Signed cycle domination numbers of digraphs *

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Abstract: The concept of signed cycle domination number of graphs introduced by B. Xu [B. Xu, On signed cycle domination in graphs, Discrete Math. 309 (2009)1007-1012] is extended to digraphs, denoted by $\gamma'_{sc}(D)$ for a digraph D. We obtain bounds on γ'_{sc}, characterize all digraphs D with $\gamma'_{sc}(D) = |A(D)|-2$ and determine the exact value of $\gamma'_{sc}(D)$ for some special classes of digraphs D. Moreover, we define a parameter $g'(m,n) = \min\{\gamma'_{sc}(D) \mid D \text{ is a digraph with } |V(D)| = n \text{ and } |A(D)| = m\}$ and obtain its value for all integers n and m satisfying $0 \leq m \leq n(n-1)$.
Keywords: Induced cycle; Signed cycle domination function on digraphs; Signed cycle domination number of digraphs

1 Introduction

All digraphs considered in this paper are finite without loops or multiple arcs. The vertex set and arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. For a vertex set X of D, the subdigraph induced by X in D is denoted by $D\langle X\rangle$ and $D - X = D\langle V(D)\backslash X\rangle$. A directed cycle C of D is said to be an *induced cycle* if $D\langle V(C)\rangle = C$, and we use $A(C)$ to denote the arc set of C.

In the last decade, some kinds of domination for graphs have been investigated such as signed domination (see [2,3,7,10]), signed k-domination (see [4]), signed total domination (see [5,14]), signed edge domination (see [8,11,12]), signed star domination (see [9,11]), signed cycle domination (see [13]), and so on. Most of those belong to *the vertex domination* of graphs, only a few results have been obtained about *edge domination* of graphs.

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For digraphs, the known results on this topic are more less (see [15] and [6]).

In this paper, we extend the concept of signed cycle domination number of graphs (introduced by B. Xu in [13]) to digraphs.

Let $D = (V(D), A(D))$ be a digraph with $A(D) \neq \emptyset$. A function $f : A(D) \to \{-1, 1\}$ is called a *signed cycle domination function* (abbreviated SCDF) on D if $\sum_{e \in A(C)} f(e) \geq 1$ holds for every induced cycle C of D. The *signed cycle domination number* $\gamma'_{sc}(D)$ of D is defined as: $\gamma'_{sc}(D) = \min\{\sum_{e \in A(D)} f(e) \mid f$ is an SCDF on $D\}$ when $A(D) \neq \emptyset$; $\gamma'_{sc}(D) = 0$ when $A(D) = \emptyset$. An SCDF f is called a *minimum signed cycle domination function* on D if $\sum_{e \in A(D)} f(e) = \gamma'_{sc}(D)$.

We present some bounds on γ'_{sc}, characterize all digraphs D with $\gamma'_{sc}(D) = |A(D)| - 2$ and determine the exact value of $\gamma'_{sc}(D)$ for some special classes of digraphs D. Moreover, we define a parameter $g'(m, n) = \min\{\gamma'_{sc}(D) \mid D$ is a digraph with $|V(D)| = n$ and $|A(D)| = m\}$ and obtain its value for all integers n and m satisfying $0 \leq m \leq n(n-1)$.

2 Terminology and preliminaries

We refer the reader to [1] for terminology and notation not defined here. Let $D = (V(D), A(D))$ be a digraph. If xy is an arc of D, then we denote it by $x \to y$. In the case when $x \to y$ and there is no arc from y to x, we write $x \mapsto y$. A directed cycle (or path) of order k is called a *k-cycle* (or *k-path*), denoted by C_k (or P_k). If an arc is contained in a 2-cycle, then we say that this arc is *bioriented*. Let $C = x_1x_2...x_\ell x_1$ be a directed cycle of a digraph D. Then we call the arc x_ix_j a *chord* of C in D if it belongs to $A(D) \setminus A(C)$ for some $i, j \in \{1, 2, ..., \ell\}$.

