On two types of $(2, k)$-distance Lucas numbers

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Abstract

In this paper we define new types of generalizations in the distance sense of Lucas numbers. These generalizations are based on introduced recently the concept of $(2, k)$-distance Fibonacci numbers. We study some properties of these numbers and present identities which generalize known identities for Lucas numbers. Moreover, we show representations and interpretations of these numbers.

Keywords: Fibonacci numbers, Lucas numbers, Perrin numbers, identities

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1 Introduction

In general we use the standard notation, see [2, 4]. The Fibonacci numbers F_n are defined by the following recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial conditions $F_0 = F_1 = 1$. The Lucas numbers L_n are defined by the same recurrence $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with $L_0 = 2$, $L_1 = 1$. In the literature there are many directions of generalizations of the Fibonacci numbers with respect to one or more parameters, see [6, 7, 8, 9, 10, 14]. Some of them generalize Fibonacci numbers in the distance sense. In [13] M. Kwaśnik and I. Włoch introduced generalized Fibonacci numbers $F(k, n)$ and generalized Lucas numbers $L(k, n)$ as follows: $F(k, n) = F(k, n-1) + F(k, n-k)$ for $n \geq k+1$ with $F(k, n) = n+1$ for $n \leq k$. Generalized Lucas numbers $L(k, n)$ were defined using numbers $F(k, n)$. Recently in [14] more comfortable recurrence for $L(k, n)$ was given, namely $L(k, n) = L(k, n-1) + L(k, n-k)$ for $n \geq 2k$ with $L(k, n) = n+1$ for $n = 0, 1, \ldots, 2k-1$. The paper [13] initiated studying of different kinds

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of distance generalizations of the numbers of the Fibonacci type, see for instance the last papers [1, 3, 15, 17]. The interest of these numbers was motivated by their graphs interpretations which are closely related with distance independent set. Independent sets and kernels are intensively studied in the graph literature, see [18, 19] and the last interesting papers of H. Galeana-Sanches and C. Hernandez-Cruz [11, 12].

In [1] it was presented the distance Fibonacci numbers $Fd(k,n)$ defined in the following way

$$Fd(k,n) = Fd(k,n-k+1) + Fd(k,n-k) \text{ for } n \geq k,$$

where $Fd(k,n) = 1$ for $n = 0, 1, \ldots, k-1$, $k \geq 1$.

Natural continuation of these numbers were distance Fibonacci numbers introduced in [17], called $(2,k)$-distance Fibonacci numbers $F_2(k,n)$ and defined recursively by the following relation

$$F_2(k,n) = F_2(k,n-2) + F_2(k,n-k) \text{ for } n \geq k \tag{1}$$

with initial conditions $F_2(k,i) = 1$ for $i = 0, 1, \ldots, k-1$.
The Table 1 includes initial words of $(2,k)$-Fibonacci numbers for special k and n.

Tab.1. $(2,k)$-distance Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$F_2(1,n)$	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
$F_2(2,n)$	1	1	2	2	4	4	8	8	16	16	32	32	64	64	128
$F_2(3,n)$	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37
$F_2(4,n)$	1	1	1	1	2	2	3	3	5	5	8	8	13	13	21
$F_2(5,n)$	1	1	1	1	1	2	2	3	3	4	5	6	8	9	12
$F_2(6,n)$	1	1	1	1	1	1	2	2	3	3	4	4	6	6	9

In this paper, being a sequel of papers [1, 3, 17] we introduce two cyclic versions of the $(2,k)$-distance Fibonacci numbers $F_2(k,n)$ which generalize the Lucas numbers L_n.

