On The Algebraic Study of Spanning Simplicial Complexes of r-cyclic Graphs $G_{n,r}$

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Abstract. In this paper, we characterize all the spanning trees of the *r-cyclic graph* $G_{n,r}$. We give the formulation of f-vectors associated to spanning simplicial complexes $\Delta_s(G_{n,r})$ and consequently, we deduce a formula for computing the Hilbert series of the Stanley-Reisner ring $k[\Delta_s(G_{n,r})]$. For the facet ideal $I_{\mathcal{F}}(\Delta_s(G_{n,r}))$, we give the characterization of all its associated primes. In particular, for the uni-cyclic graphs with the length of the cycle equal to m_1, we prove that the facet ideal $I_{\mathcal{F}}(\Delta_s(G_{n,1}))$ has linear quotients with respect to its generating set. Moreover, we prove that the $\text{projdim}(I_{\mathcal{F}}(\Delta_s(G_{n,1}))) = 1$ and $\beta_i(I_{\mathcal{F}}(\Delta_s(G_{n,1}))) = m_1$ for $i \leq 1$.

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1 Introduction

In [1], for a finite simple connected graph $G(V, E)$ the authors have introduced the concept of spanning simplicial complex $\Delta_s(G)$, which is defined on the edge set E of the graph G as follows:

$$\Delta_s(G) = \langle F_i \mid F_i \in s(G) \rangle$$

where $s(G)$ is the collection of edge-sets of all the spanning trees of G. One can always associate $\Delta_s(G)$ to any simple finite connected graph $G(V, E)$. In [1], the authors have discussed the algebraic and combinatorial properties of spanning simplicial complexes for uni-cyclic graphs. They prove that $\Delta_s(G_{n,1})$ is shifted and give the formula to compute the Hilbert series of the Stanley-Reisner ring $k[\Delta_s(G_{n,1})]$. Recently in [2], authors have discussed the algebraic properties of spanning simplicial complexes associated to friendship graphs.

In this article, we pick the class of *r-cyclic graphs* $G_{n,r}$ for the algebraic and combinatorial properties of *spanning simplicial complex* $\Delta_s(G_{n,r})$. A *r-cyclic graph* $G_{n,r}$ is a connected graph having exactly r cycles and n edges with no two cycles share a common edge. In Proposition 6, we give the characterization of $s(G_{n,r})$. We give the formulation for $f-vectors$ in Lemma 8 which enable us to device a formula to compute the *Hilbert series* of the Stanley Reisner ring $k[\Delta_s(G_{n,r})]$ given in Theorem 10. For the facet ideal $I_{\mathcal{F}}(\Delta_s(G_{n,r}))$ in Lemma 11, we give the characterization of all its associated primes.

In particular, for the uni-cyclic graphs $(G_{n,1})$ with the length of the cycle equal to m_1, we show that the facet ideal $I_{\mathcal{F}}(\Delta_s(G_{n,1}))$ has linear quotients with respect to its minimal generating set in Theorem 14. Moreover, in Theorem 16, we prove that the projdim$(I_{\mathcal{F}}(\Delta_s(G_{n,1}))) = 1$ and $\beta_i(I_{\mathcal{F}}(\Delta_s(G_{n,1}))) = m_1$ for $i \leq 1$. This paper extents the results of [1] to another class of graphs.

2 Background and notions

In this section, we cover the background of the topic and define some notions which we will follow in this paper.

Definition 1 A spanning tree of a simple connected finite graph $G(V, E)$ is a subtree of G that contains every vertex of G.

We represent the collection of all edge-sets of the spanning trees of G by $s(G)$, in other words;

$$s(G) := \{E(T_i) \subset E, \text{ where } T_i \text{ is a spanning tree of } G\}.$$

In [1], it is given that for any simple connected graph one can obtain its spanning tree by removing one edge from each cycle appearing in the graph. This method of finding the spanning trees of a connected graph is known as *cutting-down method.*

For example by using *cutting-down method* for the graph given in figure 1 we obtain:

