A note on the value about a disjoint convex partition problem *

Xinshang You, Xianglin Wei†
xinshangyou@163.com, wxlhebtu@126.com
College of Science, Hebei University of Science and Technology, 050018, China

Abstract

Let P be a planar point set with no three points collinear. A k-hole of P is a convex k-gon H such that the vertices of H are elements of P and no element of P lies inside H. In this article, we prove that for any planar 9 points set with no three points collinear, with at least 5 vertices on the boundary of the convex hull, contains a 5-hole and a disjoint 3-hole.

1 Introduction

In this paper, we only deal with the finite planar point set P in *general position*, that is no three points in P are collinear. Let P be a planar point set, denote $Ch(P)$ the *convex hull* of P. For $Q \subseteq P$ with $Ch(Q) \bigcap P = Q$, we distinguish the vertices which lie on the convex hull boundary from the remaining interior points. Let $V(Q)$ be a set of the *vertices* of Q, and $I(Q)$ be a set of the *interior points* of Q, $|Q|$ be the number of points contained in Q. We say Q is *empty* if $I(Q) = \emptyset$. A k-hole of P is a convex k-gon H such that the vertices of H are elements of P and no element of P lies inside H. A family of holes $\{H_i\}_{i\in I}$ is called pairwise disjoint, or simply disjoint, if $Ch(H_i) \cap Ch(H_j) = \emptyset$, $i \neq j; i \in I, j \in I$. Here, I is an index set. Determine the smallest integer $n(k_1, ..., k_l)$, $k_1 \leq k_2 \leq ... \leq k_l$, such that any set in general position of at least $n(k_1, ..., k_l)$ points of the plane, contains a k_i*-hole* for every i, $1 \leq i \leq l$, where the holes are disjoint. Urabe [1] showed that $n(3,4) = 7$ and Hosono and Urabe [2] showed that $n(4,4) = 9$. In [3], Hosono and Urabe also showed that $n(3,5) = 10$. The result $n(3,4) = 7$

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†Corresponding Author

and $n(4,5) \leq 14$ were reconfirmed by Wu and Ding [4]. In [5], Hosono and Urabe proved $n(4,5) \leq 13$ and $n(5,5) \geq 17$. Aichholzer et al. [6] gave this Ramsey-type theorem: Every set of 11 points in the plane, no three on a line, contains either a 6-hole or a 5-hole and a disjoint 4-hole. Bhattacharya and Das [7], [8] gave a geometric proof of this theorem. Later, they used this result to prove $n(4,5) = 12$ [9]. Also, they evaluated the upper bound on $n(5,5)$ to 19 [10].

Let R be a region in the plane. An interior point of R is an element of a given point set P in its interior, and we say R is *empty* if R contains no interior points. Let $p_1, p_2, ..., p_k$ be k points of set P. We denote $(p_1p_2...p_k)$ be a convex closed region with vertices $p_1, p_2, ..., p_k$, which are labeled consecutively; denote a k-hole H by $H = (p_1p_2...p_k)_k$ if the closed region $(p_1p_2...p_k)$ is empty. Let $l(a,b)$ be the line passing points a and b. Denote the closed half-plane with $l(a,b)$, which contains c or does not contain c by $H(c;ab)$ or $H(\overline{c};ab)$, respectively. Denote the convex cone by $\gamma(a;b,c)$ with apex a, determined by a, b and c. For $\beta = b$ or c of $\gamma(a;b,c)$, let β' be a point such that a is on the line segment $\overline{\beta\beta'}$. If we see $\gamma(a;b',c)$, it means that a lies on the segment $\overline{bb'}$. As shown in Figure 1, $\gamma(a;b',c)$, $\gamma(a;b,c')$ and $\gamma(a;b',c')$ mean the convex cone 1, 2 and 3, respectively. If $\gamma(a;b,c)$ is not empty, we define an *attack point* $\alpha(a;b,c)$, such that from the half-line ab to ac, $\gamma(a;b,\alpha(a;b,c))$ is empty. When indexing a set of t points, we identify indices modulo t.