2 Generalized Lucas numbers

Let $k \geq 1$, $n \geq 0$ be integers. The $(2,k)$-distance Lucas numbers $L_2^{(1)}(k,n)$ of the first kind are defined by the recurrence relation

$$L_2^{(1)}(k,n) = L_2^{(1)}(k,n-2) + L_2^{(1)}(k,n-k) \text{ for } n \geq k \tag{2}$$

with initial conditions

$$\begin{aligned} L_2^{(1)}(k,0) &= 2, \text{ for } k = 1,2,3, \\ L_2^{(1)}(k,0) &= k, \text{ for } k \geq 4, \\ L_2^{(1)}(k,1) &= k, \text{ for } k \geq 1, \\ L_2^{(1)}(k,n) &= 2 \text{ for } n = 2,3,\ldots,k-1. \end{aligned}$$

The Table 2 includes initial words of $(2,k)$-distance Lucas numbers of the first kind for special k and n.

Tab.2. $(2,k)$-distance Lucas numbers of the first kind

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$L_2^{(1)}(1,n)$	2	1	3	4	7	11	18	29	47	76	123	199	322	521
$L_2^{(1)}(2,n)$	2	2	4	4	8	8	16	16	32	32	64	64	128	128
$L_2^{(1)}(3,n)$	2	3	2	5	5	7	10	12	17	22	29	39	51	68
$L_2^{(1)}(4,n)$	4	4	2	2	6	6	8	8	14	14	22	22	36	36
$L_2^{(1)}(5,n)$	5	5	2	2	2	7	7	9	9	11	16	18	25	27
$L_2^{(1)}(6,n)$	6	6	2	2	2	2	8	8	10	10	12	12	20	20

It is easily seen that $L_2^{(1)}(1,n) = L_n$. For $k = 2$ we obtain known sequence with powers of 2 which double up. Moreover, $L_2^{(1)}(3,n) = Pr(n+2)$, where $Pr(n)$ is the n-th Perrin number defined as follows: $Pr(n) = Pr(n-2)+Pr(n-3)$ for $n \geq 3$ with $Pr(0) = 3, Pr(1) = 0, Pr(2) = 2$. For even k and $n \geq 0$ we have $L_2^{(1)}(k,2n) = L_2^{(1)}(k,2n+1)$. It is easily seen that for even k and $n \geq k$ the sequence of numbers $L_2^{(1)}(k,n)$ has terms repeated twice. In addition, for $k = 4$ and $n \geq 1$ $L_2^{(1)}(4,2n) = 2L_2^{(1)}(1,n)$, so in this case we obtain double Lucas sequence.

Now we give a combinatorial interpretation of the $(2,k)$-distance Lucas numbers of the first kind. It is worth mentioning that $L_2^{(1)}(2,n-1)$ for $n \geq 1$ is the number of symmetric partitions of n or equivalently the number of subsets S of the set $\{1,2,\ldots,n\}$ which satisfies the following condition: $m \in S$ implies that $n-m+1 \in S$, see [5].

Assume now that $k \geq 1$, $k \neq 2$, $n \geq 3$ are integers and $n \geq k$. Let $X = \{1,2,\ldots,n\}$. For $i,j \in X$ we define $i \oplus j$ as follows

$$i \oplus j = \begin{cases} i+j & \text{for } i+j \leq n, \\ i+j-n & \text{for } i+j > n. \end{cases}$$

In other words we say that X contains n cyclically consecutive integers.

Let $1 \leq k \leq n+1$ and $\mathcal{C}(k,n) = \{C_i; i = 1,2,\ldots,p\}$ such that $C_i = \{t_{i-1} \oplus 1, t_{i-1} \oplus 2, \ldots, t_i\}$ for $i = 1,2,\ldots,p$ and

(i). 1 is an element of C_1,
(ii). $|C_i| \in \{2, k\}$ for $i = 1, 2, \ldots, p$,
(iii). $n - 1 \leq \sum_{i=1}^{p} |C_i| \leq n$.
The family $\mathcal{C}(k, n)$ is called *cyclic k-decomposition of the first kind* of the set X.

Let us recall from [17] that $\mathcal{Y} = \{Y_t; t \in T\}$ being the family of disjoint subsets of the set X such that each subset Y_t, $t \in T$ contains consecutive integers and satisfies the following conditions:
d) $|Y_t| \in \{2, k\}$ for $t \in T$,
e) $|X \setminus \bigcup_{t \in T} Y_t| \in \{0, 1\}$,
f) if $m \in (X \setminus \bigcup_{t \in T} Y_t)$ then $m = n$.
is called *a decomposition with the rest at most one of the set X*.

