Lattices generated by partial injective maps of finite sets

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Abstract Let n be a positive integer with $n \geq 2$ and $[n] := \{1, 2, \cdots, n\}$. An m-partial injective map of $[n]$ is a pair (A, f) where A is an m-subset of $[n]$ and $f : A \to [n]$ is an injective map. Let $P = L \cup \{1\}$, where L is the set of all the partial injective maps of $[n]$. Partially ordered P by ordinary or reverse inclusion, two families of finite posets are obtained. This article proves that these posets are atomic lattices, discusses their geometricity, and computes their characteristic polynomials.

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1 Introduction

The results on the lattices generated by transitive sets of subspaces under finite classical groups may be found in Huo, Liu and Wan [4, 5, 6]. In [1], Guo discussed the lattices associated with finite vector spaces and finite affine spaces. The lattices generated by the orbits of subspaces under finite classical groups have been obtained in a series of papers by Huo and Wan [7], Guo, Li and Wang [3], Wang and Feng [9], Wang and Guo [10, 11], Guo and Nan [2, 8], Wang and Li [12], Xu et al. [13] studied the lattices generated by partial maps of finite sets. In this paper, we contiune this research, and consider the similar problem for partial injective maps of finite sets.

Let $(P, \leq)$ be a poset. We write $a < b$ whenever $a \leq b$ and $a \neq b$. For any two elements $a, b \in P$, we say a *covers* b, denoted by $b \lessdot a$, if $b < a$ and there exists

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no $c \in P$ such that $b < c < a$. If P has the minimum (respectively maximum) element, then we denote it by 0 (respectively 1), and say that P is a poset with 0 (respectively 1). A poset P is said to be a *lattice* if both $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ exist for any two elements $a, b \in P$. Let P be a finite lattice with 0. For $a \in P$, if $0 \lessdot a$, then a is called an *atom*. A lattice P with 0 is called an *atomic lattice* if $a \in P \backslash \{0\}$ is the least upper bound of some atoms. Let P be a finite poset with 0. If there is a function r from P to set of all the nonnegative integers such that

(1) r(0)=0,

(2) $r(b) = r(a) + 1$, if $a \lessdot b$.

Then r is said to be the *rank function* on P. Note that the rank function on P is unique if it exists.

Let P be a finite atomic lattice. P is said to be a *geometric lattice*, if P admits a rank function r and for any two elements $a, b \in P$

$$r(a \wedge b) + r(a \vee b) \leq r(a) + r(b).$$

Let P be a poset with 0 and 1, and P admits the rank function r. The polynomial

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{r(1) - r(a)}$$

is called the *characteristic polynomial* of P.

Let A be an m-subset of $[n] := \{1, 2, \cdots, n\}$, and $f : A \to [n]$ be an injective map. Then the pair (A, f) is said to be an *m-partial injective map* of $[n]$. In particular, we write $(A, f) = 0$ if $A = \emptyset$.

Let $P = \{(A, f) \mid (A, f)$ is a partial injective map of [n]$\} \cup \{1\}$. For any two elements $(A, f), (B, g) \in P \backslash \{1\}$, we define that 1 *includes* (A, f), and (B, g) *includes* (A, f) if $A \subseteq B$ and $g|_A = f$. Partially ordered P by ordinary or reverse inclusion, two families of finite posets are obtained, denoted by P_O and P_R, respectively.

In this paper we will prove that P_O and P_R are finite atomic lattices, discuss their geometricity and compute their characteristic polynomials.

2 The poset P_O

In this section we will prove that P_O is a finite atomic lattice and computes its characteristic polynomial. We begin with a useful lemma.

Lemma 2.1 *The poset P_O is a finite lattice.*

Proof. For any $(A, f) \in P_O \backslash \{1\}$, it is easy to see that

$$1 = (A, f) \vee 1 \text{ and } (A, f) = 1 \wedge (A, f).$$

For any $(A, f), (B, g) \in P_O \backslash \{1\}$, we assert that

$$(A, f) \vee (B, g) = \begin{cases} (A \cup B, h), h|_A = f, h|_B = g, & f|_{A \cap B} = g|_{A \cap B}, \\ 1, & f|_{A \cap B} \neq g|_{A \cap B}. \end{cases}$$

Case 1. $f|_{A \cap B} = g|_{A \cap B}$. Let (C, φ) be an upper bound of (A, f) and (B, g). Then

$$A \subseteq C, B \subseteq C \text{ and } \varphi|_A = f = h|_A, \varphi|_B = g = h|_B.$$

It follows that $A \cup B \subseteq C$ and $\varphi|_{A \cup B} = h$, i.e., $(A \cup B, h) \leq (C, \varphi)$. Hence $(A, f) \vee (B, g) = (A \cup B, h)$.