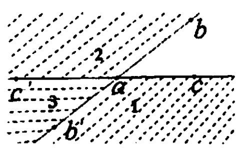

Figure 1: $\gamma(a;b',c)$ and $\gamma(a;b',c')$ mean the convex cone 1 and 3, respectively

Lemma 1. *[10] Any set P of 9 points in the plane in general position with more than 3 points on the boundary of the convex hull, contains a 5-hole.*

Figure 2 shows a 9 points set with 4 vertices, we can not find a 5-hole and a disjoint 3-hole. Figure 3 shows a 8 points set with 5 vertices, we can not find a 5-hole and a disjoint 3-hole.

2 Main Result and Proof

Theorem 2. *Any set P of 9 points in the plane in general position with more than 4 points on the boundary of the convex hull, contains a 5-hole and a disjoint 3-hole.*

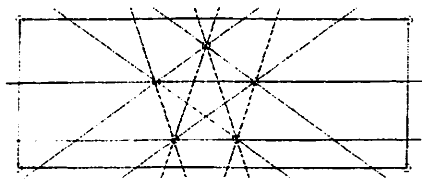

Figure 2: 9 points set with 4 vertices, no a 5-hole and a disjoint 3-hole.

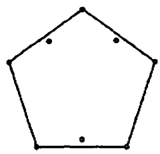

Figure 3: 8 points set with no a 5-hole and a disjoint 3-hole

Proof. Let P be a planar 9 points set in general position, with $|V(P)| \geq 5$. Obviously $|V(P)| \leq 9$. In the following, we depart two parts to discuss.

Part one: $6 \leq |V(P)| \leq 9$.

Case 1: $|V(P)| = 9$. Denote the vertices of P by v_i for $i = 1, ..., 9$ in anticlockwise. We have $(v_1v_2v_3v_4v_5)_5$ and $(v_6v_7v_8)_3$.

Case 2: $|V(P)| = 8$. Denote the vertices of P by v_i for $i = 1, ..., 8$ in anticlockwise and the remaining 1 interior point by p_1. Assume $p_1 \in (v_1v_2v_3v_4v_5)$. (Symmetrically, when $p_1 \in H(\overline{v_2}; v_1v_5)$, our conclusion is also right.) If $p_1 \in (v_1v_5v_6v_7v_8)$, we have $(v_1v_5v_6v_7v_8)_5$ and $(v_2v_3v_4)_3$. If $p_1 \in (v_2v_3v_4)$, we have $(v_1v_2p_1v_4v_5)_5$ and $(v_6v_7v_8)_3$.

Case 3: $|V(P)| = 7$. Denote the vertices of P by v_i for $i = 1, ..., 7$ in anticlockwise and the remaining 2 interior points by p_1, p_2. Consider the two open half-planes $H(v_1; p_1p_2)$ and $H(\overline{v_1}; p_1p_2)$ separated by $l(p_1, p_2)$. If either of these two half-planes contains more than 4 points of $V(P)$, the result is correct. Otherwise, we may assume that $|H(v_1; p_1p_2) \cap V(P)| = 4$. In this case $(H(v_1; p_1p_2) \cap V(P)) \cup \{p_1\}$ form a 5-hole disjoint from the 3-hole in $H(\overline{v_1}; p_1p_2) \cap V(P)$.

Case 4: $|V(P)| = 6$. Denote the vertices of P by v_i for $i = 1, ..., 6$ in anticlockwise and the remaining 3 interior points by p_1, p_2, p_3. In the following we will consider the location of p_1, p_2 and p_3.

Subcase 4.1: All of the 3 points lie in $(v_1v_2v_6)$. We have $(v_2v_3v_4v_5v_6)_5$ and $(p_1p_2p_3)_3$.