Theorem 1 *Let $n \geq 3$, $1 \leq k \leq n$, $k \neq 2$, be integers. Then the number of all cyclic k-decompositions of the first kind of the set X is equal to the number $L_2^{(1)}(k, n)$.*

Proof. It is easily seen that the Theorem holds for $n = k$. Let $n \geq k + 1$. By the definition of the family $\mathcal{C}(k, n)$ we consider two cases:
(1) $|C_1| = 2$.
Since $1 \in C_1$, we have exactly two possibilities of subsets C_1 of the form $\{1, 2\}$, $\{1, n\}$. Thus $\mathcal{C}(k, n) = \mathcal{F}(k, n) \cup C_1$, where $\mathcal{F}(k, n)$ is any decomposition with the rest at most one of the set $X \setminus C_1$. Therefore we obtain $2F_2(k, n - 2)$ families $\mathcal{F}(k, n)$ in this case.
(2) $|C_1| = k$.
Proving analogously to the case (1), we obtain $kF_2(k, n-k)$ families $\mathcal{C}(k, n)$ including the subset C_1 of the cardinality k. Finally, we obtain that the number of all families $\mathcal{C}(k, n)$ is equal to $2F_2(k, n - 2) + kF_2(k, n - k)$.

Claim.

$$2F_2(k, n - 2) + kF_2(k, n - k) = L_2^{(1)}(k, n) \quad \text{for} \quad n \geq k. \tag{3}$$

Proof (by induction on n). If $n = k$ then the result is obvious. Let $n > k$. Assume that the formula (3) is true for an arbitrary n. We will prove it for $n + 1$. By the recurrence definitions of the numbers $L_2(k, n)$ and $F_2(k, n)$ and by the induction hypothesis, we obtain

$$\begin{aligned} & L_2^{(1)}(k, n + 1) = L_2^{(1)}(k, n - 1) + L_2^{(1)}(k, n + 1 - k) = \\ = \;& 2F_2(k, n - 3) + kF_2(k, n - k - 1) + 2F_2(k, n - 1 - k) + \\ + \;& kF_2(k, n + 1 - 2k) = 2F_2(k, n - 1) + kF_2(k, n + 1 - k), \end{aligned}$$

which completes the proof. □

For $k=1$ from the above Claim we obtain the well-known identity $L_n = F_{n-1} + 2F_{n-2}$.

Now we introduce $(2,k)$−distance Lucas numbers of the second kind. Let $k \geq 1$, $n \geq 0$ be integers. The $(2,k)$−distance Lucas numbers of the second kind $L_2^{(2)}(k,n)$ are defined by the following recurrence relation

$$L_2^{(2)}(k,n) = L_2^{(2)}(k,n-2) + L_2^{(2)}(k,n-k) \text{ for } n \geq k \tag{4}$$

with initial conditions

$$\begin{aligned} L_2^{(2)}(k,0) &= 2 \text{ for } k \neq 3, \;\; L_2^{(2)}(3,0) = 1, \\ L_2^{(2)}(k,1) &= 2 \text{ for } k \neq 1, \;\; L_2^{(2)}(1,1) = 1, \\ L_2^{(2)}(k,n) &= 1 \text{ for } 2 < n \leq k-1. \end{aligned}$$

The Table 3 includes initial words of the $(2,k)$-Lucas numbers of the second kind for special k and n.

Tab.3. $(2,k)$-distance Lucas numbers of the second kind

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$L_2^{(2)}(1,n)$	2	1	3	4	7	11	18	29	47	76	123	199	322	521
$L_2^{(2)}(2,n)$	2	2	4	4	8	8	16	16	32	32	64	64	128	128
$L_2^{(2)}(3,n)$	1	2	1	3	3	4	6	7	10	13	17	23	30	40
$L_2^{(2)}(4,n)$	2	2	1	1	3	3	4	4	7	7	11	11	18	18
$L_2^{(2)}(5,n)$	2	2	1	1	1	3	3	4	4	5	7	8	11	12
$L_2^{(2)}(6,n)$	2	2	1	1	1	1	3	3	4	4	5	5	8	8

Observe, that for $k=1$ we have $L_2^{(2)}(1,n) = L_n$. Moreover, for $k=3$ we have special sequence, see [5]. Analogously as in the previous case for $k=2$ we get the same sequence with powers of 2 which double up and for $k=4$ we obtain double Lucas sequence. The sequence $L_2^{(2)}(k,n)$ for even k and $n \geq k$ also has terms repeated twice.