Case 2. $f|_{A \cap B} \neq g|_{A \cap B}$. Assume that (C, φ) is an upper bound of (A, f) and (B, g), i.e.,

$$(A, f) \leq (C, \varphi) \text{ and } (B, g) \leq (C, \varphi).$$

Then $\varphi|_{A \cap B} = f|_{A \cap B}, \varphi|_{A \cap B} = g|_{A \cap B}$, a contradiction.

On the other hand, for any $(A, f), (B, g) \in P_O \backslash \{1\}$, we assert that $(A, f) \wedge (B, g) = (D, h)$, where D is the maximum element of the set $\{C \subseteq A \cap B \mid f|_C = g|_C\}$ and $h = f|_D = g|_D$. In fact, let (C, φ) be a lower bound of (A, f) and (B, g). Then

$$C \subseteq A, C \subseteq B \text{ and } f|_C = g|_C.$$

Thus C belongs to $\{C \subseteq A \cap B \mid f|_C = g|_C\}$. Hence, $(A, f) \wedge (B, g) = (D, h)$. □

Theorem 2.2 *Let $n \geq 2$. Then P_O is a finite atomic lattice, but not a geometric lattice.*

Proof. Define $r_O(A, f) = |A|$ for any $(A, f) \in P_O \backslash \{1\}$ and $r_O(1) = n + 1$. Then r_O is the rank function on P_O.

Pick A=$\{1\}$ and

$$f : A \to [n], 1 \mapsto 1; \quad g : A \to [n], 1 \mapsto 2. \tag{1}$$

Then (A, f) and (A, g) are the atoms of P_O, and $1 = (A, f) \vee (A, g)$.

For any $(A, f) \in P_O \backslash \{1\}$ with $A = \{a_1, a_2, \cdots, a_m\}$, we have

$$(A, f) = (\{a_1\}, f|_{\{a_1\}}) \vee (\{a_2\}, f|_{\{a_2\}}) \vee \cdots \vee (\{a_m\}, f|_{\{a_m\}}).$$

Hence P_O is a finite atomic lattice.

Pick f and g as in (1). Then $(A,f)\vee(B,g)=\hat{1}$ and $(A,f)\wedge(B,g)=0$, which implies that

$$r_O((A,f)\vee(B,g))+r_O(A,f)\wedge(B,g))=n+1>2=r_O(A,f)+r_O(B,g).$$

Therefore, the desired result follows. □

Lemma 2.3 *The Möbius function on P_O is*

$$\mu_O(x,y)=\begin{cases}0, & x\not\leq y,\\ (-1)^{|B|-|A|}, & x=(A,f)\leq(B,g)=y\neq\hat{1},\\ -\sum_{i=0}^{n-m}C_{n-m}^{i}A_{n-m}^{i}(-1)^{i}, & x=(A,f)<y=\hat{1},|A|=m,\\ 1, & x=y=\hat{1},\end{cases}$$

where $C_{n-m}^{i}=\frac{(n-m)!}{(n-m-i)!i!}$ and $A_{n-m}^{i}=\frac{(n-m)!}{(n-m-i)!}$.

Proof. In order to prove that μ_O is the Möbius function on P_O, we only need to show that

$$\sum_{x\leq z\leq y}\mu_O(x,z)=0$$

for any $x,y\in P_O$ with $x<y$.

If $\hat{1}\neq y=(B,g)$, let $|B|-|A|=m$. Then

$$\sum_{x\leq z\leq y}\mu_O(x,z)=\sum_{k=0}^{m}C_m^k(-1)^k=(1-1)^m=0.$$

If $y=\hat{1}$, let $|A|=m$. Then

$$\begin{aligned}\sum_{x\leq z\leq\hat{1}}\mu_O(x,z) &= \sum_{k=0}^{n-m}C_{n-m}^{k}A_{n-m}^{k}(-1)^k+\mu(x,\hat{1})\\ &= \sum_{k=0}^{n-m}C_{n-m}^{k}A_{n-m}^{k}(-1)^k-\sum_{k=0}^{n-m}C_{n-m}^{k}A_{n-m}^{k}(-1)^k\\ &= 0.\end{aligned}$$

Hence, the function μ_O is the Möbius function on P_O. □

Theorem 2.4 *The characteristic polynomial of P_O is*

$$\chi(P_O,x)=x^{n+1}-1+\sum_{m=1}^{n}(-1)^m C_n^m A_n^m(x^{n+1-m}-1).$$

Proof. By Lemma 2.3 we obtain

$$\begin{aligned}
\chi(P_O,x) &= \sum_{u\in P_O} \mu_O(0,u)x^{r_O(1)-r_O(u)} \\
&= \sum_{m=0}^{n}(-1)^m C_n^m A_n^m x^{r_O(1)-m} + \mu_O(0,1)x^{r_O(1)-r_O(1)} \\
&= \sum_{m=0}^{n}(-1)^m C_n^m A_n^m x^{r_O(1)-m} - 1 - \sum_{m=1}^{n}(-1)^m C_n^m A_n^m \\
&= x^{n+1} - 1 + \sum_{m=1}^{n}(-1)^m C_n^m A_n^m (x^{n+1-m} - 1),
\end{aligned}$$

as desired. □

3 The poset P_R

In this section we will prove that P_R is a finite atomic lattice and compute its characteristic polynomial.