The *underlying graph* $UG(D)$ of D is the graph obtained by ignoring all orientations on the arcs of D and deleting possible multiple edges arising in this way. We say that D is *connected* if $UG(D)$ is a connected graph. A *connected component* of a digraph D is a maximal induced subdigraph of D which is connected.

A *tournament* is a digraph such that for every pair of distinct vertices, there is exactly one arc between them. An acyclic tournament is called a *transitive tournament*.

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ and $xy \in A(\overline{D})$ if and only if $xy \notin A(D)$. K_n is a digraph of order n such that for any two distinct vertices x and y, there are two mutually opposite arcs xy and yx.

For two digraphs D_1 and D_2, we define $D = D_1 \cup D_2$ to be a digraph with $V(D) = V(D_1) \cup V(D_2)$ and $A(D) = A(D_1) \cup A(D_2)$. Moreover, we define $H = D_1 + D_2$ to be a digraph with $V(H) = V(D_1) \cup V(D_2)$ and

$A(H) = A(D_1) \cup A(D_2) \cup \{xy, yx \mid$ for every vertex $x \in V(D_1)$ and $y \in V(D_2)\}$.

According to the definition of signed cycle domination number of digraphs, we can easily get the following observation.

Observation 2.1. *For any digraph D, $|A(D)| \geq \gamma'_{sc}(D) \geq -|A(D)|$. Moreover, for any two digraphs D_1 and D_2, $\gamma'_{sc}(D_1 \cup D_2) = \gamma'_{sc}(D_1) + \gamma'_{sc}(D_2)$.*

For convenience, we define $\pi(D) = \frac{1}{2}(|A(D)| - \gamma'_{sc}(D))$. Then $\gamma'_{sc}(D) = |A(D)| - 2\pi(D)$ for $\pi(D) = 0, 1, ..., |A(D)|$, and immediately, we obtain the following two useful lemmas.

Lemma 2.2. *Let D be a digraph with $A(D) \neq \emptyset$ and f be a minimum SCDF on D. Then $\pi(D) = |\{e \in A(D) \mid f(e) = -1\}|$.*

Proof. Let $E = \{e \in A(D) \mid f(e) = 1\}$ and $F = \{e \in A(D) \mid f(e) = -1\}$. Then $\pi(D) = \frac{1}{2}(|A(D)| - \gamma'_{sc}(D)) = \frac{1}{2}[(|E| + |F|) - (|E| - |F|)] = |F| = |\{e \in A(D) \mid f(e) = -1\}|$. □

Lemma 2.3. *Let e be an arc of a digraph D such that e is contained in a 2-cycle. Then for any SCDF f on D, we have $f(e) = 1$.*

To present our main results, we define the following four particular classes of digraphs:

(1) $\mathcal{T} = \{\mathcal{D} \mid D$ is a digraph and every arc of D is contained in a 2-cycle$\}$;
(2) $\mathcal{F} = \{\mathcal{D} \mid D$ is a connected digraph and every arc except one of D is contained in a 2-cycle$\}$;
(3) $\mathcal{L} = \{\mathcal{D} \mid D$ is a connected digraph as shown in Fig. 1, where $D_1 \in \mathcal{T}\}$;
(4) $\mathcal{H} = \{\mathcal{D} \mid D$ is a connected digraph as shown in Fig. 2, where $D_2 \in \mathcal{T}\}$.

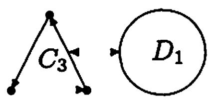

Fig.1. The arrow ↔ denotes that any arc between C_3 and D_1 is bioriented.

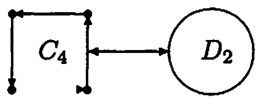

Fig.2. The arrow ↔ denotes that any arc between C_4 and D_2 is bioriented.

3 Main results

First we give two characterizations of a digraph D with $\gamma'_{sc}(D) = -|A(D)|$ and $\gamma'_{sc}(D) = |A(D)|$, respectively.

