Combinatorial identities for the r-Lah numbers

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Abstract

This paper is an orthogonal continuation of the work of Belbachir and Belkhir in sense where we establish, using bijective proofs, recurrence relations and convolution identities between lines of r-Lah triangle. It is also established a symmetric function form for the r-Lah numbers.

1 Introduction

The r-Lah numbers, denoted $\left\lfloor {n \atop k} \right\rfloor_r$, count the number of partitions of the set $\{1, 2, ..., n\}$ into k non empty ordered lists, such that the numbers $1, 2, ..., r$ are in distinct lists. They satisfy, see for instance [3, 1], the recurrence relation

$$\left\lfloor {n \atop k} \right\rfloor_r = \left\lfloor {n-1 \atop k-1} \right\rfloor_r + (n+k-1) \left\lfloor {n-1 \atop k} \right\rfloor_r, \tag{1}$$

with $\left\lfloor {n \atop k} \right\rfloor_r = \delta_{n,k}$ for $k = r$, where δ is the Kronecker delta, and $\left\lfloor {n \atop k} \right\rfloor_r = 0$ for $n < r$.

For $r = 0$ and $r = 1$, we get the classical Lah numbers.

The r-Lah numbers have the following explicit formula

$$\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!} \binom{n-r}{k-r} = \frac{(n-r)!}{(k-r)!} \binom{n+r-1}{k+r-1}. \tag{2}$$

In a previous work, the first author and Belkhir [1], established a cross recurrence formula, a triangular recurrence with rational coefficient for the Lah numbers and a vertical recurrence relation using bijective proof.

Our aim is to give some new combinatorial identities for the r-Lah numbers. All the identities given in [1] deal with relations between columns of r-Lah triangle. Our work is a dual complement to [1] in sense that we give identities explaining relations between lines of r-Lah triangle. In section

ARS COMBINATORIA 115(2014), pp. 453-458

2, using combinatorial arguments, we give symmetric function form for the r-Lah numbers. In section 3, we derive two convolution identities. A second form of expression of the r-Lah numbers in terms of Lah numbers is treated in section 4. We give in the last section, some combinatorial identities expressing r-Lah numbers in different triangles. Many basic triangular recurrences are derived.

2 r-Lah numbers and symmetric functions

The r-Stirling numbers of the first kind $\left[{n \atop k}\right]_r$, for fixed n, are elementary symmetric functions of the numbers $r, r+1, \ldots, n-1, n$ and satisfy the following identity, see for instance [2].

$$\left[{n+k \atop n}\right]_r = \sum_{r \le i_1 \le i_2 \le \cdots \le i_k \le n} i_1 (i_2+1)(i_3+2) \cdots (i_k + (k-1)), \qquad (3)$$

also, for the r-Stirling numbers of the second kind, we have

$$\left\{{n+k \atop n}\right\}_r = \sum_{r \le i_1 \le i_2 \le \cdots \le i_k \le n} i_1 i_2 \cdots i_k. \qquad (4)$$

We give a analogous property according to r-Lah numbers via symmetric functions, it is the following

Theorem 2.1 *For positive integers n, k, r we have*

$$\left\lfloor{n+k \atop n}\right\rfloor_r = \sum_{r \le i_1 \le i_2 \le \cdots \le i_k \le n} 2i_1 (2i_2+1) \cdots (2i_k + (k-1)). \qquad (5)$$

Proof. The left hand side $\left\lfloor{n+k \atop n}\right\rfloor_r$ counts the number of partition of $1, \ldots, n+k$ elements into n lists such that the r first elements are in distinct lists.

In the right hand side, we constitute n lists from the elements $1, \ldots, n$ (one by list). Now, we discuss the insertion of the remaining elements $n+1, \ldots, n+k$. To insert the element $n+1$ we have two situations:

1) If it is affected to the r first lists (containing the r first fixed elements) then there are $2r$ possibilities to do it (before or after each initial element).

