THE MAXIMUM NUMBER OF ONE-EDGE EXTENSIONS FOR GRAPHS WITH BOUNDED DEGREE

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Abstract: The maximum number of non-isomorphic one-edge extensions $M(t, n, f)$ of a graph of size t, order n, and vertex degree bounded by f for $3 \leq f \leq n-2$ is considered. An upper bound for $M(t, n, f)$ is obtained and for the case $f = n-2$ the exact value is given. Tables for all values of $M(t,n,f)$ are provided for up to $n = 12$, $\lfloor n(f-1)/2 \rfloor < t \leq \lfloor nf/2 \rfloor$, and $3 \leq f \leq n-2$. It is also noted how the general results are related to the transition digraph for the Random f-Graph Process, a Markov process pertaining to graphs with vertex degree bounded by f.

Key words: graphs with bounded degree, one-edge extensions, Random f-Graph Process

AMS subject classification: 05C20, 05C35

1. Introduction

A graph with no vertex of degree greater than f is called an *f-graph*. For each f such that $3 \leq f \leq n-2$, we consider the problem of determining $M(t, n, f)$, *the maximum number of non-isomorphic one-edge f-graph extensions of an f-graph among all f-graphs of size t and order n.*

The number of distinct edges that can be considered for extending an f-graph G with one edge is equal to $\binom{n}{2} - t$, the number of edges in G^c the complement of G. If G is an *identity graph* (i.e., G has only one automorphism), then in most cases all of its possible one-edge extensions are non-isomorphic graphs. However, some of these extensions can be isomorphic or can be $(f+1)$-graphs.

For example, the identity 3-graph of order 6 and size 6, a triangle with a pendant edge attached at one vertex and a pendant path of order 3 attached at another vertex, has all of its $\binom{6}{2} - 6 = 9$ one-edge extensions non-isomorphic but four of them are 4-graphs. Whereas the identity 4-graph of order 6 and size 7, shown in Figure 1.1, can have at most 8 non-isomorphic one-edge extensions. One of them is a 5-graph and in the set of seven 4-graph extensions two graphs are isomorphic.

We denote by $d^+(G)$, *the number of non-isomorphic one-edge f-graph extensions of a given f-graph G*. As noted above, $d^+(G)$ cannot exceed the number of edges in the complement of G, that is, $d^+(G) \le \binom{n}{2} - t$.

The problem of determining $M(t, n, f)$ for the cases $f = 2$ and $f = n - 1$ has been studied in [1] and [2], respectively. In [3] some preliminary observations concerning $M(t, n, f)$ were announced.

Here we consider the cases $3 \le f \le n - 2$ for a specific interval of t (see Section 2). An *admissible edge* for an f-graph G is an edge whose addition to G will not introduce a vertex of degree greater than f. Let $B(t, n, f)$ denote the *maximum number of admissible edges possible among all f-graphs of size t and order n*. Then, $M(t, n, f) \le B(t, n, f)$.