Now we give the combinatorial interpretation of the numbers $L_2^{(2)}(k,n)$. For $k=1$ we have known combinatorial representation of Lucas numbers. Assume that $k \geq 3$ and $n \geq 3$. Let $X = \{1,2,\ldots,n\}$ contains n cyclically consecutive integers. For fixed $k \geq 3$ and $n \geq 3$ let $C^*(k,n) = \{C_i^*; i = 1,2,\ldots,p\}$ such that $C_1^* = \{t_0 \oplus 1, t_0 \oplus 2, \ldots, t_1\}$, $C_2^* = \{t_1 \oplus 1, t_1 \oplus 2, \ldots, t_2\}$, …, $C_p^* = \{t_{p-1} \oplus 1, t_{p-1} \oplus 2, \ldots, t_p\}$ and the following conditions hold

(*i*). ($t_p = 1$ and $|C_p^*| = k$) or $t_0 = n$,

(*ii*). $|C_i^*| \in \{2,k\}$ for $i = 1,2,\ldots,p$,

(*iii*). $n-1 \leq \sum_{i=1}^{p} |C_i^*| \leq n$.

The family $C^\star(k,n)$ is called *cyclic k-decomposition of the second kind* of the set X.

Theorem 2 *Let $n \geq 3$, $3 \leq k \leq n$ be integers. Then the number of all cyclic k-decompositions of the second kind of the set X is equal to the number $L_2^{(2)}(k,n)$.*

Proof. For $n = k-1$ the Theorem is obvious. Let $n \geq k$. By the definition of the family $C^\star(k,n)$ we consider the following possibilities:
(1) $t_p = 1$ and $|C_p^\star| = k$.
Since the subset $\{n-k+2,\dots,n,1\} \in C^\star(k,n)$, we have that $C^\star(k,n) = \mathcal{F}(k,n) \cup \{n-k+2,\dots,n,1\}$, where $\mathcal{F}(k,n)$ is any decomposition with the rest at most one of the set $X \setminus \{n-k+2,\dots,n,1\}$, which is isomorphic to the set $X' = \{1,2,\dots,n-k\}$. From the combinatorial representation of the number $F_2(k,n)$ we obtain that there are $F_2(k,n-k)$ families $C^\star(k,n)$ including the subset $\{n-k+2,\dots,n,1\}$.
(2) $t_0 = n$ and $|C_1^\star| = k$.
Proving analogously to the case (1), we obtain $F_2(k,n-k)$ families $C^\star(k,n)$ including the subset $\{1,2,\dots,k\}$.
(3) $t_0 = n$ and $|C_1^\star| = 2$.
Proving analogously to the case (1), we obtain $F_2(k,n-2)$ families $C^\star(k,n)$ including the subset $\{1,2\}$.

Finally, we obtain that the number of all families $C^\star(k,n)$ is equal to $F_2(k,n-2) + 2F_2(k,n-k)$.

