HIGHER-ORDER CAUCHY OF THE SECOND KIND AND POLY-CAUCHY OF THE SECOND KIND MIXED TYPE POLYNOMIALS

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ABSTRACT. In this paper, we investigate some properties of higher-order Cauchy of the second kind and poly-Cauchy of the second mixed type polynomials with umbral calculus viewpoint. From our investigation, we derive many interesting identities of higher-order Cauchy of the second kind and poly-Cauchy of the second kind mixed type polynomials.

1. INTRODUCTION

For $\lambda \in \mathbf{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order α ($\alpha \in \mathbf{N} \cup \{0\}$) are defined by the generating function to be

$$(1) \quad \left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x|\lambda)\frac{t^n}{n!}, \quad \text{(see [2, 4, 5, 6, 13])}.$$

As is well known, the Bernoulli polynomials of order $\alpha \in \mathbf{N} \cup \{0\}$ are also defined by the generating function to be

$$(2) \quad \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad \text{(see [1, 2, 4, 7, 8])}.$$

When $x = 0$, $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ are called the Bernoulli numbers of order α. The Stirling number of the first kind is defined by

$$(3) \quad (x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (n \in \mathbf{N} \cup \{0\}).$$

From (3), we note that

$$(4) \quad (\log(1+x))^m = m!\sum_{l=m}^{\infty} S_1(l,m)\frac{x^l}{l!} = \sum_{l=0}^{\infty}\frac{m!}{(l+m)!}S_1(l+m,m)x^{l+m}.$$

It is known that the Stirling number of the second kind is given by

$$(5)\qquad (e^x-1)^m = m!\sum_{l=m}^{\infty}\frac{S_2(l+m)}{l!}x^l, \quad (m\in\mathbf{N}\cup\{0\}), \quad \text{(see [14, 15])}.$$

The poly-logarithm factorial function is defined by

$$(6)\qquad Lif_k(x)=\sum_{m=0}^{\infty}\frac{x^m}{m!(m+1)^k}, \quad (k\in\mathbf{Z}), \quad \text{(see [9, 10, 11])}.$$

The poly-Cauchy polynomials of the second kind is given by

$$(7)\qquad Lif_k\left(-\log\left(1+x\right)\right)\left(1+t\right)^x=\sum_{n=0}^{\infty}\tilde{C}_n^{(k)}(x)\frac{t^n}{n!}, \quad \text{(see [10, 11, 12])}.$$

and the Cauchy numbers of the second kind with order r ($r\in\mathbf{N}\cup\{0\}$) are defined by the generating function to be

$$(8)\qquad \left(\frac{t}{(1+t)\log\left(1+t\right)}\right)^r=\sum_{n=0}^{\infty}C_n^{(r)}\frac{t^n}{n!}, \quad \text{(see [10, 11, 12])}.$$

Now, we consider the polynomials $\tilde{A}_n^{(r,k)}(x)$ whose generating function is defined by

$$(9)\qquad \left(\frac{t}{(1+t)\log\left(1+t\right)}\right)^r Lif_k\left(-\log\left(1+t\right)\right)\left(1+t\right)^x=\sum_{n=0}^{\infty}\tilde{A}_n^{(r,k)}\frac{t^n}{n!},$$

where $r\in\mathbf{N}\cup\{0\}$ and $k\in\mathbf{Z}$.
$\tilde{A}_n^{(r,k)}(x)$ are called higher-order Cauchy of the second kind and poly-Cauchy of the second kind mixed type polynomials. When $x=0$, $\tilde{A}_n^{(r,k)}=\tilde{A}_n^{(r,k)}(0)$ are called the higher-order Cauchy of the second kind and poly-Cauchy of the second kind mixed type numbers.
Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbf{C}$ as follows:

$$(10)\qquad \mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\,\middle|\,a_k\in\mathbf{C}\right\}.$$