Subcase 4.2: Two of the 3 points, say p_1, p_2, lie in $(v_1v_2v_6)$.

If $p_3 \in (v_2v_5v_6)$, we have $(p_3v_2v_3v_4v_5)_5$ and $(v_1p_1p_2)_3$. If $p_3 \in (v_2v_3v_6)$, we have $(p_3v_3v_4v_5v_6)_5$ and $(v_1p_1p_2)_3$. If $p_3 \in (v_3v_4v_5)$, we have $(v_2v_3p_3v_5v_6)_5$ and $(v_1p_1p_2)_3$. If $p_3 \in \gamma(v_3; v_5, v_6) \cap \gamma(v_5; v_2, v_3)$: and if $(p_3; v_2, v_4')$ is not empty, we have $(v_2v_3v_4p_3p_1)_5$ and a 3-hole from the remaining 4 points, where $p_1 = \alpha(p_3; v_2, v_4')$; and if $(p_3; v_2, v_4')$ is empty, we have $(v_4v_5v_6p_2p_3)_5$ and a 3-hole from the remaining 4 points, where $p_2 = \alpha(p_3; v_6, v_4')$.

Subcase 4.3: One of the 3 points, say p_1, lies in $(v_1v_2v_6)$.

Then we will consider the location of the remaining 2 points, say p_2, p_3. Assume p_2, p_3 are in $(v_2v_5v_6)$. We have $(v_2v_3v_4v_5p_2)_5$ and $(v_1p_1v_6)_3$, where $p_2 = \alpha(v_2; v_5, v_6)$. Assume p_2, p_3 are in $(v_2v_3v_4v_5)$. If $(v_2v_3v_5)$ is empty, we have $(p_1v_2v_3v_5v_6)_5$ and $(p_2p_3v_4)_3$; if $(v_2v_3v_5)$ is not empty, let $p_2 = \alpha(v_5; v_2, v_3)$, we have $(p_1v_2p_2v_5v_6)_5$ and $(p_3v_3v_4)_3$. Assume p_2 is in $(v_2v_5v_6)$, and p_3 is in $(v_2v_3v_4v_5)$. If $p_2 \in \gamma(v_6; v_2, v_3) \cap \gamma(v_2; v_5, v_6)$: and if $p_3 \in \gamma(p_2; v_2, v_6')$, we have $(p_2v_3v_4v_5v_6)_5$ and $(v_1v_2p_1)_3$; and if $p_3 \in \gamma(p_2; v_4, v_6')$, we have $(p_2p_3v_4v_5v_6)_5$ and $(v_1v_2p_1)_3$; and if $p_3 \in \gamma(p_2; v_4, v_5)$, we have $(p_2v_2v_3v_4p_3)_5$ and $(v_1v_6p_1)_3$. If $p_2 \in \gamma(v_6; v_3, v_5) \cap \gamma(v_2; v_5, v_6)$: and if $p_3 \in \gamma(p_2; v_3, v_2)$, we have $(p_3v_3v_4v_5p_2)_5$ and $(v_1v_2p_1)_3$; and if $p_3 \in \gamma(p_2; v_3, v_5)$, we have $(p_1v_2v_3p_2v_6)_5$ and $(v_4v_5p_3)_3$.

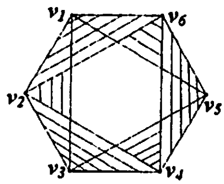

Figure 4: The shaded region is empty.

Subcase 4.4: No one of the three points lies in $(v_1v_2v_6)$. Similarly as before we can prove the result is correct, when the shaded region of Figure 4 is not empty. So in the following, we may assume the shaded region of Figure 4 is empty.