Theorem 3.1. $\gamma'_{sc}(D) = -|A(D)|$ *if and only if D has no directed cycles.*

Proof. As the sufficiency is clear, we will only prove the necessity. Assume to the contrary that there exists a directed cycle in D, then the shortest directed cycle C is just an induced cycle. Let f be a minimum

SCDF on D. Since $\gamma'_{sc}(D) = -|A(D)|$, then $f(e) = -1$ for every $e \in A(D)$. So $\Sigma_{e \in A(C)} f(e) \leq -2$. This contradicts the definition of SCDF. Therefore, D has no directed cycles.□

Theorem 3.2. *$\gamma'_{sc}(D) = |A(D)|$ if and only if $D \in \mathcal{T}$.*

Proof. The sufficiency is clear by Lemma 2.3. Now we prove the necessity. Assume to the contrary that there exists an arc e not contained in any 2-cycle. Then define $f(e) = -1$ and $f(e') = 1$ for all arcs $e' \in A(D) \setminus \{e\}$. It is easy to check that f is an SCDF on D. According to Lemma 2.2, we have $\pi(D) \geq 1$, and then, $\gamma'_{sc}(D) \leq |A(D)| - 2$. This yields a contradiction. So every arc of D is contained in a 2-cycle, i.e., $D \in \mathcal{T}$. □

The following theorem provides a characterization of a connected digraph D with $\gamma'_{sc}(D) = |A(D)| - 2$ and is proved by five cases: (1) no cycles in D, (2) no 2-cycles in D but D has an induced 3-cycle, (3) no 2-cycles in D and no induced 3-cycles in D, (4) D has a 2-cycle and an induced 3-cycle, (5) D has a 2-cycle but no induced 3-cycles. Note that $\gamma'_{sc}(D) = |A(D)| - 2$ if and only if $\pi(D) = 1$.

Theorem 3.3. *Let D be a connected digraph. Then $\gamma'_{sc}(D) = |A(D)| - 2$ if and only if $D \in \{P_2, C_3, C_4\}$, or $D \in \mathcal{L}$, or $D \in \mathcal{F}$, or $D \in \mathcal{H}$.*

Proof. (Sufficiency). It is obvious by Lemma 2.3.

(Necessity). If D has no directed cycles, then by Theorem 3.1, $\gamma'_{sc}(D) = -|A(D)|$. Combining with $\gamma'_{sc}(D) = |A(D)| - 2$, we have $|A(D)| = 1$. So $D = P_2$. Assume in the following that D has at least one directed cycle, which implies that D has an induced cycle.

Claim 1. D has no induced cycle of length more than 4.

Proof. Assume to the contrary that D has an induced cycle C of length more than 4. Then let e_1 and e_2 be two independent arcs of C. This implies that e_1 and e_2 cannot simultaneously lie in an induced cycle of length 3 or 4. Define $f(e_1) = f(e_2) = -1$ and $f(e) = 1$ for all arcs $e \in A(D) \setminus \{e_1, e_2\}$. It is not difficult to check that f is an SCDF on D. So $\pi(D) \geq 2$, which leads to a contradiction.□

Case 1. D has no 2-cycles but D has an induced 3-cycle.

Let $C_3 = u_1u_2u_3u_1$ be an induced 3-cycle. Then $V(D) \setminus \{u_1, u_2, u_3\} = \emptyset$. In fact, if $V(D) \setminus \{u_1, u_2, u_3\} \neq \emptyset$, then by the connectivity of D, there exists a vertex $v \in V(D) \setminus \{u_1, u_2, u_3\}$ such that v is adjacent to C_3. Assume without loss of generality that $v \rightarrow u_1$. Since D has no 2-cycles, we have $v \mapsto u_1$. Define $f(vu_1) = f(u_3u_1) = -1$ and $f(e) = 1$ for all arcs $e \in A(D) \setminus \{vu_1, u_3u_1\}$. Clearly, f is an SCDF on D, and then, $\pi(D) \geq 2$. This contradicts the assumption of this theorem. Therefore, $D = C_3$.