Let $\mathbb{P}=\mathbf{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, and the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle=\langle L|p(x)\rangle+\langle M|p(x)\rangle$, $\langle cL|p(x)\rangle=c\langle L|p(x)\rangle$, where c is a complex constant. For $f(t)\in\mathcal{F}$, let $\langle f(t)|x^n\rangle=a_n$. Then, by (10), we see that

$$(11)\qquad \langle t^k|x^n\rangle=n!\delta_{n,k}, \quad \text{(see [14, 15])},$$

where $\delta_{n,k}$ is the Kronecker's symbol.
Let us assume that $f_L(t)=\sum_{k=0}^{\infty}\langle L|x^k\rangle\frac{t^k}{k!}$. By (11), we see that $\langle f_L(t)|x^n\rangle=\langle L|x^n\rangle$. That is, $f_L(t)=L$. Additionally, the map $L\longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the

algebra of the formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought as a formal power series and a linear functional. $\mathcal{F}$ is called the umbral algebra. The umbral calculus is the study of umbral algebra.
The order of the power series $f(t)(\neq 0)$ is the smallest integer for which a_k does not vanish. The order of $f(t)$ is denoted by $O(f(t))$. If $O(f(t)) = 0$, then $f(t)$ is called an invertible series. If $O(f(t)) = 1$, then $f(t)$ is said to be a delta series.
For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$(12) \qquad f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x)\rangle \frac{x^k}{k!}, \quad \text{(see [14])}.$$

Thus, by (12), we get
(13)

$$p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{l=k}^{\infty} \frac{\langle t^l|p(x)\rangle}{(l-k)!} x^{l-k}, \quad p^{(k)}(0) = \langle t^k|p(x)\rangle = \langle 1|p^{(k)}(x)\rangle.$$

From (13), we note that

$$(14) \qquad t^k p(x) = p^{(k)}(x), \quad e^{yt}p(x) = p(x+y), \quad \langle e^{yt}|p(x)\rangle = p(y).$$

For $O(f(t)) = 1$, $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$.
Let $p(x) \in \mathbb{P}$ and $f(t) \in \mathcal{F}$. Then we see that

$$(15) \qquad \langle f(t)|xp(x)\rangle = \langle \partial_t f(t)|p(x)\rangle = \langle f'(t)|p(x)\rangle, \quad \text{(see [14])}.$$

For $s_n(x) \sim (g(t), f(t))$, we have the following equations:

$$(16) \qquad s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\left(\bar{f}(t)\right)^{-1} \bar{f}(t)^j \middle| x^n \right\rangle x^j,$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = t$,

$$(17) \qquad \frac{1}{g\left(\bar{f}(t)\right)} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad \text{for all } x \in \mathbf{C},$$

$$(18) \quad s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) p_{n-k}(y), \quad \text{where } p_{n-k}(y) = g(t)s_{n-k}(y),$$

and
(19)

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)} s_n(x), \quad f(t)s_n(x) = ns_{n-1}(x), \quad \text{(see [3, 5, 9, 14])}.$$

For $s_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$, we have

(20)
$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x),$$

where

(21)
$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\left(\bar{f}(t)\right)}{g\left(\bar{f}(t)\right)} l\left(\bar{f}(t)\right)^m \middle| x^n \right\rangle, \quad \text{(see [14])}.$$

In this paper, we consider higher-order Cauchy of the second kind and poly-Cauchy of the second kind mixed type polynomials and we investigate some properties of those polynomials with umbral calculus viewpoint. From our investigation, we can derive many interesting identities related to higher-order Cauchy of the second kind and poly-Cauchy of the second kind mixed type polynomials.

2. Higer-order Cauchy of the second kind and poly-Cauchy of the second kind mixed type polynomials

From (3) and (17), we note that $\hat{A}_n^{(r,k)}(x)$ is the Sheffer sequence for the pair $\left(\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)}, e^t - 1\right)$. That is,

(22)
$$\hat{A}_n^{(r,k)}(x) \sim \left(\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)}, e^t - 1\right).$$

Komatsu considered the number $\hat{A}_n^{(r,k)}$, which was denoted by $\hat{T}_{r+1}^{(k)}(n)$ (see [10 − 12]).
By (22), we easily see that

(23)
$$\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)} \hat{A}_n^{(r,k)}(x) \sim \left(1, e^t - 1\right),$$

and we see that $(x)_n \sim (1, e^t - 1)$.
From the uniqueness of Sheffer sequence, we note that
(24)
$$\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)} A_n^{(r,k)}(x) = (x)_n = \sum_{m=0}^{n} S_1(n,m) x^m \sim \left(1, e^t - 1\right).$$