If $\gamma(v_6; v_3, v_4)$ or $\gamma(v_6; v_2, v_3)$ is empty, we can easy to see the result is correct. So we may assume $\gamma(v_6; v_3, v_4)$ and $\gamma(v_6; v_2, v_3)$ are not empty, without loss of generality, suppose $p_1 \in \gamma(v_6; v_3, v_4)$ and $p_2, p_3 \in \gamma(v_6; v_2, v_3)$. Assume $p_1 \in \gamma(v_4; v_1, v_6)$. If $\gamma(p_1; v_6, v_4')$ or $\gamma(p_1; v_3, v_5')$ is not empty, the result is right. If $\gamma(p_1; v_6, v_4')$ and $\gamma(p_1; v_3, v_5')$ are empty, we have $(v_2v_3v_4p_1p_2)_5$ and $(v_1v_5v_6)_3$ when $\gamma(p_1; v_2, v_5')$ is empty and $p_2 = \alpha(p_1; v_5', v_2)$, we have $(v_1v_6v_5p_1p_2)_5$ and $(v_2v_3v_4)_3$ when $\gamma(p_1; v_1, v_4')$ is empty and $p_2 = \alpha(p_1; v_4', v_1)$, we have $(v_1v_2p_2p_1p_3)_5$ and $(v_3v_4v_5)_3$ when $p_2 \in \gamma(p_1; v_2, v_5')$, $p_3 \in \gamma(p_1; v_1, v_4')$. Assume $p_1 \in \gamma(v_4; v_1, v_2)$. The result is also right by the similar reason for $p_1 \in \gamma(v_4; v_1, v_6)$.

Part two: $|V(P)| = 5$.

By Lemma 1, we know P contains a 5-hole F, denoted by $(v_1v_2v_3v_4v_5)_5$. Name the remaining 4 points p_1, p_2, p_3, p_4. Denote the convex cone $E_i = \gamma(v_i; v_{i+1}, v_{i-1}')$ for $1 \le i \le 5$, and the triangular zone $F_i = E_i \cap H(v_i; v_{i+1}v_{i+2})$ for $1 \le i \le 5$. If any triangular zone F_i contains at least three of the remaining 4 points p_1, p_2, p_3, p_4, then the conclusion is right. So we may assume

$|F_i| \le 2$, that is to say, every triangular zone F_i contains at most two of the remaining 4 points p_1, p_2, p_3, p_4. Let $Q = \{v_1, v_2, v_3, v_4, v_5\} \cap V(P)$. In the following, we will consider the value of $|Q|$. Since $|V(P)| = 5$, so $|Q| \ge 1$. If $|Q| = 5$, then we can prove F is not a 5-hole, a contradiction. So $|Q| \le 4$.

Case 1: $|Q| = 4$. Assume $v_1, v_2, v_3, v_4 \in Q$.

If F_1, F_2 and F_3 are not empty, then we have $|Q| \ge 5$. So suppose F_1, F_2 and F_3 are empty. Assume $|F_5| = 0$. Then $p_1, p_2, p_3, p_4 \in \gamma(v_4; v_5, v_3')$, we have the 5-hole F and a 3-hole from $\{p_1, p_2, p_3, p_4\}$. Assume $|F_5| = 1$. Let $p_1 \in F_5$, we have the 5-hole F and $(p_2p_3p_4)_3$. Assume $|F_5| = 2$. Let $p_1, p_2 \in F_5$ and $p_1 = \alpha(v_5; v_1, v_4')$, we have $(p_1v_1v_2v_3v_5)_5$ and $(v_4p_3p_4)_3$.

Case 2: $|Q| = 3$. Assume $v_1, v_2, v_3 \in Q$.