$s(G) = \{\{e_{22}, e_{23}, e_{24}, e_{12}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{22}, e_{23}, e_{24}, e_{11}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{22}, e_{23}, e_{24}, e_{11}, e_{12}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{23}, e_{24}, e_{11}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{23}, e_{24}, e_{11}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{23}, e_{24}, e_{11}, e_{12}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{24}, e_{12}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{24}, e_{11}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{24}, e_{11}, e_{12}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{23}, e_{12}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{23}, e_{11}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{23}, e_{11}, e_{12}, e_1, e_2, e_3, e_4\}\}$

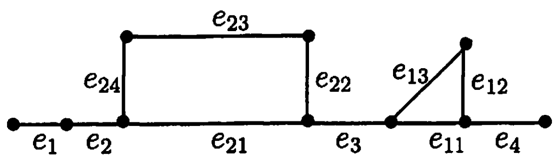

Fig. 1 . $G_{11,2}$

Definition 2 A Simplicial complex Δ over a finite set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$, with the property that $\{i\} \in \Delta$ for all $i \in [n]$, and if $F \in \Delta$ then Δ will contain all the subsets of F (including the empty set). An element of Δ is called a face of Δ, and the dimension of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The maximal faces of Δ under inclusion are called facets of Δ. The dimension of the simplicial complex Δ is :

$$\dim\Delta = \max\{\dim F | F \in \Delta\}.$$

We denote the simplicial complex Δ with facets $\{F_1, \ldots, F_q\}$ by

$$\Delta = \langle F_1, \ldots, F_q \rangle$$

Definition 3 For a simplicial complex Δ having dimension d, its $f-vector$ is a $d+1$-tuple, defined as:

$$f(\Delta) = (f_0, f_1, \dots, f_d)$$

where f_i denotes the number of $i-dimensional$ faces of Δ.

Definition 4 (Spanning Simplicial Complex)
For a simple finite connected graph $G(V,E)$ with $s(G) = \{E_1, E_2, \dots, E_s\}$, we define a simplicial complex $\Delta_s(G)$ on E such that the facets of $\Delta_s(G)$ are precisely the elements of $s(G)$, we call $\Delta_s(G)$ as the *spanning simplicial complex* of $G(V,E)$. In other words;

$$\Delta_s(G) = \langle E_1, E_2, \dots, E_s\rangle.$$

For example; the spanning simplicial complex of the graph G given in figure 1 is:

$\Delta_s(G) = \langle\{e_{22}, e_{23}, e_{24}, e_{12}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{22}, e_{23}, e_{24}, e_{11}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{22}, e_{23}, e_{24}, e_{11}, e_{12}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{23}, e_{24}, e_{11}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{23}, e_{24}, e_{11}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{23}, e_{24}, e_{11}, e_{12}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{24}, e_{12}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{24}, e_{11}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{24}, e_{11}, e_{12}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{23}, e_{12}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{23}, e_{11}, e_{13}, e_1, e_2, e_3, e_4\}, \{e_{21}, e_{22}, e_{23}, e_{11}, e_{12}, e_1, e_2, e_3, e_4\}\rangle$

We conclude this section with the definition of *r-cyclic graph* $G_{n,r}$;

Definition 5 An *r-cyclic graph* $G_{n,r}$ is a connected graph having n edges and containing exactly r cycles $\{C^1_{m_1}, C^2_{m_2}, \dots, C^r_{m_r}\}$ of lengths $m_1 \le m_2 \le \dots \le m_r$ such that no two cycles share a common edge.

3 Spanning trees of $G_{n,r}$ and Stanley-Reisner ring $\Delta_s(G_{n,r})$

Throughout this paper, for $G_{n,r}$, we fix the edge-labeling $\{e_{11}, \dots, e_{1m_1}, e_{21}, \dots, e_{2m_2}, \dots, e_{r1}, \dots, e_{rm_r}, e_1 \dots e_t\}$ such that $\{e_{i1}, \dots,$

$e_{im_i}\}$ is the edge-set of ith-cycle in $G_{n,r}$ for $1 \le i \le r$ and $t = n - \sum_{i=1}^{r} m_i$. In the following result, we give the characterization of $s(G_{n,r})$.