By (24), we get

(25)

$$
\begin{aligned}
A_n^{(r,k)}(x) &= \left(\frac{e^t-1}{te^t}\right)^r Lif_k(-t)(x)_n = \sum_{m=0}^{n} S_1(n,m)\left(\frac{e^t-1}{te^t}\right)^r Lif_k(-t)x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\left(\frac{e^t-1}{te^t}\right)^r \sum_{l=0}^{\infty}\frac{(-t)^l}{l!(l+1)^k}x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l(m)_l}{l!(l+1)^k}\left(\frac{e^{-t}-1}{-t}\right)^r x^{m-l} \\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l(m)_l}{l!(l+1)^k} \\
&\quad \times \sum_{a=0}^{m-l}\frac{r!}{(a+r)!}S_2(a+r,r)(-1)^a(m-l)_a x^{m-l-a}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{a=0}^{m-l}(-1)^m\frac{\binom{m}{l}\binom{m-l}{a}}{\binom{a+r}{r}(l+1)^k}S_1(n,m)S_2(a+r,r)(-x)^{m-l-a} \\
&= \sum_{j=0}^{n}\left\{\sum_{m=j}^{n}\sum_{l=0}^{m-j}(-1)^m\frac{\binom{m}{l}\binom{m-l}{j}}{\binom{m-l-j+r}{r}(l+1)^k}S_1(n,m)S_2(m-l-j+r,r)\right\}(-x)^j.
\end{aligned}
$$

Therefore, by (25), we obtain the following theorem.

Theorem 1. *For $n, r \geq 0$, $k \in \mathbf{Z}$, we have*

$$
\begin{aligned}
\tilde{A}_n^{(r,k)}(x) = \sum_{j=0}^{n}\Bigg\{\sum_{m=j}^{n}\sum_{l=0}^{m-j}(-1)^m&\frac{\binom{m}{l}\binom{m-l}{j}}{\binom{m-l-j+r}{r}(l+1)^k} \\
&\times S_1(n,m)S_2(m-l-j+r,r)\Bigg\}(-x)^j.
\end{aligned}
$$

From (16) and (22), we note that
(26)

$$\begin{aligned}
&\tilde{A}_n^{(r,k)}(x)\\
&=\sum_{j=0}^{n}\frac{1}{j!}\left\langle\left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t))(\log(1+t))^j \middle| x^n\right\rangle x^j\\
&=\sum_{j=0}^{n}\frac{1}{j!}\sum_{l=0}^{n-j}\frac{j!}{(l+j)!}S_1(l+j,j)(n)_{l+j}\\
&\times\left\langle\left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t)) \middle| x^{n-l-j}\right\rangle x^j,
\end{aligned}$$

and

$$\begin{aligned}
(27)\qquad &\left\langle\left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t)) \middle| x^{n-l-j}\right\rangle\\
&=\sum_{a=0}^{\infty}\frac{\tilde{A}_a^{(r,k)}}{a!}\left\langle t^a \middle| x^{n-l-j}\right\rangle\\
&=\tilde{A}_{n-l-j}^{(r,k)}.
\end{aligned}$$

By (26) and (27), we get

$$\begin{aligned}
(28)\qquad \tilde{A}_n^{(r,k)}(x) &= \sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\binom{n}{l+j}S_1(l+j,j)\tilde{A}_{n-l-j}^{(r,k)}\right\}x^j\\
&=\sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\tilde{A}_l^{(r,k)}\right\}x^j.
\end{aligned}$$

Therefore, by (28), we obtain the following theorem.

