $\sigma_d(\mathbb{P}_n[h_n]) = nt + 1 = J(d,t,n+d)$, by the definition of $J(d,t,n)$.

Assume that for an arbitrary subsequence of h_n holds $\sigma_d(\mathbb{P}_m[h_m^*]) = J(d,t,m+d)$, where $m < n$. Let $I \subset V(\mathbb{P}_n[h_n])$ be a d-independent set of the graph $\mathbb{P}_n[h_n]$. We consider the following cases:

1. $(u_n, y_p) \notin I$, for each $p = 1,\dots,t$.
   Then $I = I^*$, where I^* is an arbitrary d-independent set of the graph $\mathbb{P}_n[h_n] \setminus \bigcup_{p=1}^{t} (u_n, y_p)$ which is isomorphic to $\mathbb{P}_{n-1}[h_{n-1}]$. By induction's assumption there are $J(d,t,n-1+d)$ d-independent sets containing no vertex $(u_n, y_p) \notin I$.

2. There is $1 \leq p \leq t$ such that $(u_n, y_p) \in I$.
   The definition of $\mathbb{P}_n[h_n]$ implies that for arbitrary two vertices from $V(H_n^c)$ the distance between them is at most 2. So only one vertex from the copy H_n^c can belong to d-independent set I. If $(u_n, y_p) \in I$, then $(u_n, y_q) \notin I$ for $q \neq p = 1,\dots,t$ and $(u_{n-i}, y_j) \notin I$ for $i = 1,\dots,d$ and $j = 1,\dots,t$. This means that $I = I' \cup \{(u_n, y_p)\}$, where I' is an arbitrary d-independent set of the graph $\mathbb{P}_{n-d}[h_{n-d}] \setminus \bigcup_{i=0}^{d} \bigcup_{j=1}^{t} \{(x_{n-i}, y_j)\}$ which is isomorphic to $\mathbb{P}_{n-d}[h_{n-d}^*]$. By induction's hypothesis there are $J(d,t,n)$ d-independent sets I containing the vertex (u_n, y_p). Since the vertex $(u_n, y_p) \in I$ can be chosen on t ways, so the total number of d-independent set including a vertex from the copy H_n^c is equal to $t \cdot J(d,t,n)$.

From the above cases we obtain that

$$\sigma_d(\mathbb{P}_n[h_n]) = J(d,t,n+d-1) + t \cdot J(d,t,n) = J(d,t,n+d).$$

Thus the theorem is proved. □

Using this graph interpretation we give the direct formula for the Jacobsthal number $J(d,t,n)$.

Theorem 7. *Let $d \geq 2$, $t \geq 1$, $n \geq 1$ be integers. Then*

$$J(d,t,n) = \sum_{p \geq 0} \binom{n-d-(p-1)(d-1)}{p} t^p.$$

Proof. If $n = 1$, then $p = 0$ or $p = 1$ and $J(d,t,n) = 1 + (n-d)t$.

Let $n \geq 2$. Let $j(d,n,p)$ be the number of p-elements d-independent set in the graph $\mathbb{P}_n$ with the numbering of $V(\mathbb{P}_n)$ with the natural fashion. We shall show that

$$j(d,n,p) = \binom{n-(p-1)(d-1)}{p}.$$

Let $(a_1, ..., a_n)$ be a binary sequence associated with the graph $\mathbb{P}_n$ and a d-independent set of $I \subset V(\mathbb{P}_n)$ such that

$$a_i = \begin{cases} 0 & \text{if} \quad u_i \notin I \\ 1 & \text{if} \quad u_i \in I. \end{cases}$$

It is clear that for each a_i, a_j such that $a_i = a_j = 1$ holds $|i-j| \geq d$. In the other words there are at least $d-1$ zeros between two consecutive 1's. Hence p 1's we can put on $n-(p-1)(d-1)$ places in the sequence $(a_1, ..., a_n)$. From the combinatorial statements we can do it on $\binom{n-(p-1)(d-1)}{p}$ ways. In remaining places we give 0's. Next we extend this sequence by adding $d-1$ zeros between every two consecutive 1's. Consequently $j(d,t,n) = \binom{n-(p-1)(d-1)}{p}$.