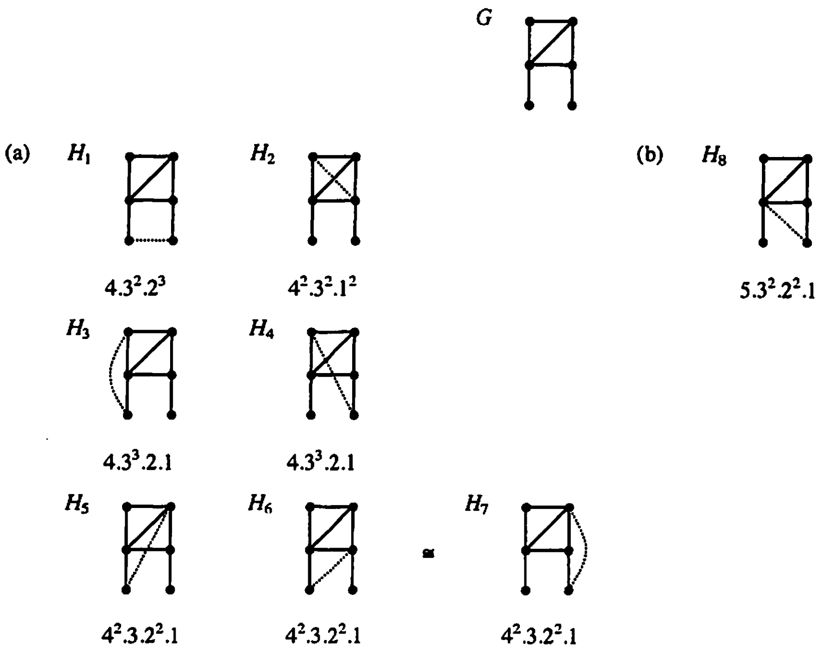


Figure 1.1. The 4-graph G with $n = 6$ and $t = 7$;
(a) six distinct 4-graph extensions of G; (b) one 5-graph extension of G

In Section 2 we obtain lemmas related to $B(t, n, f)$ and bounds for the number of vertices of degree f in an f-graph. The exact value of $M(t, n, f)$ when $f = n - 2$ is given in Section 3. The main Theorem for $B(t, n, f)$ as a bound for $M(t, n, f)$ is derived in Section 4. In Section 5 open problems are posed and our motivation is given for the study of graphs with bounded degree.

2. The number of admissible edges and bounds for the number of vertices of degree f

We call a vertex in an f-graph *orexic*, if its degree is strictly less than f. We make the following observations.

(a) If $0 \leq t \leq \lfloor n(f-1)/2 \rfloor$, there exist f-graphs of order n and size t such that every vertex is orexic. In this case $B(t,n,f) = \binom{n}{2} - t$.

(b) If $t = \lfloor nf/2 \rfloor$, then $B(t,n,f) = 0 = M(t,n,f)$ and

(c) If $t > \lfloor nf/2 \rfloor$, there are no f-graphs.

Thus, here we focus on the t-interval $\lfloor n(f-1)/2 \rfloor < t \leq \lfloor nf/2 \rfloor$.

The value of $B(t,n,f)$ is given in Lemma 2.1.

Lemma 2.1. Let G denote the set of f-graphs of order n and size $t \leq \lfloor nf/2 \rfloor$. For a $G \in G$ let $w(G)$ denote the number of orexic vertices in G and $W(G)$ denote the subgraph of G induced by the orexic vertices in G. Then

$$B(t,n,f) = \max_{G \in G}\left\{\binom{w(G)}{2} - |E(W(G))|\right\}.$$

Proof. Let G be an f-graph of order n, size t, and having $w = w(G)$ orexic vertices. The vertices of an admissible edge must both be orexic prior to its insertion in G. There are $\binom{w}{2}$ pairs of orexic vertices in G. However, some of these pairs may already be edges in G and consequently are not available for addition. These are precisely the edges in $W(G)$. Thus, the number of admissible edges for a given G is $\binom{w(G)}{2} - |E(W(G))|$. The maximum of this expression over all $G \in G$ is the value of $B(t,n,f)$. ■

For f such that $3 \leq f \leq n-2$, Lemmas 2.2 and 2.3 below, are used to prove Theorem 4.1.

Lemma 2.2. Let $G(w)$ denote the set of f-graphs of order n, size t, and having w orexic vertices. Then,

$$\max_{G \in G(w)}\left\{\binom{w}{2} - |E(W)|\right\} = \binom{w}{2} - \min_{G \in G(w)}\{|E(W)|\}.$$

Proof. Since w is fixed, $|E(W)|$ is the only variable part in the elements of the set $\left\{\binom{w}{2} - |E(W)|\right\}$. Therefore, the maximum of $\left\{\binom{w}{2} - |E(W)|\right\}$ for $G \in G(w)$ is equal to $\binom{w}{2} - \min_{G \in G(w)}\{|E(W)|\}$. ■

Lemma 2.3 specifies bounds for the number of vertices of degree f in an f-graph of order n and size t and is crucial for proving Theorems 3.1, 4.1, and 4.2.