Case 2. D has no 2-cycles and no induced 3-cycles.

In this case, D has an induced 4-cycle by Claim 1. Let $C_4 = u_1u_2u_3u_4u_1$ be such one. If $V(D) \setminus \{u_1, u_2, u_3, u_4\} \neq \emptyset$, then by the connectivity of D, there exists a vertex $v \in V(D) \setminus \{u_1, u_2, u_3, u_4\}$ such that v is adjacent to C_4. Assume without loss of generality that $v \rightarrow u_1$. Define $f(vu_1) =$

$f(u_4u_1) = -1$ and $f(e) = 1$ for all arcs $e \in A(D) \setminus \{vu_1, u_4u_1\}$. Note that D has no 2-cycles. So f is an SCDF on D, and hence, $\pi(D) \geq 2$. This yields a contradiction. Therefore, $V(D) \setminus \{u_1, u_2, u_3, u_4\} = \emptyset$ and $D = C_4$.

Case 3. D has a 2-cycle and an induced 3-cycle.

Let $C_3 = u_1u_2u_3u_1$ be an induced 3-cycle. Since D has a 2-cycle and is connected, then $V(D) \setminus \{u_1, u_2, u_3\} \neq \emptyset$ and there is a vertex in $V(D) \setminus \{u_1, u_2, u_3\}$ adjacent to C_3.

Claim 2. Every arc between C_3 and $D - V(C_3)$ is contained in a 2-cycle.

Proof. Assume without loss of generality that wu_1 is an arc between C_3 and $D - V(C_3)$. If wu_1 is not contained in any 2-cycle, then we define $f(wu_1) = f(u_3u_1) = -1$ and $f(e) = 1$ for all arcs $e \in A(D) \setminus \{wu_1, u_3u_1\}$. Obviously, f is an SCDF on D. So $\pi(D) \geq 2$, a contradiction.□

Claim 3. Every arc in $D - V(C_3)$ is contained in a 2-cycle, i.e., $D - V(C_3) \in \mathcal{T}$.

Proof. Let v_1v_2 be an arbitrary arc in $D-V(C_3)$. If v_1v_2 is not contained in any 2-cycle, then we define $f(v_1v_2) = f(u_1u_2) = -1$ and $f(e) = 1$ for all arcs $e \in A(D) \setminus \{v_1v_2, u_1u_2\}$. It follows from Claim 2 that u_1u_2 and v_1v_2 cannot lie in an induced 4-cycle simultaneously. So f is an SCDF on D, and then, $\pi(D) \geq 2$, which leads to a contradiction.□

From the discussion above, we conclude that $D \in \mathcal{L}$.

Case 4. D has a 2-cycle but no induced 3-cycles.

Subcase 4.1 D has no induced 4-cycles.

If every arc of D is contained in a 2-cycle, then it follows from Theorem 3.2 that $\gamma'_{sc}(D) = |A(D)|$. This contradicts the assumption of this theorem. So there exists an arc not contained in any 2-cycle.

If there are at least two arcs which are not in any 2-cycle, say e_1 and e_2, then define $f(e_1) = f(e_2) = -1$ and $f(e) = 1$ for all arcs $e \in A(D) \setminus \{e_1, e_2\}$. Clearly, f is an SCDF on D, which implies that $\pi(D) \geq 2$, a contradiction.

Therefore, every arc except one of D is contained in a 2-cycle, i.e., $D \in \mathcal{F}$.

Subcase 4.2 D has an induced 4-cycle.

Let $C_4 = u_1u_2u_3u_4u_1$ be an induced 4-cycle. Since D has a 2-cycle and D is connected, we know $V(D) \setminus \{u_1, u_2, u_3, u_4\} \neq \emptyset$ and there exists a vertex in $V(D) \setminus \{u_1, u_2, u_3, u_4\}$ adjacent to C_4.