Theorem 2. *For $n,r\geq 0$ and $k\in\mathbf{Z}$, we have*

$$\tilde{A}_n^{(r,k)}(x)=\sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\tilde{A}_l^{(r,k)}\right\}x^j.$$

It is known that

$$(29)\qquad \left(\frac{t}{\log(1+t)}\right)^n(1+t)^{x-1}=\sum_{k=0}^{\infty}B_k^{(k-n+1)}(x)\frac{t^k}{k!}.$$

In particular, for $x=1-r$, $n=r$, we have

$$(30)\qquad \left(\frac{t}{(1+t)\log(1+t)}\right)^r=\sum_{k=0}^{\infty}B_k^{(k-r+1)}(1-r)\frac{t^k}{k!}.$$

By (26) and (30), we get

(31)

$$\begin{aligned}\tilde{A}_n^{(r,k)}(x) &= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l+j}S_1(l+j,j)\sum_{a=0}^{\infty}\frac{B_a^{(a-r+1)}(1-r)}{a!}\\ &\quad\times\left\langle Lif_k\left(-\log\left(1+t\right)\right)\middle|t^a x^{n-l-j}\right\rangle x^j\\ &= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l+j}S_1(l+j,j)\sum_{a=0}^{n-l-j}B_a^{(a-r+1)}(1-r)\frac{1}{a!}(n-l-j)_a\\ &\quad\times\left\langle Lif_k\left(-\log\left(1+t\right)\right)\middle|x^{n-l-j-a}\right\rangle x^j\\ &= \sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\sum_{a=0}^{n-j-l}\binom{n}{l+j}\binom{n-j-l}{a}S_1(l+j,j)B_a^{(a-r+1)}(1-r)\right.\\ &\quad\left.\times\,\tilde{C}_{n-j-l-a}^{(k)}\right\}x^j,\end{aligned}$$

where $\tilde{C}_n^{(k)}$ are the poly-Cauchy numbers of the second kind.
Therefore, by (31), we obtain the following theorem.

Theorem 3. *For $n, r \geq 0$ and $k \in \mathbf{Z}$, we have*

$$\begin{aligned}\tilde{A}_n^{(r,k)}(x) &= \sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\sum_{a=0}^{n-j-l}\binom{n}{l+j}\binom{n-j-l}{a}S_1(l+j,j)B_a^{(a-r+1)}(1-r)\right.\\ &\quad\left.\times\,\tilde{C}_{n-j-l-a}^{(k)}\right\}x^j.\end{aligned}$$

By (29), we easily see that

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty}B_n^{(n)}\frac{t^n}{n!}. \tag{32}$$

Thus, by (26) and (32), we get

(33)

$$\begin{aligned}\tilde{A}_n^{(r,k)} &= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l+j}S_1(l+j,j)\\ &\quad\times\left\langle Lif_k\left(-\log(1+t)\right)\middle|\left(\frac{t}{(1+t)\log(1+t)}\right)^r x^{n-l-j}\right\rangle x^j\\ &=\sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\sum_{a=0}^{n-j-l}\sum_{a_1+\cdots+a_r=a}\binom{n}{l+j}\binom{n-j-l}{a}\binom{a}{a_1,\cdots,a_r}S_1(l+j,j)\right.\\ &\quad\left.\times\left(\prod_{i=1}^{r}B_{a_i}^{(a_i)}\right)\tilde{C}_{n-j-l-a}^{(k)}\right\}x^j.\end{aligned}$$

Therefore, by (33), we obtain the following corollary.

Corollary 4. *For $n \geq 0$, $r \in \mathbf{N}$ and $k \in \mathbf{Z}$, we have*

$$\begin{aligned}\tilde{A}_n^{(r,k)}(x) &= \sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\sum_{a=0}^{n-j-l}\sum_{a_1+\cdots+a_r=a}\binom{n}{l+j}\binom{n-j-l}{a}\binom{a}{a_1,\cdots,a_r}\right.\\ &\quad\left.\times S_1(l+j,j)\left(\prod_{i=1}^{r}B_{a_i}^{(a_i)}\right)\tilde{C}_{n-j-l-a}^{(k)}\right\}x^j.\end{aligned}$$

From (18) and (19), we can derive
(34)

$$\tilde{A}_n^{(r,k)}(x+y) = \sum_{j=0}^{n}\binom{n}{j}\tilde{A}_j^{(r,k)}(x)(y)_{n-j},\quad \left(e^t-1\right)\tilde{A}_n^{(r,k)}(x) = n\tilde{A}_{n-1}^{(r,k)}(x).$$