Lemma 2.3. If G is an f-graph of order n, size $t = \lfloor n(f-1)/2 \rfloor + j$ with $0 < j < \lfloor nf/2 \rfloor - \lfloor n(f-1)/2 \rfloor$, and having x vertices of degree f, then

$$2j \le x \le \left\lfloor \frac{2j + n(f-1)}{f} \right\rfloor \text{ when } n(f-1) \text{ is even and}$$

$$2j - 1 \le x \le \left\lfloor \frac{2j - 1 + n(f-1)}{f} \right\rfloor \text{ when } n(f-1) \text{ is odd.}$$

Proof. Let $n(f-1)$ be even. Then,

$$2t = 2(n(f-1)/2 + j) = \sum_{i=1}^{n} \deg(v_i) \le xf + (n-x)(f-1)$$

so that

$$n(f-1) + 2j \le xf + (n-x)(f-1) = xf + n(f-1) - x(f-1)$$

which when simplified yields $2j \le x$. An upper bound for x is obtained as follows.

$$xf \le \sum_{i=1}^{n} \deg(v_i) = 2(n(f-1)/2 + j) = n(f-1) + 2j$$

so that

$$x \le \frac{2j + n(f-1)}{f}.$$

Let $n(f-1)$ be odd. Then,

$$2t = 2(n(f-1)/2 - 1/2 + j) = \sum_{i=1}^{n} \deg(v_i) \le xf + (n-x)(f-1)$$

so that

$$n(f-1) - 1 + 2j \le xf + (n-x)(f-1) = xf + n(f-1) - x(f-1)$$

which when simplified yields $2j - 1 \le x$.

Next note, $xf \le \sum_{i=1}^{n} \deg(v_i) = 2(n(f-1)/2 - 1/2 + j) = n(f-1) + 2j - 1$, so that

$$x \le \frac{2j - 1 + n(f-1)}{f}. \blacksquare$$

3. The value of $M(t, n, f)$ when $f = n - 2$

The following theorem determines the exact value of $M(t, n, n-2)$.

Theorem 3.1. Let t, n, f, and j be integers such that $n \geq 5$, $f = n - 2$ and $t = n(f-1)/2 + j$, where $1 \leq j \leq \lfloor n/2 \rfloor$. Then

$$M(t,n,n-2) = \begin{cases} n-2-2j & \text{when } 1 \leq j \leq \lfloor n/2 \rfloor - 2 \\ 1 & \text{when } j = \lfloor n/2 \rfloor - 1 \\ 0 & \text{when } j = \lfloor n/2 \rfloor \end{cases}$$

Proof. We consider three cases.

(i) $1 \leq j \leq \lfloor n/2 \rfloor - 2$

For each j we construct the complement H of an $(n-2)$-graph G with $d^{+}(G) = M(t, n, n-2) = n - 2 - 2j$. The graph H is a forest having order n and j components. The sizes of H and G, respectively, are equal to $n - j$ and $\binom{n}{2} - (n - j) = \frac{n(n-3)}{2} + j$.

Note that if $f = n - 2$, then $n(f-1)$ is always even. Thus, from Lemma 2.3, the number x of vertices of degree f in G of order n and size $t = n(f-1)/2 + j$ must satisfy

$$2j \leq x \leq \lfloor (2j + n(f-1))/f \rfloor.$$

It will be shown that for a given j it is sufficient to consider only $x = 2j$ and $x = 2j + 1$ to obtain $M(t, n, n-2)$.

First consider $x = 2j + 1$. We define a class $F(n, j)$ of graphs. Each graph H in $F(n, j)$ is a forest that consists of a tree T and $j - 1$ copies of K_2.

Thus, T is of order $n - 2j + 2$ and is constructed from the path P of order $n - 2j + 1$ by adding a pendant vertex at distance one from an endvertex of P.

Each vertex of degree $n - 2$ in G corresponds to a vertex of degree 1 in H. The number of these vertices in H is $2(j-1) + 3 = 2j + 1 = x$.

An edge in H is admissible, if it is not incident to a vertex of degree 1. The number of remaining edges in H is $(j-1) + 3 = j + 2$. Therefore, the number of admissible edges in H is equal to $(n-j) - (j+2) = n - 2 - 2j$. The deletion of an edge in H corresponds to the insertion of an edge in G. Since there are no equivalent admissible edges in H, this leads to $n - 2 - 2j$ non-isomorphic one-edge extensions of G.