Lemma 6 Characterization of $s(G_{n,r})$
Let $G_{n,r}$ be the *r-cyclic graph* with the edge set $E = \{e_{11}, \ldots, e_{1m_1}, e_{21}, \ldots, e_{2m_2}, \ldots, e_{r1}, \ldots, e_{rm_r}, e_1 \ldots e_t\}$. A subset $E(T_{(1i_1,2i_2,\ldots,ri_r)})$ of E will belong to $s(G_{n,r})$ if and only if $E(T_{(1i_1,2i_2,\ldots,ri_r)}) = E \setminus \{e_{1i_1}, e_{2i_2}, \ldots, e_{ri_r}\}$ for some $i_j \in \{1, \ldots, m_j\}$ and $1 \le j \le r$. In particular; $s(G_{n,r}) = \{\hat{E}_{(1i_1,2i_2,\ldots,ri_r)} = E \setminus \{e_{1i_1}, e_{2i_2}, \ldots, e_{ri_r}\}$ for $i_j \in \{1, \ldots, m_j\}$ and $1 \le j \le r\}$.

Proof. As $G_{n,r}$ contains r-cycles of lengths $m_1, m_2, \ldots, m_r$, so its spanning trees will be obtained by removing exactly one edge from each cycle of $G_{n,r}$ follows from *cutting-down method* [6]. For some $j \in \{1, 2, \ldots, r\}$ by removing e_{ji_j} from C_{m_j} for each $1 \le j \le r$ with $i_j = \{1, 2, \ldots, m_j\}$, one obtains a spanning tree of $G_{n,r}$. In other words the edge-set $E(T)$ of a spanning tree T will be of the form: $E(T) = \{\ E \setminus \{e_{1i_1}, e_{2i_2}, \ldots, e_{ri_r}\}$ for some $i_j \in \{1, \ldots, m_j\}$ for all $j \in \{1, \ldots, r\}\}$. Which yields the desired result for $s(G_{n,r})$.

Here we recall an elementary result from [1] ;

Proposition 7 For a simplicial complex Δ over $[n]$ of dimension d, if $f_t = \binom{n}{t+1}$ for some $t \le d$ then $f_i = \binom{n}{i+1}$ for all $0 \le i < t$.

We will always consider $\binom{n}{k} = 0$, whenever $k > n$ or $k < 0$.

Our next result is the characterization of the f-vector of $\Delta_s(G_{n,r})$.

Proposition 8 Let $\Delta_s(G_{n,r})$ be the spanning simplicial complex of *r-cyclic graph* $G_{n,r}$ with cycles of lengths $m_1 \le m_2 \le \cdots \le m_r$, then the $\dim(\Delta_s(G_{n,r}) = n - r - 1$ with f-vector

$$f_i = \binom{n}{i+1} + \sum_{k=1}^{r}(-1)^k \left[\sum_{(j_1,\dots,j_r)=1,\ j_1\neq\cdots\neq j_r}^{r} \binom{n - \sum_{s=1}^{k} m_{j_s}}{i+1-\sum_{s=1}^{k} m_{j_s}} \right]$$

where $0 \leq i \leq n-r-1$.

Proof. Let E be the edge-set of $G_{n,r}$, then from Lemma 6 ;

$$s(G_{n,r}) = \{\hat{E}_{(1i_1,2i_2,\dots,ri_r)} = E \setminus \{e_{1i_1}, e_{2i_2}, \dots, e_{ri_r}\}$$

for $i_j \in \{1,\dots,m_j\}$ and $1 \leq j \leq r\}$. Therefore, by Definition 4 we have;

$$\Delta_s(G_{n,r}) = \langle \hat{E}_{(1i_1,2i_2,\dots,ri_r)} \text{ for } i_j \in \{1,\dots,m_j\} \text{ and } 1 \leq j \leq r\rangle.$$

Since each facet $\hat{E}_{(1i_1,2i_2,\dots,ri_r)}$ is of the same dimension $n-r-1$ (as $\mid \hat{E}_{(1i_1,2i_2,\dots,ri_r)} \mid = n-r$), therefore $\Delta_s(G_{n,r})$ will be of dimension $n-r-1$. Also, it is clear from the definition of $\Delta_s(G_{n,r})$ that $\Delta_s(G_{n,r})$ contains all those subsets of E that do not contain $\{e_{i1},\dots,e_{im_i}\}$ for all $1 \leq i \leq r$.