By (19) and (22), we get

(35)

$$\begin{aligned}\tilde{A}_{n+1}^{(r,k)}(x) &= x\tilde{A}_n^{(r,k)}(x-1) - r\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{a=0}^{m-l}\frac{(-1)^{m-a}\binom{m}{l}\binom{m-l}{a}}{(a+2)(a+1)(l+1)^k}S_1(n,m)\\ &\quad\times B_{m-l-a}^{(1-r)}(2-x) - \sum_{m=0}^{n}\sum_{a=0}^{m}\frac{(-1)^m\binom{m}{a}}{(a+2)^k}S_1(n,m)B_{m-a}^{(-r)}(1-x).\end{aligned}$$

From (11), we note that

(36)

$$\begin{aligned}\tilde{A}_n^{(r,k)}(y) &= \left\langle \sum_{m=0}^{\infty} \tilde{A}_m^{(r,k)}(y)\frac{t^m}{m!} \middle| x^n \right\rangle \\ &= \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t))(1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t))(1+t)^y \middle| xx^{n-1} \right\rangle.\end{aligned}$$

By (15) and (36), we get

(37)

$$\begin{aligned}\tilde{A}_n^{(r,k)}(y) &= \left\langle \partial_t \left(\left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t))(1+t)^y \right) \middle| x^{n-1} \right\rangle \\ &= \left\langle \left(\partial_t \left(\frac{t}{(1+t)\log(1+t)}\right)^r\right) Lif_k(-\log(1+t))(1+t)^y \middle| x^{n-1} \right\rangle \\ &\quad + \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r (\partial_t(Lif_k(-\log(1+t))))(1+t)^y \middle| x^{n-1} \right\rangle \\ &\quad + \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t))(\partial_t(1+t)^y) \middle| x^{n-1} \right\rangle \\ &= y\tilde{A}_{n-1}^{(r,k)}(y-1) \\ &\quad + \left\langle \left(\partial_t \left(\frac{t}{(1+t)\log(1+t)}\right)^r\right) Lif_k(-\log(1+t))(1+t)^y \middle| x^{n-1} \right\rangle \\ &\quad + \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r (\partial_t(Lif_k(-\log(1+t))))(1+t)^y \middle| x^{n-1} \right\rangle.\end{aligned}$$

Now, we observe that

$$\begin{aligned}(38)\qquad \frac{\log(1+t)-t}{t^2}x^l &= \sum_{a=0}^{l} \frac{(-1)^{a-1}}{a+2} t^a x^l = \sum_{a=0}^{l} \frac{(-1)^{a-1}}{a+2}(l)_a x^{l-a} \\ &= \sum_{a=0}^{l} \frac{(-1)^{l-a-1}}{l-a+2}\binom{l}{a}(l-a)!x^a.\end{aligned}$$

By (38), we get

(39)

$$\left\langle \left(\partial_t \left(\frac{t}{(1+t)\log(1+t)}\right)^r\right) Lif_k(-\log(1+t))(1+t)^y \middle| x^{n-1}\right\rangle$$

$$= r\left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^{r+1} \frac{\log(1+t)-t}{t^2} Lif_k(-\log(1+t))(1+t)^y \middle| x^{n-1}\right\rangle$$

$$= r\sum_{a=0}^{n-1} \frac{(-1)^{n-a}(n-a-1)!}{n-a+1}\binom{n-1}{a}$$

$$\times \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^{r+1} Lif_k(-\log(1+t))(1+t)^y \middle| x^a\right\rangle$$

$$= r\sum_{a=0}^{n-1} \frac{(-1)^{n-a}(n-a-1)!}{n-a+1}\binom{n-1}{a}\tilde{A}_a^{(r+1,k)}(y)$$

$$= r\sum_{a=0}^{n-1} \frac{(-1)^{a+1}a!}{a+2}\binom{n-1}{a}\tilde{A}_{n-1-a}^{(r+1,k)}(y).$$

It is not difficult to show that

(40)

$$(Lif_k(-\log(1+t)))' = \frac{Lif_{k-1}(-\log(1+t)) - Lif_k(-\log(1+t))}{(1+t)\log(1+t)}.$$

By (40), we get

(41)

$$\left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r (\partial_t Lif_{k-1}(-\log(1+t)))(1+t)^y \middle| x^{n-1}\right\rangle$$

$$= \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r \frac{Lif_{k-1}(-\log(1+t)) - Lif_k(-\log(1+t))}{(1+t)\log(1+t)}\right.$$

$$\left.\times (1+t)^y \middle| \frac{1}{n}tx^n\right\rangle$$

$$= \frac{1}{n}\left(\tilde{A}_n^{(r+1,k-1)}(y) - \tilde{A}_n^{(r+1,k)}(y)\right).$$

Therefore, by (37), (39) and (41), we obtain the following theorem.