Next consider the case $x = 2j$. Let K be a graph of order $n \geq 5$ and size $n - j$ with x vertices of degree 1, r vertices of degree 2, and s vertices of degree at least 3. Then, $n = x + r + s$ and $2(n-j) \geq 1x + 2r + 3s$. Since $x = 2j$, we have

$$2(n-j) \geq 2j + 2r + 3s \text{ so that } n \geq 2j + r + (3/2)s = x + r + (3/2)s.$$

Combining this with $n = x + r + s$ we get $x + r + s \geq x + r + (3/2)s$ or $0 \geq s/2$. Thus, the number s of vertices of degree at least 3 must be 0. Therefore, K must be the union of paths and cycles.

If K is the complement of G, then any two edges deleted from a cycle in K when added to G yield isomorphic one-edge extended graphs. Thus, $d^+(K^c)$ is less than the number of admissible edges. If K is a union of paths (only paths of order at least 4 contribute admissible edges), then it has at most the maximum of 0 or $n - j - 2j$ admissible edges. The latter value occurs when all j paths have order at least 4. Thus,

$$d^+(K^c) \leq \max\{0, n - 3j\}.$$

For $j \geq 2$, we have $n - 3j = n - j - 2j \leq n - 2 - 2j$.

If $j = 1$, that is, K is a single path and $n \geq 5$, then K has $n - 1 - 2 = n - 3$ admissible edges, which is greater than $n - 2 - 2(1) = n - 4$. However, for $n \geq 5$, all paths contain equivalent admissible edges. Thus, $d^+(G) \leq n - 4 = n - 2 - 2(1)$.

It is easy to show that if $x > 2j + 1$, then the number of admissible edges in a complement of G is not greater than $n - 2j - 1$ but at least two of these edges are equivalent. Thus, $d^+(G) \leq n - 2 - 2j$.

(ii) $j = \lfloor n/2 \rfloor - 1$

There exist $(n-2)$-graphs G with exactly two nonadjacent vertices that have degree strictly less than $n - 2$. Thus, here $M(t, n, n-2) = 1$. When n is even, let G be the complement of a forest consisting of P_4, a path of order 4, and $(n-4)/2$ copies of a K_2. Here the only admissible edge is contained in P_4. When n is odd, let G be the complement of a graph from $F(n, j)$, a forest containing a tree of order 5 and $(n-5)/2$ copies of a K_2. This forest has exactly one admissible edge.

(iii) $j = \lfloor n/2 \rfloor$

When n is even, G is regular and the complement of G is isomorphic to $n/2$ copies of a K_2. When n is odd, G is almost-regular (i.e., G has $n - 1$ vertices of degree $n - 2$ and one vertex of degree $n - 3$) and the complement of G is a forest consisting of P_3 and $(n-3)/2$ copies of a K_2. In both cases G is unique and $d^+(G) = 0$. ■

4. An upper bound for $M(t, n, f)$

The following theorem provides a bound $B(t, n, f, x)$ for $d^+(G)$ for any f-graph G of size t, order n, and having x vertices of degree f. The theorem also provides the bound $B(t, n, f)$ for $M(t, n, f)$. Note, that for an f-graph of a given size and order to be realizable the number of vertices of degree f is restricted by the bounds given in Lemma 2.3.

Theorem 4.1. Let $B(t, n, f, x)$ denote the maximum number of admissible edges possible among all f-graphs having size t, order n, x vertices of degree f, and consequently $w = n - x$ orexic vertices. Further let $3 \leq f \leq n - 2$ and $\lfloor n(f-1)/2 \rfloor < t < \lfloor nf/2 \rfloor$. Then

$$\text{(I)} \qquad B(t,n,f,x) = \begin{cases} \binom{w}{2} - t + xf & \text{if } w \geq f \text{ and } w(f-1) > xf \\ \binom{w}{2} & \text{if } x \geq f-1 \text{ and } w(f-1) \leq xf \\ \binom{w}{2} - t + xw + \lfloor x(f-w)/2 \rfloor & \text{if } x < f-1 \text{ and } w < f \end{cases}$$

$$\text{(II)} \qquad B(t,n,f) = \max_{a \leq x \leq b}\{B(t,n,f,x)\},$$

where $t = \lfloor n(f-1)/2 \rfloor + j$ with $0 < j < \lfloor nf/2 \rfloor - \lfloor n(f-1)/2 \rfloor$,