By a similar argument as in the proof of Claim 2 and Claim 3, we deduce that every arc between C_4 and $D - V(C_4)$ is contained in a 2-cycle and $D - V(C_4) \in \mathcal{T}$, respedctily. So $D \in \mathcal{H}$. This completes the proof of Theorem 3.3.□

Note that $\gamma'_{sc}(D) \leq |A(D)| - 4$ if and only if $\pi(D) \geq 2$. So by Theorem 3.2 and 3.3 we can easily obtain the following corollary.

Corollary 3.4. *If D is a connected digraph satisfying $D \notin \{P_2, C_3, C_4\}$, $D \notin \mathcal{T}$, $D \notin \mathcal{L}$, $D \notin \mathcal{F}$ and $D \notin \mathcal{H}$, then $\gamma'_{sc}(D) \leq |A(D)| - 4$.*

One could generalize Theorem 3.3 by removing the requirement that D is connected.

Theorem 3.5. *For any digraph D, $\gamma'_{sc}(D) = |A(D)| - 2$ if and only if $D = D_1 \cup D_2$, where $D_2 \in \mathcal{T}$, and $D_1 \in \{P_2, C_3, C_4\}$, or $D_1 \in \mathcal{L}$, or $D_1 \in \mathcal{F}$, or $D_1 \in \mathcal{H}$. Here, D_2 is permitted to be non-existent, and D_2 exists if and only if D is disconnected.*

Proof. Let $D'_1, D'_2, ..., D'_t$ be the connected components of D. Then $D = D'_1 \cup D'_2 \cup ... \cup D'_t$. Note that $t \geq 2$ if and only if D is disconnected. It follows from Observation 2.1 that $\pi(D) = \pi(D'_1) + \pi(D'_2) + ... + \pi(D'_t)$. Recall that $\gamma'_{sc}(D) = |A(D)| - 2$ if and only if $\pi(D) = 1$. We may assume without loss of generality that $\pi(D'_1) = 1$ and $\pi(D'_i) = 0$ for $i = 2, ..., t$. Let $D_1 = D'_1$, and if $t \geq 2$, let $D_2 = D'_2 \cup ... \cup D'_t$. Then $D = D_1 \cup D_2$. According to Theorem 3.3, $\pi(D_1) = 1$ if and only if $D_1 \in \{P_2, C_3, C_4\}$, or $D_1 \in \mathcal{L}$, or $D_1 \in \mathcal{F}$, or $D_1 \in \mathcal{H}$, and from Theorem 3.2, we know $\pi(D_2) = \pi(D'_2) + ... + \pi(D'_t) = 0$ if and only if $D_2 \in \mathcal{T}$. Note that when $t = 1$, D_2 does not exist. So, D_2 exists if and only if D is disconnected.□

Theorem 3.1, Theorem 3.2 and Theorem 3.5 characterize all digraphs D with $\pi(D) = |A(D)|$, $\pi(D) = 0$ and $\pi(D) = 1$, respectively. It is natural to pose the following problem.

Problem 3.6. *Characterize all digraphs D with $\gamma'_{sc}(D) = |A(D)| - 4$, i.e., $\pi(D) = 2$.*

Next we give another sharp lower bound on γ'_{sc} in terms of the order and the size of a digraph.

Theorem 3.7. *Let D be a digraph with $|V(D)| = n$ and $|A(D)| = m$. Then $\gamma'_{sc}(D) \geq m - n^2 + n$ and the equality holds if and only if D is a transitive tournament.*

Proof. First we show the inequality. If D has no 2-cycles, then $\pi(D) \leq m \leq \frac{n(n-1)}{2}$; if D has at least one 2-cycle, then by Lemma 2.3, we have $\pi(D) \leq \frac{n(n-1)}{2}$. Hence, $\gamma'_{sc}(D) = m - 2\pi(D) \geq m - n^2 + n$.