Theorem 3.1 *Let $n \geq 2$. Then P_R is a finite atomic lattice, but not a geometric lattice.*

Proof. Similar to the proof of Lemma 2.1, P_R is a finite lattice. Define $r_R(A,f) = n+1-|A|$ for any $(A,f) \in P_R\backslash\{1\}$ and $r_R(1) = 0$. Then r_R is the rank function on P_R.

Pick

$$f:[n]\to[n], i\mapsto i \text{ and } g:[n]\to[n], i\mapsto i+1 (1\leq i\leq n-1), n\mapsto 1. \quad (2)$$

Then $0 = ([n],f)\vee([n],g)$. For $(A,f)\in P_R\backslash\{1\}$ with $A=\{a_1,a_2,\cdots,a_m\}$, let $[n]=\{a_1,a_2,\cdots,a_m,a_{m+1}\cdots,a_n\}$ and $[n]=\{f(a_1),\cdots,f(a_m),b_{m+1},\cdots,b_n\}$. Pick $([n],g),([n],h)\in P_R$ such that

$$g:[n]\to[n], a_i\mapsto f(a_i)(1\leq i\leq m), a_j\mapsto b_j(m+1\leq j\leq n);$$

$$h:[n]\to[n], a_i\mapsto f(a_i)(1\leq i\leq m), a_j\mapsto b_{j+1}(m+1\leq j\leq n-1), a_n\mapsto b_{m+1}.$$

Then $([n],g)$ and $([n],h)$ are atoms of P_R and $(A,f) = ([n],g)\vee([n],h)$. Hence, P_R is a finite atomic lattice.

Pick f, g as in (2). Then

$$([n], f) \vee ([n], g) = 0, ([n], f) \wedge ([n], g) = \mathbb{1},$$

which implies that

$$r_R(([n], f) \vee ([n], g)) + r_R([n], f) \wedge ([n], g)) = n + 1 > 2 = r_R([n], f) + r_R([n], g).$$

Therefore, the desired result follows. □

Lemma 3.2 *The Möbius function on P_R is*

$$\mu_R(x, y) = \begin{cases} 0, & x \not\leq y, \\ (-1)^{|A|-|B|}, & \mathbb{1} \neq x = (A, f) \leq y = (B, g), \\ -\sum_{k=0}^{n-m} C_{n-m}^k A_{n-m}^k (-1)^k, & \mathbb{1} = x \leq y = (B, g), |B| = m, \\ 1, & x = y = \mathbb{1} . \end{cases}$$

Proof. If $x \neq \mathbb{1}$, let $|A| - |B| = m$. Then

$$\sum_{x \leq z \leq y} \mu_R(x, z) = \sum_{k=0}^{m} C_m^{m-k} (-1)^k = (1 - 1)^m = 0.$$

If $x = \mathbb{1}$, let $|B| = m$. Then

$$\begin{aligned} \sum_{\mathbb{1} \leq z \leq y} \mu_R(\mathbb{1}, z) &= \mu_R(\mathbb{1}, \mathbb{1}) + \sum_{\mathbb{1} < z \leq y} \mu_R(\mathbb{1}, z) \\ &= 1 + \sum_{i=0}^{n-m} C_{n-m}^i A_{n-m}^i \left(- \sum_{k=0}^{n-m-i} C_{n-m-i}^k A_{n-m-i}^k (-1)^k \right) \\ &= 0. \end{aligned}$$

Hence, the function μ_R is the Möbius function on P_R. □

Theorem 3.3 *The characteristic polynomial of P_R is*

$$\chi(P_R, x) = x^{n+1} + \sum_{m=0}^{n} (-1)^m C_n^m A_n^m x^m.$$

Proof. By Lemma 3.3 we obtain

$$\begin{aligned} \chi(P_R, x) &= \sum_{u \in P_R} \mu_R(\mathbb{1}, u) x^{r_R(0) - r_R(u)} \\ &= \sum_{m=0}^{n} (-1)^m C_n^m A_n^m x^m + \mu_R(\mathbb{1}, \mathbb{1}) x^{r_R(0) - r_R(\mathbb{1})} \\ &= x^{n+1} + \sum_{m=0}^{n} (-1)^m C_n^m A_n^m x^m, \end{aligned}$$

as desired. □

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