If F_1 and F_2 are not empty, then we have $|Q| \ge 4$. So suppose F_1 and F_2 are empty. Assume $|F_5| = 0$. If $|F_3| = 0$, we have the 5-hole F and a 3-hole from $\{p_1, p_2, p_3, p_4\}$. If $|F_3| = 1$, let $p_1 \in F_3$, we have the 5-hole F and $(p_2p_3p_4)_3$. If $|F_3| = 2$, let $p_1, p_2 \in F_3$ and $p_1 = \alpha(v_4; v_3, v_5')$, we have $(p_1v_4v_1v_2v_3)_5$ and $(v_5p_3p_4)_3$. Assume $|F_5| = 1$. Let $p_1 \in F_5$. If $|F_3| = 0$, we have the 5-hole F and $(p_2p_3p_4)_3$. If $|F_3| = 1$, let $p_2 \in F_3$, we have $(p_1v_1v_2v_3p_2)_5$ and a 3-hole from $\{v_4, v_5, p_3, p_4\}$. If $|F_3| = 2$, let $p_2, p_3 \in F_3$, we have $(p_1v_1v_2v_3v_5)_5$ and $(v_4p_2p_3)_3$. Assume $|F_5| = 2$. Let $p_1, p_2 \in F_5$ and $p_1 = \alpha(v_5; v_1, v_4')$. If $|F_3| = 0$, we have $(p_1v_1v_2v_3v_4)_5$ and $(v_5p_3p_4)_3$. If $|F_3| = 1$, let $p_3 \in F_3$, we have $(v_1v_2v_3p_3v_4)_5$ and $(v_5p_1p_2)_3$. If $|F_3| = 2$, we have $(p_1v_1v_2v_3v_5)_5$ and $(v_4p_3p_4)_3$.

Case 3: $|Q| = 2$. Assume $v_1, v_2 \in Q$.

If F_1 is not empty, then we have $|Q| \ge 3$. So suppose F_1 is empty.

Subcase 3.1: $|F_2| = 0$.

At first assume $|F_5| = 0$. If $\gamma(v_4; v_3, v_5') \cap H(v_3; v_1v_2)$ has at most 1 point or at least 3 points, our conclusion is right. So we may suppose $\gamma(v_4; v_3, v_5') \cap H(v_3; v_1v_2)$ has 2 points, say p_1, p_2. Let $p_1 = \alpha(v_3; v_4', v_2')$, $p_3, p_4 \in \gamma(v_4; v_3', v_5) \cap H(v_4; v_1v_2)$, and $p_3 = \alpha(v_5; v_4', v_1')$. If $p_1 \in \gamma(v_3; v_5', v_2') \cap \gamma(v_4; v_3, v_5')$, $p_3 \in \gamma(v_4; p_1', v_3')$, we have the 5-hole F and a 3-hole from $\{p_1, p_2, p_3, p_4\}$. For other locations of p_1 and p_3, the proof are similar.

Secondly assume $|F_5| = 1$. Let $p_1 \in F_5$. If $\gamma(v_4; v_3, v_5') \cap H(v_3; v_1v_2)$ is empty, we have the 5-hole F and $(p_2p_3p_4)_3$. If $\gamma(v_4; v_3, v_5') \cap H(v_3; v_1v_2)$ has 1 point, say p_2, we have $(p_1v_1v_2v_3v_5)_5$ and $(v_4p_3p_4)_3$. If $\gamma(v_4; v_3, v_5') \cap H(v_3; v_1v_2)$ has 2 points, say p_2, p_3, we have $(p_1v_1v_2v_3v_5)_5$ and $(v_4p_2p_3)_3$.