Now we show that $\gamma'_{sc}(D) = m - n^2 + n$ if and only if D is a transitive tournament. As the sufficiency is clear, we will only prove the necessity. If $A(D) = \emptyset$, then $m = 0$ and $\gamma'_{sc}(D) = 0$. Combining with $\gamma'_{sc}(D) = m - n^2 + n$, we have $n = 1$. So, D is a transitive tournament with only one vertex. Assume now that $A(D) \neq \emptyset$. Let f be a minimum SCDF on D and $E = \{e \in A(D) \mid f(e) = 1\}$, $F = \{e \in A(D) \mid f(e) = -1\}$. It is clear that $|F| = \pi(D) = \frac{1}{2}(m - \gamma'_{sc}(D)) = \frac{1}{2}(m - m + n^2 - n) = \frac{n(n-1)}{2}$. Define D_1 to be a subdigraph of D with $V(D_1) = V(D)$ and $A(D_1) = F$. Then $|V(D_1)| = n$ and $|A(D_1)| = \frac{n(n-1)}{2}$.

Now we prove that D_1 has no directed cycles. Assume to the contrary that D_1 has a directed cycle, then the shortest directed cycle C in D_1 is just an induced cycle of D_1. Since $f(e) = -1$ for every arc $e \in A(C) \subseteq F$, the cycle C can not be an induced cycle of D. That is to say the cycle C

has at least one chord belonging to E. Then there exists an induced cycle C' in D with only one arc in E and all the other arcs from C. This implies $\Sigma_{e\in A(C')}f(e) \le 0$, which contradicts the assumption that f is an SCDF on D. So D_1 has no directed cycles. Combining with $|V(D_1)| = n$ and $|A(D_1)| = \frac{n(n-1)}{2}$, we deduce that D_1 is a transitive tournament of order n. It follows that $D = D_1$ is a transitive tournament by Lemma 2.3. □

Corollary 3.8. *For any digraph D with $|V(D)| = n$ and $|A(D)| = m$, if D is not a transitive tournament, then $\gamma'_{sc}(D) \ge m - n^2 + n + 2$.*

Proof. It follows from Theorem 3.7 that $\gamma'_{sc}(D) \ge m - n^2 + n + 1$. This implies that $\pi(D) = \frac{1}{2}(m - \gamma'_{sc}(D)) \le \frac{1}{2}[m - (m - n^2 + n + 1)] = \frac{n(n-1)}{2} - \frac{1}{2}$. Since $\pi(D)$ is an integer, we have $\pi(D) \le \frac{n(n-1)}{2} - 1$. So, $\gamma'_{sc}(D) = m - 2\pi(D) \ge m - n^2 + n + 2$.□

Theorem 3.9. *If D is a digraph of order n and not a transitive tournament, then $\gamma'_{sc}(D) + \gamma'_{sc}(\overline{D}) \ge n - n^2 + 4$.*

Proof. Since D is not a transitive tournament, then $\overline{D}$ is not a transitive tournament, too. According to Corollary 3.8, we have $\gamma'_{sc}(D) + \gamma'_{sc}(\overline{D}) \ge |A(D)| - n^2 + n + 2 + |A(\overline{D})| - n^2 + n + 2 = n(n-1) - 2n^2 + 2n + 4 = n - n^2 + 4$. □

It is very difficult to determine $\gamma'_{sc}(D)$ for a general digraph D, but for some special classes of digraphs, we can easily determine their signed cycle domination numbers.

Theorem 3.10. *(1)For any digraph D of order n, $\gamma'_{sc}(D + K_1) = 2n + \gamma'_{sc}(D)$. (2) For any transitive tournament T of order n, $\gamma'_{sc}(T + K_1) = \frac{5n-n^2}{2}$.*

Proof.(1) If $A(D) = \emptyset$, then $\gamma'_{sc}(D) = 0$ and $D + K_1 \in \mathcal{T}$. Theorem 3.2 implies that $\gamma'_{sc}(D + K_1) = 2n = 2n + \gamma'_{sc}(D)$. Assume now that $A(D) \ne \emptyset$. Let $H = D + K_1$ and $M = A(H) \setminus A(D)$. Obviously, $|M| = 2n$. Let f_1 be a minimum SCDF on D and define an SCDF f on H as follows: $f(e_1) = 1$ for all $e_1 \in M$ and $f(e_2) = f_1(e_2)$ for all $e_2 \in A(D)$. Clearly, f is a minimum SCDF on H and $\gamma'_{sc}(D + K_1) = 2n + \gamma'_{sc}(D)$.