$a = 2j,\ b = \left\lfloor \frac{2j + n(f-1)}{f} \right\rfloor$ when $n(f-1)$ is even, and

$a = 2j - 1,\ b = \left\lfloor \frac{2j - 1 + n(f-1)}{f} \right\rfloor$ when $n(f-1)$ is odd.

Proof. (I). Graphs are constructed to show that the stated expressions for $B(t, n, f, x)$ and $B(t, n, f)$ are the maximums for the parameters t, n, f, and x. The proof is divided into three mutually exclusive and all encompassing cases:

(i) $w \geq f$ and $w(f-1) > xf$, (ii) $x \geq f-1$ and $w(f-1) \leq xf$, and (iii) $x < f-1$ and $w < f$.

In particular, the graphs constructed will have x vertices of degree f, $w = n - x$ orexic vertices and the maximum number of admissible edges for these values of x and w.

In each case we start with n vertices, color x vertices black and $w = n - x$ vertices white. After the construction the black vertices will have degree f and the white vertices will be orexic.

(i) $w \geq f$ and $w(f-1) > xf$

Label the black vertices b_s with $1 \leq s \leq x$ and the white vertices r_u with $1 \leq u \leq w$. Join each black vertex b_s to f distinct white vertices as follows

b_s is joined to r_v, where $v = ((s-1)f + k - 1) \pmod{w} + 1$, $1 \leq k \leq f$.

Since the number of edges going from the black vertices to the white vertices is xf and the maximum number of edges that can be placed on the white vertices is $w(f-1)$, the condition $w(f-1) > xf$ and the construction restricts the degree of any white vertex to be no greater than $f-1$.

Next insert $t - xf$ edges on the white vertices so that each white vertex has degree no greater than $f - 1$. This is possible since $t - xf < w(f - 1)$ is always true.

The result is an f-graph of order n, size t, having x independent vertices of degree f and w orexic vertices.

Since the number of black-vertex to white-vertex edges (*black-white edges*) is maximized, the total number of black to black edges (*black edges*) and white to white edges (*white edges*) is minimized. To see that the number of white edges, $t - xf$, is minimum, assume there is a black edge. Then, the total of black edges and black-white edges would be less than xf. Thus, more than $t - xf$ edges would have to be white edges to reach size t. Therefore, by Lemma 2.2, $\binom{w}{2} - t + xf$ is the maximum number of admissible edges for the values t, n, f, and x, in this case.

(ii) $x \geq f - 1$ and $w(f - 1) \leq xf$

Label the black vertices b_s with $1 \leq s \leq x$ and the white vertices r_u with $1 \leq u \leq w$. Join each white vertex r_u to $f - 1$ distinct black vertices by using an assignment scheme analogous to that in Case (i). Specifically, join each white vertex to $f - 1$ distinct black vertices as follows

r_u is joined to b_v, where $v = ((u - 1)(f - 1) + k - 1) \pmod{x} + 1, \quad 1 \leq k \leq f - 1$.

Similarly to Case (i), here the condition $w(f - 1) \leq xf$ assures that the construction can be carried out so that the degree of any black vertex will not exceed f.

If $t - w(f - 1) \geq 0$, insert $t - w(f - 1)$ black edges so that each black vertex has degree at most f. This is possible because $t - w(f - 1) < xf$ is always true. This graph now has t edges. If not all black vertices have degree f at this point or if $t - w(f - 1) < 0$, then it is necessary to delete some black-white edges and introduce black edges so that there are x vertices of degree f. This construction produces an f-graph of order n, size t, having w independent orexic (white) vertices and x (black) vertices of degree f. Thus, W, the subgraph induced by the w orexic vertices satisfies $|E(W)| = 0$ and by Lemma 2.2, the number of admissible edges $\binom{w}{2}$ for this constructed graph is maximum and $d^+(G) \leq \binom{w}{2}$ for all G in this case.