2) Else, it belongs to a list i_1 ($r+1 \le i_1 \le n$), we must consider all the possible situations of the element already in the list i_1 and we get two other situations: a) the initial element stay in the list i_1 and we have 2 possibilities. b) Or move it to the previous lists (we move the elements only from right to left to avoid the double counting of situations), and we have $2(i_1 - 1)$ ways to do it. Thus from a) and b) we get $2i_1$ possibilities. We sum over all the possible insertion of the element $n+1$ in the lists $r+1, \ldots, n$, we get $\sum_{r+1 \le i_1 \le n} 2i_1$ possibilities.

From 1) and 2), $2r + \sum_{r+1 \le i_1 \le n} 2i_1 = \sum_{r \le i_1 \le n} 2i_1$.

To insert the element $n+2$, we consider the elements of the lists $1, \ldots, i_1$ as fixed ones due to the insertion of the previous element $n+1$ where we consider all the situations. We have the same two situations as before:

1) If we add the element $n+2$ to the lists $1, \ldots, i_1$ we have $2i_1 + 1$ possibilities (one possibility added by the element $n+1$).

2) Else, it belongs to a list i_2 $(i_1 + 1 \le i_2 \le n)$, we have $2i_2 + 1$ possibilities (2 possibilities to insert $n+2$ before or after the initial element of the list i_2 or $2(i_2 - 1) + 1$ possibilities of the initial elements of the list i_2 to move to the previous lists), that gives $\sum_{i_1+1 \le i_2 \le n} (2i_2 + 1)$ possibilities.

Thus, from 1) and 2), $\sum_{r \le i_1 \le n} 2i_1 \left((2i_1 + 1) + \sum_{i_1+1 \le i_2 \le n} (2i_2 + 1)\right) = \sum_{r \le i_1 \le i_2 \le n} 2i_1 (2i_2 + 1)$ possibilities to insert the elements $n+1$ and $n+2$.

We carry on by the same process for the remaining $k-2$ elements. So, for the last element $n+k$ we consider the elements of the lists $1, \ldots, i_{k-1}$ as fixed ones and we have $2i_{k-1} + k - 1$ possibilities to insert the element $n+k$ in these lists. Or insert it to a list i_k $(i_{k-1} + 1 \le i_k \le n)$, and we have $2i_k + k - 1$ possibilities (2 possibilities to insert $n+k$ before or after the initial element of the list i_k and $2(i_k - 1) + k - 1$ possibilities of the initial elements of the list i_k to move to the previous lists), that gives $\sum_{r \le i_1 \le i_2 \le \cdots \le i_k \le n} 2i_1 (2i_2 + 1) \cdots (2i_k + (k-1))$ possibilities. □

3 Convolution identities

The r-Lah numbers satisfy a Vandermonde's convolution type formula. This relation, for fixed s, expresses the r-Lah numbers $\left\lfloor {n \atop k} \right\rfloor_r$ in terms of $\left\lfloor {n-s \atop k-i} \right\rfloor_r$, $i = 1, \ldots, s$. It can be seen as an horizontal recurrence relation.

Theorem 3.1 *Let n, k, r and s be positive integers such that $r \leqslant k \leqslant n$ and $r \leqslant n - s$, we have*

$$\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!} \sum_{i=0}^{s} \frac{(k-i+r-1)!}{(n-s+r-1)!} \binom{s}{i} \left\lfloor {n-s \atop k-i} \right\rfloor_r. \tag{6}$$

Note that the coefficients of $\left\lfloor {n-s \atop k-i} \right\rfloor_r$ are of rational type.

Proof. We have $\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!} \binom{n-r}{k-r}$, Vandermonde's formula gives $\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!} \sum_{i=0}^{s} \binom{s}{i} \binom{n-s-r}{k-i-r}$. Thus we get the result. □

Corollary 3.1.1 *For $s = 1$, we obtain a rational coefficients recurrence relation*

$$\left\lfloor {n \atop k} \right\rfloor_r = (n+r-1) \left\lfloor {n-1 \atop k} \right\rfloor_r + \frac{(n+r-1)}{(k+r-1)} \left\lfloor {n-1 \atop k-1} \right\rfloor_r. \tag{7}$$

Under the restriction $r = 0$, we get relation (5) of [1].