Now, let F be any subset of E of order $i+1$ such that it does not contain any C_i in it. The total number of such F is indeed f_i. We use inclusion exclusion to find this number, therefore we have $f_i =$ Total number of subsets of E of order $i+1$ not containing ant C_i. By Inclusion Exclusion Principle we have $f_i = \Big($ Total number of subsets of E of order $i+1\Big) - \Big(\sum_{i_1=1}^{r}$ Total number of subsets of E containing $C_{i_1}\Big) + \Big(\sum_{(i_1,i_2)=1,\ i_1\neq i_2}^{r}$ Total number of subsets of E containing both $C_{i_1}$and$C_{i_2}\Big) + \cdots(-1)^r\Big(\sum_{(i_1,\dots,i_r)=1,\ i_1\neq\cdots\neq i_r}^{r}$ Total number of subsets of E containing each $C_{j_1},\dots,C_{j_r}\Big)$.

This implies that $f_i = \binom{n}{i+1} - \left[\sum\limits_{i_1=1}^{r} \binom{n-m_{i_1}}{i+1-m_{i_1}}\right] +$
$\left[\sum\limits_{(i_1,i_2)=1,\ i_1\neq i_2}^{r} \binom{n-m_{i_1}-m_{i_2}}{i+1-m_{i_1}-m_{i_2}}\right] + \cdots$
$+ (-1)^r \left[\sum\limits_{(i_1,\ldots,i_r)=1,\ i_1\neq\cdots\neq i_r}^{r} \binom{n-m_{i_1}-m_{i_2}-\cdots-m_{i_r}}{i+1-m_{i_1}-m_{i_2}-\cdots-m_{i_r}}\right]$
$\Longrightarrow$

$$f_i = \binom{n}{i+1} + \sum_{k=1}^{r}(-1)^k \left[\sum_{(i_1,\cdots,i_k)=1,\ i_1\neq\cdots\neq i_k}^{r} \binom{n-\sum\limits_{s=1}^{k} m_{i_s}}{i+1-\sum\limits_{s=1}^{k} m_{i_s}}\right]$$

Corollary 9 *Let $\Delta_s(G_{n,2})$ be a spanning simplicial complex of a 2-cyclic graph of lengths $m_1 \leq m_2$, then the* $\dim(\Delta_s(G_{n,2})) = n-3$ *with $f-$vectors $f(\Delta_s(G_{n,2})) = (f_0, f_1, \cdots, f_{n-3})$*

$$f_i = \binom{n}{i+1} - \left[\binom{n-m_1}{i+1-m_1} + \binom{n-m_2}{i+1-m_2}\right] +$$
$$+ \left[\binom{n-m_1-m_2}{i+1-m_1-m_2}\right], \text{ where } 0 \leq i \leq n-3.$$

For a simplicial complex Δ over $[n]$, one would associate to it the Stanley-Reisner ideal, that is, the monomial ideal $I_{\mathcal{N}}(\Delta)$ in $S = k[x_1, x_2, \ldots, x_n]$ generated by monomials corresponding to non-faces of this complex (here we are assigning one variable of the polynomial ring to each vertex of the complex). It is well known that the Stanley-Reisner ring $k[\Delta] = S/I_{\mathcal{N}}(\Delta)$ is a standard graded algebra. We refer the readers to [7] and [8] for more details about graded algebra A, the Hilbert function $H(A,t)$ and the Hilbert series $H_t(A)$ of a graded algebra.

Our main result of this section is as follows;

Theorem 10 Let $\Delta_s(G_{n,r})$ be the spanning simplicial complex of $G_{n,r}$, then the Hilbert series of the Stanley-Reisner ring $k[\Delta_s(G_{n,r})]$ is given by,

$$H(k[\Delta_s(G_{n,r})], t) = 1 + \sum_{i=0}^{d} \frac{\binom{n}{i+1} t^{i+1}}{(1-t)^{i+1}} + \sum_{i=0}^{d}\sum_{k=1}^{r} (-1)^k \times$$

$$\times \left[\sum_{(i_1,\cdots,i_k)=1,\ i_1\neq\cdots\neq i_k}^{r} \binom{n-\sum_{s=1}^{k} m_{i_s}}{i+1-\sum_{s=1}^{k} m_{i_s}} \right] \frac{t^{i+1}}{(1-t)^{i+1}}$$

Proof. From [8], we know that if Δ is a simplicial complex of dimension d and $f(\Delta) = (f_0, f_1, \dots, f_d)$ its f-vector, then the Hilbert series of Stanley-Reisner ring $k[\Delta]$ is given by

$$H(k[\Delta], t) = 1 + \sum_{i=0}^{d} \frac{f_i t^{i+1}}{(1-t)^{i+1}}.$$

By substituting the values of f_i's from Proposition 8 in the above expression, we get the desired result.