Thirdly assume $|F_5| = 2$. Let $p_1, p_2 \in F_5$ and $p_1 = \alpha(v_5; v_1, v_4')$. If $\gamma(v_4; v_3, v_5') \cap H(v_3; v_1v_2)$ is empty, we have $(p_1v_1v_2v_3v_5)_5$ and $(v_4p_3p_4)_3$. If $\gamma(v_4; v_3, v_5') \cap H(v_3; v_1v_2)$ has 1 point, say p_3: and if $p_4 \in \gamma(v_4; v_3', v_5')$, we have $(p_1v_1v_2v_3v_5)_5$ and $(v_4p_3p_4)_3$; and if $p_4 \in \gamma(v_5; p_1', v_4) \cap H(v_5; v_3v_4)$, we have $(p_1v_1v_4p_4v_5)_5$ and $(v_2v_3p_3)_3$; and if $p_4 \in \gamma(v_5; p_1', v_4')$, we have $(p_1v_1v_2v_3v_4)_5$ and $(p_4p_2v_5)_3$. If $\gamma(v_4; v_3, v_5') \cap H(v_3; v_1v_2)$ has 2 points, say p_3, p_4, we have $(p_1v_1v_2v_3v_5)_5$ and $(v_4p_3p_4)_3$.

Subcase 3.2: $|F_2| = 1$. Let $p_1 \in F_2$.

If $|F_5| = 0$, by the Subcase 3.1, we know our conclusion is right. If $|F_5| = 1$, let $p_2 \in F_5$, we have the 5-hole F and $(p_1p_3p_4)_3$ when the shaded region of Figure 5 is empty; we have $(p_2v_1v_2p_1v_5)_5$ and a 3-hole from the points in the shaded region when the shaded region of Figure 5 is not empty. If $|F_5| = 2$, let $p_2, p_3 \in F_5$, we have $(p_1v_3v_4v_1v_2)_5$ and $(p_2p_3v_5)_3$.

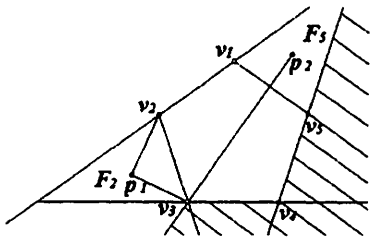

Figure 5:

Subcase 3.3: $|F_2| = 2$. Let $p_1, p_2 \in F_2$.

Let $p_1 = \alpha(v_3; v_2, v_4')$. If $|F_5| \leq 1$, we know our conclusion is right similarly as before. If $|F_5| = 2$, let $p_3, p_4 \in F_5$, we have $(p_1v_3v_4v_1v_2)_5$ and $(v_5p_3p_4)_3$.

Case 4: $|Q| = 1$. Without loss of generality, suppose $v_2 \in Q$. And p_1, p_2, p_3, p_4 are all the vertices of P in clockwise.

Subcase 4.1: $\gamma(v_1; v_2, v_4')$ is not empty (Similarly, when $\gamma(v_3; v_2, v_5')$ is not empty, the result is also right).

Let $p_1 = \alpha(v_1; v_2, v_4')$. We have $(p_1v_2v_3v_4v_1)_5$ and $(v_5p_2p_3)_3$.

Subcase 4.2: $\gamma(v_1; v_2, v_4')$ and $\gamma(v_3; v_2, v_5')$ are empty.

At first assume $p_1 \in \gamma(v_1; v_3', v_4')$. Then $p_2 \in \gamma(p_1; v_5, v_2')$. Suppose $p_2 \in \gamma(p_1; v_5, v_2') \cap H(p_1; v_4v_5)$. We have $(p_1v_1v_4v_5p_2)_5$ and $(v_2v_3p_4)_3$. Suppose $p_2 \in \gamma(v_5; v_4', v_1')$. If $p_3 \in \gamma(p_1; v_5, v_2')$, we have the 5-hole F and $(p_1p_2p_3)_3$; if $p_3 \in \gamma(v_4; v_1', p_2) \cap H(v_4; p_1v_5)$, we have $(p_1v_1v_4p_3v_5)_5$ and $(v_2v_3p_4)_3$; if $p_3 \in \gamma(v_4; v_1', p_2') \cap H(v_4; v_2v_3)$, we have $(v_1v_2v_3p_3v_4)_5$ and $(v_5p_1p_2)_3$; if $p_3 \in \gamma(v_3; v_1', v_2') \cap H(\overline{v_3}; v_4v_5)$, we have $(v_1v_3p_4p_3v_4)_5$ and $(v_5p_1p_2)_3$ when $p_4 \in \gamma(v_3; v_1', v_2')$, or we have $(p_1v_2p_4v_3v_1)_5$ and $(v_4v_5p_2)_3$ when $p_4 \in \gamma(v_3; v_1', v_5')$; if $p_3 \in \gamma(v_4; p_2', v_2') \cap H(\overline{v_4}; v_2v_3)$, we have $(P_1v_2p_4p_3v_3)_5$ and $(v_4v_5p_2)_3$. Suppose $p_2 \in \gamma(v_5; p_1', v_1')$. If $p_2 \in F_4$, we have $(p_2v_5v_1v_2v_3)_5$ and $(v_4p_3p_4)_3$; if $p_2 \in \gamma(v_5; p_1', v_1') \cap H(\overline{v_5}; v_3v_4)$, we have the 5-hole F and $(p_2p_3p_4)_3$.