Claim.

$$F_2(k,n-2) + 2F_2(k,n-k) = L_2^{(2)}(k,n) \quad \text{for} \quad n \geq k. \tag{5}$$

Proof (by induction on n). If $n = k$ then the result is obvious. Let $n > k$ and assume that the formula (5) holds for an arbitrary n. We will show that it is true for $n+1$. Using the definitions of the numbers $L_2^{(2)}(k,n)$ and $F_2(k,n)$ and by the induction hypothesis, we have

$$\begin{aligned} & L_2^{(2)}(k,n+1) = L_2^{(2)}(k,n-1) + L_2^{(2)}(k,n-k+1) = \\ = \; & F_2(k,n-3) + 2F_2(k,n-k-1) + F_2(k,n-k-1) + \\ + \; & 2F_2(k,n-2k+1) = F_2(k,n-3) + F_2(k,n-k-1) + \\ + \; & 2(F_2(k,n-k-1) + F_2(k,n-2k+1)) = \\ = \; & F_2(k,n-1) + 2F_2(k,n-k+1), \end{aligned}$$

which ends the proof. □

It is worth mentioning that $(2,k)$-distance Lucas numbers of the first and the second kind have graph interpretation closely related to the concept of H-matchings. Let G and H be two graphs. An *H-matching* M of G is a subgraph of G such that all components of M are isomorphic to H. If M is also an induced subgraph of G, then H-matching is called induced. Problem of counting H-matchings in some graphs was considered in [14]. Let $V(C_n) = \{x_1, x_2, \dots, x_n\}$, $n \geq 3$, be the vertex set of the graph C_n numbering in the natural fashion. Graph interpretation of the numbers $L_2^{(1)}(k,n)$ and $L_2^{(2)}(k,n)$ concern the special coverings of C_n by subgraphs P_i, $i \in \{2,k\}$. In the first case the number $L_2^{(1)}(k,n)$ is equal to the number of $\{P_2, P_k\}$-matching of C_n such that the vertex x_1 is an arbitrary vertex of the subgraph P_i which belongs to $\{P_2, P_k\}$-matching of C_n . The number $L_2^{(2)}(k,n)$ is equal to the number of all such $\{P_2, P_k\}$-matchings of the graph C_n such that x_1 is a pendant vertex of a subgraph P_i, $i \in \{2,k\}$.

3 Identities

In this section we present the list of identities for the distance Lucas numbers of the first kind $L_2^{(1)}(k,n)$ and the second kind $L_2^{(2)}(k,n)$, which generalize known identities for Lucas and Perrin numbers.

Theorem 3 *For* $n \geq 1$

$$(i) \quad \sum_{i=1}^{n} L_2^{(1)}(k, ki+m) = L_2^{(1)}(k, nk+m+2) - 2 \text{ for } k \geq 3 \text{ and } 0 \leq m \leq k-3$$

$$(ii) \quad \sum_{i=1}^{n} L_2^{(1)}(3, 3i+1) = L_2^{(1)}(3, 3n+3) - 5.$$

Proof.
(*i*) (by induction on n). For $n = 1$ we have

$$\begin{aligned} L_2^{(1)}(k, k+m+2) - 2 &= L_2^{(1)}(k, k+m) + L_2^{(1)}(k, m+2) - 2 = \\ &= L_2^{(1)}(k, k+m) + 2 - 2 = L_2^{(1)}(k, k+m). \end{aligned}$$

Assume that the formula (6) holds for an arbitrary n. We will prove that

$$\sum_{i=1}^{n+1} L_2^{(1)}(k, ki+m) = L_2^{(1)}(k, (n+1)k+m+2) - 2.$$

By the induction hypothesis and recurrence relation (2), we obtain

$$\begin{aligned}
\sum_{i=1}^{n+1} L_2^{(1)}(k, ki+m) &= \sum_{i=1}^{n} L_2^{(1)}(k, ki+m) + L_2^{(1)}(k,(n+1)k+m) = \\
&= L_2^{(1)}(k, nk+m+2) - 2 + L_2^{(1)}(k, nk+k+m) = \\
&= L_2^{(1)}(k, nk+m+2+k) - 2 = L_2^{(1)}(k,(n+1)k+m+2) - 2,
\end{aligned}$$

which ends the proof.
(*ii*) analogously as in (*i*). □

Note that by (*i*) for $k = 3$ we obtain the well-known identity for the Perrin numbers:

$$\sum_{i=1}^{n} Pr(3i) = Pr(3n+2) - 2.$$

Theorem 4 *Let* $n \geq 1$*. Then*

(*iv*) $\sum_{i=1}^{n} L_2^{(2)}(k, ki+m) = L_2^{(2)}(k, nk+m+2) - 1$ *for* $k \geq 3$ *and* $0 \leq m \leq k-3$

(*v*) $\sum_{i=1}^{n} L_2^{(2)}(k, ki+m) = L_2^{(2)}(k, nk+m+2) - 3$ *for* $k \geq 3$ *and* $k-2 \leq m \leq k-1$,

(*vi*) $\sum_{i=1}^{n} L_2^{(2)}(k, ki+k) = L_2^{(1)}(k, nk+k+2) - 4$ *for* $k \geq 1, k \neq 2$.