Let consider the graph $\mathbb{P}_n[h_n]$, where $h_n = (H_i)_{i \in \{1,...,n\}}$ is the sequence of arbitrary vertex disjoint graphs. From the fact that at most one vertex from each H_i^c, $i \in \{1, ..., n\}$ can belong to an d-independent set of $\mathbb{P}_n[h_n]$ we conclude that are $\binom{n-d-(p-1)(d-1)}{p} t^p$ p-element d-independent subsets of $\mathbb{P}_n[h_n]$ and consequently $\sigma_d(\mathbb{P}_n[h_n]) = \sum_{p \geq 0} \binom{n-d-(p-1)(d-1)}{p} t^p$. Using Theorem 6 we obtain that $J(d,t,n) = \sum_{p \geq 0} \binom{n-d-(p-1)(d-1)}{p} t^p$. If $d = 2$, then the formula follows from Theorem 5 which ends the proof. □

Let $n \geq 0$, $t \geq 1$, $d \geq 2$ be integers. The nth distance Jacobsthal-Lucas number $JL(d,t,n)$ is defined by the following recurrence relation.

$$JL(d,t,n) = JL(d,t,n-1) + t \cdot JL(d,t,n-d) \text{ for } n \geq d \qquad (1)$$

with the initial conditions
$JL(d,t,0) = d$ and
$JL(d,t,i) = 1$ for $i = 1, ..., d-1$.

If $t = 1$ and $d = 2$, then this definition gives the Lucas numbers L_n. For $t = 1$ and an arbitrary $d \geq 2$ we obtain the generalized Lucas numbers $L(d,n)$ introduced in [10].

If $t = 2$ and $d = 2$, then $JL(2, 2, n)$ gives the classical Jacobsthal-Lucas numbers j_n.

The following Table presents initial words of distance Jacobsthal-Lucas sequences for $d = 3$

n	0	1	2	3	4	5	6	7	8	9	10	11
$JL(3,1,n)$	3	1	1	4	5	6	10	15	21	31	46	67
$JL(3,2,n)$	3	1	1	7	9	11	25	43	65	115	201	331
$JL(3,3,n)$	3	1	1	10	13	16	46	85	133	271	526	925
$JL(3,4,n)$	3	1	1	13	17	21	73	141	225	517	1081	1981
$JL(3,5,n)$	3	1	1	16	21	26	106	211	341	871	1926	3631
$JL(3,6,n)$	3	1	1	19	25	31	145	295	481	1351	3121	6007

Table 4. The distance Jacobsthal-Lucas sequences $JL(3, t, n)$.

In case $d = 2$ we can observe the following result.

Observation. Let $n \geq 3$, $t \geq 1$ be integers. Then

$$\sigma(\mathbb{C}_n[K_t]) = j_n.$$

We give the graph interpretation of the distance Jacobsthal-Lucas numbers $JL(d, t, n)$ for a general case $d \geq 3$.

Theorem 8. *Let $n \geq 3$, $t \geq 1$, $d \geq 3$ be integers. Then for an arbitrary sequence $h_n = (H_i)_{i\in\{1,\dots,n\}}$ of vertex disjoint graphs on t vertices*

$$\sigma_d(\mathbb{C}_n[h_n]) = JL(d, t, n).$$

Proof. Let n, t, d be as in the statement of the theorem. If $3 \leq n \leq 2d-1$, then every nonempty d-independent set of $\mathbb{C}_n[h_n]$ has exactly one vertex, so $\sigma_d(\mathbb{C}_n[h_n]) = nt + 1 = JL(d, t, n)$. Assume that for an arbitrary subsequence h_n^* of the sequence h_n holds $\sigma_d(\mathbb{C}_m[h_m^*]) = JL(d, t, m)$, where $m < n$.