4 Associated primes of the facet ideal $I_{\mathcal{F}}(\Delta_s(G_{n,r}))$

In this section, we give the characterization of all associated primes of the facet ideal $I_{\mathcal{F}}(\Delta_s(G_{n,r}))$ of spanning simplicial complex $\Delta_s(G_{n,r})$ of r-cyclic graph with cycles $C^1_{m_1}, C^2_{m_2}, \dots, C^r_{m_r}$ of lengths $m_1 \leq m_2 \leq \cdots \leq m_r$.

Associated to a simplicial complex Δ over $[n]$, one defines *the facet ideal* $I_{\mathcal{F}}(\Delta) \subset S$, which is generated by square-free monomials $x_{i1} \dots x_{is}$, where $\{v_{i1}, \dots, v_{is}\}$ is a facet of Δ.

Lemma 11 If $\Delta_s(G_{n,r})$ be the spanning simplicial complex of the r-cyclic graph $G_{n,r}$, then

$$I_{\mathcal{F}}(\Delta_s(G_{n,r})) = \left(\bigcap_{e_t \notin C^i_{m_i};1\leq i\leq r} (x_t) \right) \bigcap \left(\bigcap_{1\leq j\leq r;1\leq l<m\leq m_j} (x_{jl}, x_{jm}) \right)$$

Proof. Let us consider the spanning simplicial complex $\Delta_s(G_{n,r})$ of the r-cyclic graph $G_{n,r}$ having r-cycles $C^1_{m_1}, C^2_{m_2}, \dots, C^r_{m_r}$ with lengths $m_1 \leq m_2 \leq \dots \leq m_r$. Let $I_{\mathcal{F}}(\Delta_s(G_{n,r}))$ be the

facet ideal of $\Delta_s(G_{n,r})$.
From [4, Proposition 1.8], we know that a minimal prime ideal of the facet ideal $I_{\mathcal{F}}(\Delta)$ has one-to-one correspondence with the minimal vertex cover of the simplicial complex. Therefore, in order to compute the primary decomposition of the facet ideal $I_{\mathcal{F}}(\Delta_s(G_{n,r}))$; it is sufficient to compute all the minimal vertex cover of $\Delta_s(G_{n,r})$. From the definition of $\Delta_s(G_{n,r})$ and Lemma 6, we have $\{e_t\}$ a minimal vertex cover of $\Delta_s(G_{n,r})$ (with $e_t \notin C^i_{m_i}$ for all $i \in \{1, \dots, r\}$) as $\{e_t\} \in \hat{E}_{(1i_1,2i_2,\dots,ri_r)}$ for any $i_j \in \{1, \dots, m_j\}$ and $1 \leq j \leq r$. Moreover, $\{e_{jl}, e_{jm}\}$ is also a minimal vertex cover of $\Delta_s(G_{n,r})$ (with $1 \leq j \leq r$ and $1 \leq l < m \leq m_j$), because $\{e_{jl}\} \in \hat{E}_{(1i_1,\dots,ji_j,\dots,ri_r)}$ for all $1 \leq j \leq r$ if and only if $i_j \neq l$ with$(1 \leq l < m_j)$ and $\{e_{jm}\} \in \hat{E}_{(1i_1,\dots,jl,\dots,ri_r)}$ for any $1 \leq l < m \leq m_j$. Hence $\{e_{jl}, e_{jm}\}$ is a minimal vertex cover of $\Delta_s(G_{n,r})$.

5 Facet ideal of uni-cyclic graph has linear quotients

For a uni-cyclic graph $G_{n,1}$ containing the only cycle of length $m_1(\leq n)$, we have

$$I_{\mathcal{F}}(\Delta_s(G_{n,1})) = (\hat{x}_{11}, \dots, \hat{x}_{1m_1}),$$

where

$$\hat{x}_{1i} = \frac{x_{11}, \dots, x_{1m_1}, x_1 \dots, x_{n-m_1}}{x_{1i}}, \quad \forall\, 1 \leq i \leq m_1$$

in the polynomial ring $S = k[x_{11}, \dots, x_{1m_1}, x_1 \dots, x_{n-m_1}]$ follows from Definition 4 and Lemma 6.