Proof.
(*iv*) (by induction on n). It is easy to check that the formula (*iv*) is true for $n = 1$. Assume that the formula (*iv*) holds for an arbitrary n. We will prove that it holds for $n+1$. Using the induction hypothesis and the recurrence (4), we have

$$\begin{aligned}
\sum_{i=1}^{n+1} L_2^{(2)}(k, ki+m) &= \sum_{i=1}^{n} L_2^{(2)}(k, ki+m) + L_2^{(2)}(k, nk+k+m) \\
&= L_2^{(2)}(k, nk+m+2) - 1 + L_2^{(2)}(k, nk+k+m) = \\
&= L_2^{(2)}(k, nk+k+m+2) - 1,
\end{aligned}$$

which completes the proof.
(*v*), (*vi*) analogously as in (*iv*). □

For $k=1$ by (vi) we get the well-known identity for Lucas numbers

$$\sum_{i=1}^{n} L_{i+1} = L_{n+3} - 4.$$

Theorem 5 *For $n > k$ and $k \geq 3$*

$$L_2^{(2)}(k,n) = F_2(k,n) + F_2(k,n-k).$$

Proof. By the definition of the numbers $F_2(k,n)$ and (5), we have

$$\begin{aligned}
&F_2(k,n) + F_2(k,n-k) = \\
&= F_2(k,n-2) + F_2(k,n-k) + F_2(k,n-k-2) + F_2(k,n-2k) = \\
&= F_2(k,n-2) + F_2(k,n-k-2) + F_2(k,n-2k) + \\
&+F_2(k,n-k-2) + F_2(k,n-2k) = \\
&= F_2(k,n-4) + 2F_2(k,n-k-2) + F_2(k,n-k-2) + 2F_2(k,n-2k) = \\
&= L_2^{(2)}(k,n-2) + L_2^{(2)}(k,n-k) = L_2^{(2)}(k,n).
\end{aligned}$$

□

By the Table 4 we can observe that for $k \geq 3$ and $m = 0, 1, \ldots, k-1$ we have the following result:

$$L_2^{(2)}(k, k+m) = 3 + \left\lfloor \frac{m}{2} \right\rfloor.$$

Tab.4. Some $(2,k)$-distance Lucas numbers of the second kind

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$L_2^{(2)}(3,n)$	3	3	4										
$L_2^{(2)}(4,n)$		3	3	4	4								
$L_2^{(2)}(5,n)$			3	3	4	4	5						
$L_2^{(2)}(6,n)$				3	3	4	4	5	5				
$L_2^{(2)}(7,n)$					3	3	4	4	5	5	6		
$L_2^{(2)}(8,n)$						3	3	4	4	5	5	6	6

For both kinds of $(2,k)$-distance Lucas numbers we get.

Theorem 6 *For $n > 2k-2$ and $k \geq 1$ and $j = 1, 2$*

$$L_2^{(j)}(k,n) = L_2^{(j)}(k,n-2) + L_2^{(j)}(k,n-k+2) - L_2^{(j)}(k,n-2k+2). \quad (6)$$

Proof. We will prove the formula (6) for numbers $L_2^{(1)}(k,n)$. The same proof works for numbers $L_2^{(2)}(k,n)$.