Let $n \geq 2d$ and let I be an arbitrary d-independent set of the graph $\mathbb{C}_n[h_n]$. Let $V(\mathbb{C}_n[h_n]) \supset D = \bigcup_{i=1}^{d-1} V(H_i^c)$.

We distinguish the following cases:

1. $|D \cap I| = \emptyset$
   Then $I = I^*$, where I^* is an arbitrary d-independent set of the graph $\mathbb{C}_n[h_n] \setminus D$ which is isomorphic to $\mathbb{P}_{n-(d-1)}\left[h_{n-(d-1)}^*\right]$. By Theorem 6 there are $J(d, t, n+1)$ d-independent sets containing no vertex from the set D.

2. $|D \cap I| \neq \emptyset$
   Since the subset D induces in $\mathbb{C}_n[h_n]$ subgraph with diameter equals to $d-2$ so $|D \cap I| = 1$. Assume that $|V(H_i^c) \cap D| = 1$, where $1 \leq l \leq d-1$. Let $(u_l, y_p) \in I \cap D$, where $1 \leq p \leq t$. Then $N_d[(u_l, y_p)] \cap I = \emptyset$ where $N_d[x]$ is the d-distance close neighbourhood of the vertex x. This implies that $I = I' \cup \{(u_l, y_p)\}$ where I' is an arbitrary d-independent set of the graph $\mathbb{C}_n \setminus N_d[(u_l, y_p)]$ which is isomorphic to $\mathbb{P}_{n-(2d-1)}\left[h^*_{n-(2d-1)}\right]$. The Theorem 6 implies that there are $J(d,t,n-d+1)$ d-independent sets containing the vertex (u_l, y_p).

Since the vertex (u_l, y_p) can be chosen on $t(d-1)$ ways hence from the above cases we obtain that

$$\sigma_d\left(\mathbb{C}_n[h_n]\right) = J(d,t,n+1) + t(d-1)J(d,t,n-d+1).$$

Claim.

$$J(d,t,n+1) + t(d-1)J(d,t,n-d+1) = JL(d,t,n) \text{ for } n \geq d-1. \quad (2)$$

Proof. (Proof of Claim by induction on n).
If $n = d-1, \ldots, 2d-1$ then the result follows by simple calculations. Let $n \geq 2d$ and suppose that the Claim is true for $m \leq n$.
From the definition $JL(d,t,n) = JL(d,t,n-1) + t \cdot JL(d,t,n-d)$.
Using induction's assumption for the right side of this equation we have

$$\begin{aligned}
&JL(d,t,n-1) + t \cdot JL(d,t,n-d) = \\
&= J(d,t,n) + t(d-1)J(d,t,n-d) + t\,[J(d,t,n-d+1)+ \\
&\quad +t(d-1)J(d,t,n-2d+1)] = \\
&= J(d,t,n) + t \cdot J(d,t,n-d+1) + t(d-1)\,[J(d,t,n-d)+ \\
&\quad +t \cdot J(d,t,n-2d+1)] = \\
&= J(d,t,n+1) + t(d-1)J(d,t,n-d+1),
\end{aligned}$$

which ends the proof of the Claim.

Consequently $\sigma_d(\mathbb{C}_n[h_n]) = JL(d,t,n)$ and the theorem is proved. □

The identity (2) can be used to obtain the direct formula the number $JL(d,t,n)$. Using it in the Theorem 7 it immediately follows.

Theorem 9. *Let $d \geq 2$, $t \geq 1$, $n \geq 1$ be integers. Then*

$$JL(d,t,n) = \sum_{p\geq 0} \binom{n-p(d-1)}{p} t^p + t(d-1) \sum_{q \geq 0} \binom{n-d-q(d-1)}{q} t^q.$$

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