(iii) $x < f - 1$ and $w < f$

Here neither of the constructions given in Cases (i) and (ii) is possible. In particular, the maximum number of black-white edges is bounded by xw, the size of a complete bipartite graph $K_{x,w}$. Thus, here start with a $K_{x,w}$, with the black and the white vertices defining the vertex bipartition, so as to join the black and white vertices with a maximum number of edges.

Since the x black vertices of $K_{x,w}$ have degree $w < f$, insert an $(f - w)$-regular graph of order x on the black vertices to obtain x vertices of degree f. This can be done provided $x(f - w)$ is even. If $x(f - w)$ is odd, insert an $(f - w)$-almost-regular graph of order x on the black vertices to obtain $x - 1$ vertices of degree f and one vertex of degree $f - 1$ (a *k-almost-regular graph* has one vertex of degree $k - 1$ and the remaining vertices of degree k). Follow this by reassigning a black-white edge with a black vertex of degree f to be a black edge that makes the black vertex of degree $f - 1$ obtain degree f, thereby obtaining x vertices of degree f. In both cases the constructions yield x vertices of degree f, w orexic vertices, and $xw + \lfloor x(f - w)/2 \rfloor$ edges.

In order to obtain an f-graph of size t, insert $t - xw - \lfloor x(f - w)/2 \rfloor$ edges on the w white independent vertices maintaining vertex degree less than f. Since the number of black-white and black edges is maximized, it follows that the size of W, namely, $t - xw - \lfloor x(f - w)/2 \rfloor$ is minimum.

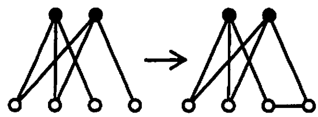
Therefore, by Lemma 2.2, the number of admissible edges for the constructed graph, $\binom{w}{2} - t + xw + \lfloor x(f - w)/2 \rfloor$ is maximum in this case.

In each of the cases, (i), (ii), and (iii), the number of admissible edges determined is the value of $B(t, n, f, x)$.

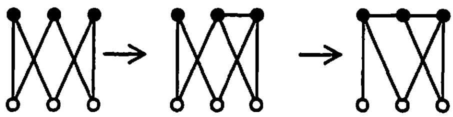
(II). By Lemma 2.3 the number of vertices of degree f lies between a and b. Thus, $B(t,n,f) = \max_{a \le x \le b}\{B(t,n,f,x)\}$.
This completes the proof of Theorem 4.1. ■

In Figure 4.1 examples of constructions defined in Theorem 4.1 are shown.

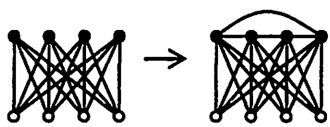
(*i*)



(*ii*)



(*iii*) $x(f-w)$ is even



(*iii*) $x(f-w)$ is odd

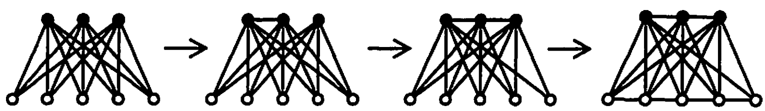


Figure 4.1. Examples of constructions
(*i*) $n = 6, t = 7, f = 3, x = 2$, (*ii*) $n = 6, t = 7, f = 3, x = 3$, and
(*iii*) $n = 8, t = 20, f = 6, x = 4$ for $x(f-w)$ even and $n = 8, t = 20, f = 6, x = 3$ for $x(f-w)$ odd

The following example provides small order graphs that illustrate Theorem 4.1.

Example. In Figure 4.2 we show all the 3-graphs of order 6 and size 7. Partitioning these graphs in accordance with x, their number of vertices of degree $f = 3$ and letting $b = b(G)$ denote the number of admissible edges in graph G we have

$x = 4$: G_1, G_2, G_3	$b = 1, 1, 0$	$d^+ = 1, 1, 0$
$x = 3$: G_4, G_5, G_6, G_7	$b = 3, 2, 2, 3$	$d^+ = 3, 1, 2, 2$
$x = 2$: G_8, G_9, G_{10}, G_{11}	$b = 4, 4, 5, 4$	$d^+ = 3, 2, 2, 1$.