Using twice the recurrence (2), we obtain

$$\begin{aligned}&L_2^{(1)}(k,n-2)+L_2^{(1)}(k,n-k+2)=\\&=L_2^{(1)}(k,n-2)+L_2^{(1)}(k,n-k)+L_2^{(1)}(k,n-2k+2)=\\&=L_2^{(1)}(k,n)+L_2^{(1)}(k,n-2k+2).\end{aligned}$$

Hence we get the formula (6). □

In [17] it was proved the following result.

Theorem 7 [17] *Let $n \geq 2$, $k \geq 2$ be integers. Then $F_2(k,n) = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \binom{i+\lfloor \frac{n-ik}{2} \rfloor}{i}$.*

By Theorem 5 and Theorem 7, we get the direct formula for numbers $L_2^{(2)}(k,n)$.

Theorem 8 *Let $n \geq 2$, $k \geq 3$ be integers. Then*

$$L_2^{(2)}(k,n) = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \binom{i+\lfloor \frac{n-ik}{2} \rfloor}{i} + \sum_{i=0}^{\lfloor \frac{n-k}{k} \rfloor} \binom{i+\lfloor \frac{n-k-ik}{2} \rfloor}{i}.$$

Analogously, using the equality (3) and Theorem 7, we get

Theorem 9 *Let $n \geq 2$, $k \geq 3$ be integers. Then*

$$L_2^{(1)}(k,n) = 2\sum_{i=0}^{\lfloor \frac{n-2}{k} \rfloor} \binom{i+\lfloor \frac{n-2-ik}{2} \rfloor}{i} + k\sum_{i=0}^{\lfloor \frac{n-k}{k} \rfloor} \binom{i+\lfloor \frac{n-k-ik}{2} \rfloor}{i}.$$

Similarly to the classical Lucas numbers, $(2,k)$-distance Lucas numbers $L_2^{(j)}(k,n)$, $j=1,2$, can be extended to negative integers n. Let $k \geq 1$, $k \neq 2$, $n \geq 0$ be integers. Then for $j=1,2$

$$L_2^{(j)}(k,-n) = L_2^{(j)}(k,-n+k) - L_2^{(j)}(k,-n+k-2) \text{ for } n \geq 1 \qquad (7)$$

with the same initial conditions as for $L_2^{(j)}(k,n)$.

For example, the Table 5 includes initial words of $(2,k)$-distance Lucas numbers of the second kind for special k and negative n.

Tab.5. $(2, k)$-distance Lucas numbers $L_2^{(2)}(k, n)$ for negative n

n	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1
$L_2^{(2)}(1, n)$	123	-76	47	-29	18	-11	7	-4	3	-1	2	1
$L_2^{(2)}(3, n)$	3	0	-2	3	-2	1	1	-1	2	0	1	2
$L_2^{(2)}(4, n)$	-11	-11	7	7	-4	-4	3	3	-1	-1	2	2
$L_2^{(2)}(5, n)$	6	1	-4	-3	1	3	2	-1	-1	0	2	2

Proving analogously as Theorem 4, we get the following relations for $(2, k)$-distance Lucas numbers $L_2^{(2)}(k, n)$ for negative integers.

Theorem 10 *Let $n \geq 0$. Then*

(i) $\sum_{i=1}^{n} L_2^{(2)}(k, -ki + k) = -L_2^{(2)}(k, -nk + 2) + 1$ *for* $k \geq 3$,

(ii) $\sum_{i=1}^{n} L_2^{(2)}(k, -ki) = -L_2^{(2)}(k, -nk - k + 2) - 1$ *for* $k \geq 4$,

(iii) $\sum_{i=1}^{n} L_2^{(2)}(3, -3i) = -L_2^{(2)}(k, -3n - 1)$.

4 Concluding remarks

In this paper we introduce special generalizations of the well-known Lucas numbers L_n in the distance sense. These generalizations are closely related to the concept of distance independent sets. We express the generalized Lucas numbers in terms of an integer parameter k determining any element of the sequence of these numbers by adding two elements of the sequence: one of the distance 2 and the second in the distance k, it was chosen in such way to generalize the numbers L_n. Many natural related generalizations of the Fibonacci numbers and the Lucas numbers remain for further study, in particular interesting relations between different distance generalizations of the Fibonacci numbers.

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