Remark 3.2 *For $s = n - r$ in relation (6), we get the classical explicit form of r-Lah numbers given by (2).*

The following result improve the precedent one in sense that the coefficients are integers.

Theorem 3.3 *Let s, r, k and n nonnegative integers such that $r \leqslant k \leqslant n$ and $r \leqslant n - s$, we have*

$$\left\lfloor {n \atop k} \right\rfloor_r = \sum_{j=0}^{s} \frac{(n+k-j-1)!}{(n+k-s-1)!} \binom{s}{j} \left\lfloor {n-s \atop k-j} \right\rfloor_r . \qquad (8)$$

Proof. We divide the n elements into two groups : a first one with s elements $\{1, \ldots, s\}$ and second one with $n - s$ elements . With the first group we can constitute j lists $(0 \leqslant j \leqslant s)$ and with the second group we can constitute $k - j$ lists such that $1, \ldots, r$ are in distinct lists (it is possible because $r \leqslant n - s$). The r fixed elements must be chosen from the elements of the second group. We have $\left\lfloor {n-s \atop k-j} \right\rfloor_r$ possibilities to constitute the $k - j$ lists. It remains to count how to constitute the j remaining ones. We have $\binom{s}{j}$ possibilities to choose j elements from the first group with one element by list. Then, we order the remaining $s - j$ elements into the k lists, so the first one has $(n - s + k)$ choices ($n - s$ ways after each ordered element and k ways as head list), the second one has $(n - s + k + 1)$ choices (one possibility added by the previews insertion) and so on.... The last element $s - j$ has $(n - s + k + (s - j - 1)) = (n + k - j - 1)$ choices. It gives $\frac{(n+k-j-1)!}{(n+k-s-1)!} = (n - s + k)(n - s + k + 1) \cdots (n + k - j - 1)$ possibilities. We conclude by summing. □

Remark 3.4 *For $s = 1$, we obtain the well known recurrence relation (1), and for $s = n - r$ we get again the explicit formula (2).*

4 Relation between r-Lah and Lah numbers

It is established [1], by combinatorial approach, that the r-Lah numbers can be expressed in terms of Lah numbers as follows

$$\left\lfloor {n+r \atop k+r} \right\rfloor_r = \sum_{s=0}^{n-k} \sum_{i_1+\cdots+i_r=s} (i_1+1)! \cdots (i_r+1)! \binom{n}{i_1, \ldots, i_r, n-s} \left\lfloor {n-s \atop k} \right\rfloor .$$
(9)

To prove the relation above, the authors consider the r first lists containing the r first elements and i_j $(1 \leqslant j \leqslant r)$ other elements. So the operation of counting the different situations was done in two steps : first we choose the i_j elements, then arrange the elements of each lists.

Now, we give an other formulation expressing r-Lah numbers in terms of Lah numbers without counting a multi-sum with a combinatorial argument.

Theorem 4.1 *Let r, k and n positive integers such that, $r \leqslant k \leqslant n$, we have*

$$\left\lfloor {n \atop k} \right\rfloor_r = \sum_{s=0}^{n-k} \frac{(s+2r-1)!}{(2r-1)!} \binom{n-r}{s} \left\lfloor {n-r-s \atop k-r} \right\rfloor. \tag{10}$$

Proof. The r first elements can be considered as representing of the r first lists. Because we have to constitute k lists, let us consider the s $(0 \leqslant s \leqslant n-k)$ elements that will belong to the r first lists. We have $\binom{n-r}{s}$ possibilities to choose them. Then, we insert the s elements to the r lists and we have $2r$ possibilities for the first one, $2r+1$ possibilities for the second and so on ..., until the last element s, it has $(s+2r-1)$ possibilities. This gives $2r\,(2r+1)\cdots(2r+s-1) = \frac{(s+2r-1)!}{(2r-1)!}$ possibilities. Finally, we constitute the remaining $k-r$ lists with the remaining $n-r-s$ elements and we have $\left\lfloor {n-r-s \atop k-r} \right\rfloor$ possibilities. □