Theorem 5. *For $n \geq 1$, $r,k \in \mathbf{Z}$ with $r \geq 1$, we have*

$$\tilde{A}_n^{(r,k)}(x) = x\tilde{A}_{n-1}^{(r,k)}(x-1) + r\sum_{a=0}^{n-1} \frac{(-1)^{a+1}a!}{a+2}\binom{n-1}{a}\tilde{A}_{n-1-a}^{(r+1,k)}(x) + \frac{1}{n}\left(\tilde{A}_n^{(r+1,k-1)}(x) - \tilde{A}_n^{(r+1,k)}(x)\right).$$

Now, we compute the following equation (42) in two different ways:

$$\left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t))(\log(1+t))^m \middle| x^n \right\rangle. \tag{42}$$

On the one hand,

$$\begin{aligned}
&\left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t))(\log(1+t))^m \middle| x^n \right\rangle \\
&= \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m} \\
&\quad \times \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t)) \middle| x^{n-l-m} \right\rangle \\
&= \sum_{l=0}^{n-m} m!\binom{n}{l+m} S_1(l+m,m)\tilde{A}_{n-l-m}^{(r,k)} \\
&= \sum_{l=0}^{n-m} m!\binom{n}{l} S_1(n-l,m)\tilde{A}_l^{(r,k)}.
\end{aligned} \tag{43}$$

On the other hand, (42) is

(44)

$$\begin{aligned}
&\left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t))(\log(1+t))^m \middle| xx^{n-1} \right\rangle \\
&= \left\langle \partial_t\left(\left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t))(\log(1+t))^m\right) \middle| x^{n-1} \right\rangle
\end{aligned}$$

$$= \left\langle \left(\partial_t \left(\frac{t}{(1+t)\log(1+t)}\right)^r\right) Lif_k(-\log(1+t))(\log(1+t))^m \middle| x^{n-1} \right\rangle$$

$$+ \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r (\partial_t Lif_k(-\log(1+t)))(\log(1+t))^m \middle| x^{n-1} \right\rangle$$

$$+ \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t))(\partial_t(\log(1+t))^m) \middle| x^{n-1} \right\rangle$$

Here, we observe that

(45)

$$\left\langle \left(\partial_t \left(\frac{t}{(1+t)\log(1+t)}\right)^r\right) Lif_k(-\log(1+t))(\log(1+t))^m \middle| x^{n-1} \right\rangle$$

$$= r\left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^{r+1} \frac{\log(1+t)-t}{t^2} Lif_k(-\log(1+t)) \times (\log(1+t))^m \middle| x^{n-1} \right\rangle$$

$$= r\sum_{l=0}^{n-m-1} \frac{m!}{(l+m)!} S_1(l+m,m)(n-1)_{l+m} \times \sum_{a=0}^{n-1-m-l} \frac{(-1)^{a+1} a! \binom{n-l-m-1}{a}}{a+2} \tilde{A}^{(r+1,k)}_{n-l-m-a-1}$$

$$= r\sum_{l=0}^{n-m-1} \sum_{a=0}^{n-1-l-m} \frac{(-1)^{a+1} a! m!}{a+2} \binom{n-1}{l+m} \binom{n-l-m-1}{a} \times S_1(l+m,m) \tilde{A}^{(r+1,k)}_{n-l-m-a-1},$$

and

(46)

$$\left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r (\partial_t Lif_k(-\log(1+t)))(\log(1+t))^m \middle| x^{n-1} \right\rangle$$

$$= \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r \left(\frac{Lif_{k-1}(-log(1+t)) - Lif_k(-log(1+t))}{(1+t)\log(1+t)}\right) \times (\log(1+t))^m \middle| x^{n-1} \right\rangle$$

$$= \sum_{l=0}^{n-m-1} \frac{m!}{(l+m)!} S_1(l+m,m)(n-1)_{l+m} \frac{1}{n-l-m} \left(\tilde{A}_{n-l-m}^{(r+1,k-1)} - \tilde{A}_{n-l-m}^{(r+1,k)} \right)$$
$$= \sum_{l=0}^{n-m-1} \frac{m!}{n-l-m} \binom{n-1}{l+m} S_1(l+m,m) \left(\tilde{A}_{n-l-m}^{(r+1,k-1)} - \tilde{A}_{n-l-m}^{(r+1,k)} \right).$$