Secondly assume $p_1 \in \gamma(v_1; v_2', v_3') \cap H(v_1; v_4v_5)$. Similarly as before, we know the conclusion is also right when $p_4 \in \gamma(v_3; v_1', v_5')$. So we can assume $p_4 \in \gamma(v_3; v_1', v_2') \cap H(v_3; v_4v_5)$. Suppose $p_2 \in \gamma(p_1; v_5, v_2') \cap H(p_1; v_4v_5)$. If $p_3 \in H(p_2; p_1v_5)$, we have the 5-hole F and $(p_1p_2p_3)_3$; if $p_3 \in \gamma(v_5; v_4, p_1') \cap H(v_5; v_1v_4)$, we have $(p_1v_1v_4p_3v_5)_5$ and $(v_2v_3p_4)_3$; if $p_3 \in \gamma(v_4; v_1', v_5') \cap H(v_4; v_2v_3)$, we have $(v_1v_2v_3p_3v_4)_5$ and $(p_1p_2v_5)_3$; if $p_3 \in \gamma(v_3; v_2', v_4')$, we have $(p_1v_1v_3v_4v_5)_5$ and $(v_2p_3p_4)_3$. Suppose $p_2 \in \gamma(p_1; v_5, v_2') \cap \gamma(v_4; v_5, v_3')$. If $p_3 \in H(p_2; p_1v_5)$, we have the 5-hole F and $(p_1p_2p_3)_3$; if $p_3 \in \gamma(v_4; p_2, v_1') \cap$

$H(v_4;p_1v_5)$, we have $(p_1v_1v_4p_3v_5)_5$ and $(v_2v_3p_4)_3$; if $p_3 \in \gamma(v_4;p'_2,v'_1) \cap H(v_4;v_2v_3)$, we have $(v_1v_2v_3p_3v_4)_5$ and $(v_5p_1p_2)_3$; if $p_3 \in \gamma(v_4;p'_2,v'_1) \cap H(\overline{v_4};v_2v_3)$, we have$(p_1v_1v_3v_4v_5)_5$ and $(v_2p_3p_4)_3$. Suppose $p_2 \in \gamma(v_4;v'_3,v'_5) \cap H(\overline{v_4};p_1v_5)$. We have the 5-hole F and $(p_2p_3p_4)_3$.

Thirdly assume $p_1 \in \gamma(v_5;v'_1,v'_2) \cap H(\overline{v_1};v_4v_5)$. If $p_4 \in \gamma(v_4;v'_3,v'_5) \cap H(\overline{v_3};v_4v_5)$, then we have the 5-hole F and a 3-hole from $\{p_1,p_2,p_3,p_4\}$. If $p_4 \in \gamma(v_3;v'_1,v'_5)$ or $p_4 \in \gamma(v_3;v'_2,v'_1) \cap H(v_3;v_4v_5)$, the proof is similar. □

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