Corollary 4.1.1 *For $r = 1$, in the relations (9) and (10), we get the vertical recurrence relation for the Lah numbers*

$$\left\lfloor {n \atop k} \right\rfloor = \sum_{i=0}^{n-k} (i+1)! \binom{n-1}{i} \left\lfloor {n-i-1 \atop k-1} \right\rfloor. \tag{11}$$

5 Expression of the r-Lah numbers in terms of the $(r \pm s)$-Lah numbers

The r-Lah numbers satisfy the following horizontal recurrence relations. They express an element $\left\lfloor {n \atop k} \right\rfloor_r$ of r-Lah triangle in terms of the elements of the same line from the $(r+s)$-Lah triangle and $(r-s)$-Lah triangle.

Theorem 5.1 *The r-Lah numbers satisfy*

$$\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!} \sum_{i=0}^{s} \frac{(k+i+(r+s)-1)!}{(n+(r+s)-1)!} \binom{s}{i} \left\lfloor {n \atop k+i} \right\rfloor_{r+s}, \tag{12}$$

$$\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n-r)!}{(k-r)!} \sum_{i=0}^{s} \binom{s}{i} \frac{(k+i-r+s)!}{(n-r+s)!} \left\lfloor {n \atop k+i} \right\rfloor_{r-s}, \quad (r \geq s). \tag{13}$$

Proof. From (2), $\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!} \binom{n-r}{k-r}$, Vandermonde's formula gives $\left\lfloor {n \atop k} \right\rfloor_r = \frac{(n+r-1)!}{(k+r-1)!} \sum_{i=0}^{s} \binom{s}{i} \binom{n-r-s}{k+i-r-s}$, thus we get the result. The same approach gives the second relation. □

An expression of the Lah numbers in terms of the s-Lah numbers can be deduced from (12) for $r = 1$.

Corollary 5.1.1 *For $s \geq 1$, we get*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \frac{n!}{k!} \sum_{i=0}^{s-1} \binom{s-1}{i} \frac{(k+i+s-1)!}{(n+s-1)!} \left\lfloor \begin{matrix} n \\ k+i \end{matrix} \right\rfloor_s, \tag{14}$$

And for $s = 1$, in relations (12) and (13), we get

Corollary 5.1.2 *Triangular recurrence relations*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = (k+r+1) \frac{(k+r)}{(n+r)} \left\lfloor \begin{matrix} n \\ k+1 \end{matrix} \right\rfloor_{r+1} + \frac{(k+r)}{(n+r)} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_{r+1}, \tag{15}$$

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_{r+1} = (k-r+1) \frac{(k-r)}{(n-r)} \left\lfloor \begin{matrix} n \\ k+1 \end{matrix} \right\rfloor_r + \frac{(k-r)}{(n-r)} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r. \tag{16}$$

Using (7) in (15), we get a recurrence relation of order 3 with integer coefficients which improve the quality of the recurrence relation.

Corollary 5.1.3 *The following recurrence of order three holds*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rfloor_{r+1} + 2(k+r) \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_{r+1} + (k+r+1)(k+r) \left\lfloor \begin{matrix} n-1 \\ k+1 \end{matrix} \right\rfloor_{r+1}.$$

As a special case of (13), for $s = r$, we get

Corollary 5.1.4 *Expression of the r-Lah numbers in terms of the Lah numbers*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r = \frac{(n-r)!}{n!\,(k-r)!} \sum_{i=0}^{r} (k+i)! \binom{r}{i} \left\lfloor \begin{matrix} n \\ k+i \end{matrix} \right\rfloor.$$

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