Finally, we easily see that

(47)

$$\left\langle \left(\frac{t}{(1+t)\log(1+t)} \right)^r Lif_k(-\log(1+t)) \left(\partial_t (\log(1+t))^m \right) \middle| x^{n-1} \right\rangle$$
$$= m \left\langle \left(\frac{t}{(1+t)\log(1+t)} \right)^r Lif_k(-\log(1+t)) (1+t)^{-1} (\log(1+t))^{m-1} \middle| x^{n-1} \right\rangle$$
$$= m \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1,m-1)(n-1)_{l+m-1} \tilde{A}_{n-l-m}^{(r,k)}(-1)$$
$$= \sum_{l=0}^{n-m} m! \binom{n-1}{l+m-1} S_1(l+m-1,m-1) \tilde{A}_{n-l-m}^{(r,k)}(-1).$$

Therefore, by (43), (44), (45), (46) and (47), we obtain the following theorem.

Theorem 6. *For $n-1 \geq m \geq 1$, we have*

$$\sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \tilde{A}_l^{(r,k)}$$
$$= r \sum_{l=0}^{n-m-1} \sum_{a=0}^{n-l-m-1} \frac{(-1)^{a+1} a!}{a+2} \binom{n-1}{l+m} \binom{n-l-m-1}{a} S_1(l+m,m) \tilde{A}_{n-l-m-a-1}^{(r+1,k)}$$
$$+ \sum_{l=0}^{n-m-1} \frac{1}{n-l-m} \binom{n-1}{l+m} S_1(l+m,m) \left(\tilde{A}_{n-l-m}^{(r+1,k-1)} - \tilde{A}_{n-l-m}^{(r+1,k)} \right)$$
$$+ \sum_{l=0}^{n-m} \binom{n-1}{l+m-1} S_1(l+m-1,m-1) \tilde{A}_{n-l-m}^{(r,k)}(-1).$$

For $s_n(x) \sim (g(t), f(t))$, we note that

$$\frac{d}{dx} s_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} \left\langle \bar{f}(t) \middle| x^{n-l} \right\rangle s_l(x). \tag{48}$$

From (22) and (48), we can derive the following equation (49):

$$\begin{aligned}(49)\qquad \frac{d}{dx}\tilde{A}_n^{(r,k)}(x) &= \sum_{l=0}^{n-1}\binom{n}{l}\left\langle \log(1+t) \middle| x^{n-l}\right\rangle \tilde{A}_l^{(r,k)}(x)\\ &= \sum_{l=0}^{n-1}\binom{n}{l}\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\left\langle t^{m+1} \middle| x^{n-l}\right\rangle \tilde{A}_l^{(r,k)}(x)\\ &= \sum_{l=0}^{n-1}\binom{n}{l}\frac{(-1)^{n-l-1}}{n-l}(n-l)!\tilde{A}_l^{(r,k)}(x)\\ &= (-1)^n n!\sum_{l=0}^{n-1}\frac{(-1)^{l+1}}{(n-l)l!}\tilde{A}_l^{(r,k)}(x).\end{aligned}$$

For $\tilde{A}_n^{(r,k)}(x) \sim \left(\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)}, e^t-1\right)$, $B_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^s, t\right)$, let us assume that

$$(50)\qquad \tilde{A}_n^{(r,k)}(x) = \sum_{m=0}^{n} C_{n,m}B_m^{(s)}(x), \quad (r,s \in \mathbf{N}).$$