Definition 12 Let $I \subset S$ be a graded ideal. We say that I has *linear quotients*, if there exists a system of homogeneous generators $f_1, f_2, \dots, f_m$ of I such that the colon ideal $(f_1, \dots, f_{i-1}) : f_i$ is generated by linear forms for all i.

It is important to recall a result from [7] which tells that when a monomial ideal has linear quotients with respect to its minimal generating set;

Lemma 13 Let I be a square-free monomial ideal with $G(I) = \{u_1, u_2, \ldots, u_m\}$, and let $F_i = \text{supp}(u_i)$ for $i = 1, \ldots, m$. Then I has linear quotients with respect to $u_1, u_2, \ldots, u_m$ if and only if for all $j < i$ there exists an integer l with $x_l \in F_j \setminus F_i$ and an integer $k < i$ such that $F_k \setminus F_i = \{x_l\}$

After giving the description about the facet ideal and linear quotients, we are ready to give the following result;

Theorem 14 The facet ideal $I_{\mathcal{F}}(\Delta_s(G_{n,1})) \subset S$ of the spanning simplicial complex of *uni-cyclic graph* $\Delta(G_{n,r})$, has liner quotients with respect to the monomial generator $G(I_{\mathcal{F}}(\Delta_s(G_{n,1}))) = \{\hat{x}_{11}, \hat{x}_{12}, \ldots, \hat{x}_{1m_1}\}$.

Proof. Let us take the facet ideal $I_{\mathcal{F}}(\Delta_s(G_{n,1})) \subset S$ of uni-cyclic graph with $G(I_{\mathcal{F}}(\Delta_s(G_{n,1}))) = \{\hat{x}_{11}, \hat{x}_{12}, \ldots, \hat{x}_{1m_1}\}$ and denote $F_i = \text{supp}(\hat{x}_{1i})$ for all $i \in \{1, \ldots, m_1\}$. It is clear from the construction that

$$F_j \setminus F_i = \{x_{1i}\} \qquad \text{for any} \qquad j < i.$$

Also

$$F_k \setminus F_i = \{x_{1i}\} \qquad \text{for all} \qquad k < i.$$

From Lemma 13, it is clear that there exist $l(= i)$, hence proved.

Definition 15 For a monomial ideal I with the minimal generating set $G(I) = \{u_1, u_2, \ldots, u_m\}$. We define

$$L_k = (u_1, \ldots, u_{k-1}) : u_k \ \text{ for each } \ k \leq m_1$$

and r_k to be the cardinality of the minimal set of generators of L_k.

Theorem 16 For the facet ideal $I_{\mathcal{F}}(\Delta_s(G_{n,1})) \subset S$ of the spanning simplicial complex of *uni-cyclic graph*, we have $\operatorname{projdim}(I_{\mathcal{F}}(\Delta_s(G_{n,1}))) = 1$ and

$$\beta_i(I_{\mathcal{F}}(\Delta_s(G_{n,1}))) = m_1 \quad \text{for all } i \leq 1.$$

Proof. For the minimal generating set $G(I_{\mathcal{F}}(\Delta_s(G_{n,1}))) = \{\hat{x}_{11}, \hat{x}_{12}, \dots, \hat{x}_{1m_1}\}$, we have

$$L_k = (\hat{x}_{11}, \dots, \hat{x}_{1k-1}) : \hat{x}_{1k} = x_{1k} \text{ for all } \quad k \leq m_1$$

Therefore, $I_{\mathcal{F}}(\Delta_s(G_{n,1}))$ has linear quotient in one degree and $r_k = 1$ for each $k \leq m_1$. The result immediately follows from [7, Corrolary 8.2.2], which says that for any graded ideal I with linear quotients generated in one degree we have;

$$\beta_i(I) = \sum_{k=1}^{m_1} \binom{r_k}{i} \quad \text{and } \operatorname{projdim}(I) = \max\{r_1, r_2, \dots, r_{m_1}\}.$$

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