(2) Theorem 3.1 and (1) imply that $\gamma'_{sc}(T + K_1) = 2n - \frac{n(n-1)}{2} = \frac{5n-n^2}{2}$.□

Now we define a new parameter for digraphs: $g'(m, n) = \min\{\gamma'_{sc}(D) \mid D$ is a digraph with $|V(D)| = n$ and $|A(D)| = m\}$. It is natural to pose the following problem.

Problem 3.11. *Determine the exact value of $g'(m, n)$ for all integers n and m satisfying $0 \le m \le n(n-1)$.*

In [13], B. Xu defined a similar parameter and posed a similar problem for graphs. Up to now, his problem remains unsolved. For digraphs, however, Problem 3.11 is fully solved as follows.

For $n = 1$ and $n = 2$, it is clear that $g'(0, 1) = g'(0, 2) = 0$, $g'(1, 2) = -1$ and $g'(2, 2) = 2$. So we only need to consider Problem 3.11 for $n \ge 3$.

Theorem 3.12. *If $n \geq 3$ and $0 \leq m \leq \frac{n(n-1)}{2}$, then $g'(m,n) = -m$.*

Proof. Construct a digraph D with $|V(D)| = n$ and $|A(D)| = m$ as follows: Let $V(D) = \{x_1, x_2, ..., x_n\}$ and add arbitrarily m arcs of the form $x_i x_j$ for some $1 \leq i < j \leq n$. This construction is reasonable since $m \leq \frac{n(n-1)}{2}$. Then D contains no directed cycles. It follows from Theorem 3.2 that $\gamma'_{sc}(D) = -m$ and then $g'(m,n) = -m$.□

Theorem 3.13. *If $n \geq 3$ and $\frac{n(n-1)}{2} + 1 \leq m \leq n(n-1)$, then $g'(m,n) = 3m - 2n(n-1)$.*

Proof. First we construct a digraph D with $|V(D)| = n$ and $|A(D)| = m$ as follows: Let $V(D) = \{x_1, x_2, ..., x_n\}$ and $A(D) = \{x_i x_j \mid i$ and j are all integers satifying $1 \leq i < j \leq n\} \cup A$, where A consists of $(m - \frac{n(n-1)}{2})$ arcs of the form $x_j x_i$ for some $1 \leq i < j \leq n$.

From the construction above we know D has exactly $(m - \frac{n(n-1)}{2})$ 2-cycles and no other induced cycles. Define an SCDF f on D: $f(e_1) = 1$ for all arcs e_1 contained in a 2-cycle and $f(e_2) = -1$ for all arcs e_2 not contained in any 2-cycle. Obviously, f is a minimum SCDF on D and then $\gamma'_{sc}(D) = 3m - 2n(n-1)$. So, $g'(m,n) \leq 3m - 2n(n-1)$ holds.

On the other hand, for any digraph D with $|V(D)| = n$ and $|A(D)| = m$, since $m \geq \frac{n(n-1)}{2} + 1$, the digraph D has at least $(m - \frac{n(n-1)}{2})$ 2-cycles. Lemma 2.3 implies that $\pi(D) \leq m - 2[m - \frac{n(n-1)}{2}] = n(n-1) - m$. It follows that $\gamma'_{sc}(D) = m - 2\pi(D) \geq m - 2[n(n-1) - m] = 3m - 2n(n-1)$. According to the arbitrariness of D, we have $g'(m,n) \geq 3m - 2n(n-1)$.

In conclusion, we have $g'(m,n) = 3m - 2n(n-1)$. □

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