By (21), we get

(51)

$$\begin{aligned}C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r \left(\frac{t}{\log(1+t)}\right)^s Lif_k(-\log(1+t))\right.\\ &\quad \left.\times (\log(1+t))^m \middle| x^n\right\rangle\\ &= \frac{1}{m!}\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\\ &\quad \times\left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^{r+s} Lif_k(-\log(1+t))(1+t)^s \middle| x^{n-l-m}\right\rangle\\ &= \sum_{l=0}^{n-m}\binom{n}{l+m}S_1(l+m,m)\left\langle \sum_{a=0}^{\infty}\tilde{A}_a^{(r+s,k)}(s)\frac{t^a}{a!} \middle| x^{n-l-m}\right\rangle\\ &= \sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\tilde{A}_l^{(r+s,k)}(s).\end{aligned}$$

Therefore, by (50) and (51), we obtain the following theorem.

Theorem 7. *For $n \geq 0$, $r, s \in \mathbf{N}$, we have*

$$\tilde{A}_n^{(r,k)}(x) = \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \tilde{A}_l^{(r+s,k)}(s) \right\} B_m^{(s)}(x).$$

For $\tilde{A}_n^{(r,k)}(x) \sim \left(\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)}, e^t - 1\right)$, $H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right)$, let us assume that

$$\tilde{A}_n^{(r,k)}(x) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\lambda), \tag{52}$$

where $\lambda \in \mathbf{C}$ with $\lambda \neq 1$, $r, s \in \mathbf{N}$ and $k \in \mathbf{Z}$.
From (21), we have

(53)

$$\begin{aligned}
C_{n,m} = \frac{1}{m!} \Bigg\langle & \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t)) \left(1 + \frac{t}{1-\lambda}\right)^s \\
& \times (\log(1+t))^m \Bigg| x^n \Bigg\rangle \\
= \frac{1}{m!} & \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m} \\
& \times \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t)) \left(1 + \frac{t}{1-\lambda}\right)^s \Bigg| x^{n-l-m} \right\rangle \\
= & \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m) \sum_{a=0}^{n-l-m} \binom{s}{a} \left(\frac{1}{1-\lambda}\right)^a (n-l-m)_a \\
& \times \left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^r Lif_k(-\log(1+t)) \Bigg| x^{n-l-m-a} \right\rangle \\
= & \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m) \sum_{a=0}^{n-l-m} \binom{s}{a} \left(\frac{1}{1-\lambda}\right)^a (n-l-m)_a \tilde{A}_{n-l-m-a}^{(r,k)} \\
= & \sum_{l=0}^{n-m} \sum_{a=0}^{l} \frac{\binom{n}{l}\binom{s}{a}\binom{l}{a} a!}{(1-\lambda)^a} S_1(n-l,m) \tilde{A}_{l-a}^{(r,k)}.
\end{aligned}$$

Therefore, by (52) and (53), we obtain the following theorem.

Theorem 8. *For $n \geq 0$, $r, s \in \mathbf{N}$, $k \in \mathbf{Z}$ and $\lambda \in \mathbf{C}$ with $\lambda \neq 1$, we have*

$$\tilde{A}_n^{(r,k)}(x) = \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \sum_{a=0}^{l} \frac{\binom{n}{l}\binom{s}{a}\binom{l}{a} a!}{(1-\lambda)^a} S_1(n-l,m) \tilde{A}_{l-a}^{(r,k)} \right\} H_m^{(s)}(x|\lambda).$$

Let us consider the following two Sheffer sequences:

$$\tilde{A}_n^{(r,k)}(x) \sim \left(\left(\frac{te^t}{e^t-1} \right)^r \frac{1}{Lif_k(-t)}, e^t - 1 \right)$$

and

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l \sim \left(1, e^t - 1\right).$$

Suppose that

$$\tilde{A}_n^{(r,k)}(x) = \sum_{m=0}^{n} C_{n,m} (x)_m. \tag{54}$$

By (21), we get

$$C_{n,m} = \binom{n}{m} \tilde{A}_{n-m}^{(r,k)}. \tag{55}$$

Therefore, by (54) and (55), we get

$$\tilde{A}_n^{(r,k)}(x) = \sum_{m=0}^{n} \binom{n}{m} \tilde{A}_{n-m}^{(r,k)}(x)_m, \tag{56}$$

where $r, n \geq 0$ and $k \in \mathbf{Z}$.

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