Thus, for $x = 4$, 3, and 2, $B(7, 6, 3, 4) = 1$, $B(7, 6, 3, 3) = 3$, and $B(7, 6, 3, 2) = 5$. These numbers are as predicted by Theorem 4.1(I) and $\max\{1, 3, 5\} = 5$ is as predicted by Theorem 4.1(II). Further note, from $t = \lfloor n(f-1)/2 \rfloor + j$, we obtain $j = 1$ and that $B(7, 6, 3) = 5$ is realized by G_{10} with $x = 2j = 2$ and $M(7, 6, 3) = 3$ is realized by two graphs, G_8 (with $x = 2j = 2$) and G_4 (with $x = 2j + 1 = 3$).

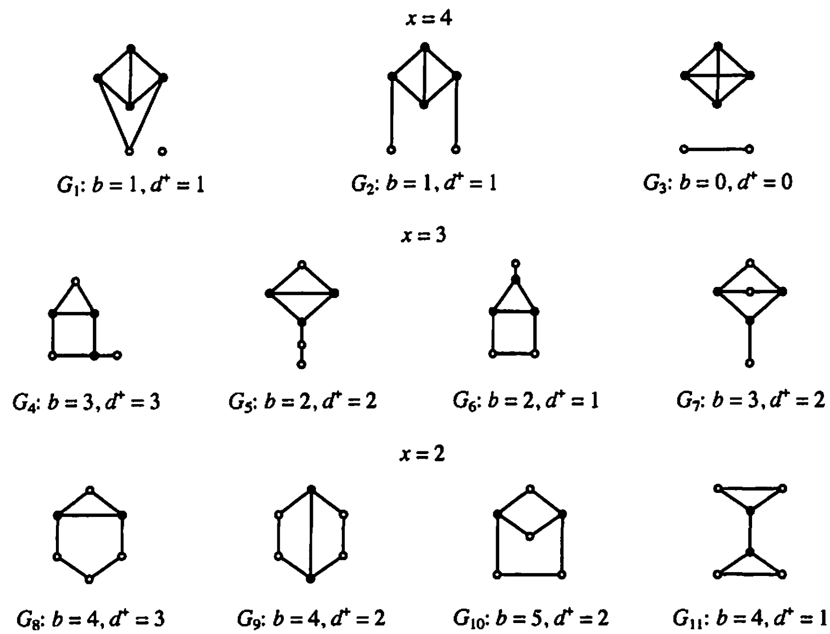


Figure 4.2. All 3-graphs with $n = 6$ and $t = 7$ ($j = 1$) partitioned into x-classes, $x = 4, 3$, and 2

In Table A (see Appendix), the values of $B(t, n, f)$ (from Theorem 4.1) and $M(t, n, f)$ (from an exact algorithm) for $6 \leq n \leq 12$ are given.

For the size range $\lfloor n(f-1)/2 \rfloor < t < \lfloor nf/2 \rfloor$, Theorem 4.1(I) provides an upper bound, $B(t, n, f, x)$, for $d^+(G)$ for an f-graph G of order n and size t when x, the number of vertices of degree f is given and realizable for the specified t. If x is not given, the following Theorem 4.2 provides an upper bound for $B(t, n, f)$ and consequently for $M(t, n, f)$ for the stated range of t, independent of the number of vertices of degree f.

Theorem 4.2. Let n, f, and t be integers such that $n \geq 5$, $3 \leq f \leq n-2$, and $t = \lfloor n(f-1)/2 \rfloor + j$ with $0 < j < \lfloor nf/2 \rfloor - \lfloor n(f-1)/2 \rfloor$. Then

$$B(t,n,f) \leq \begin{cases} \binom{n-2j}{2} & \text{when } n(f-1) \text{ is even} \\ \binom{n-2j+1}{2} & \text{otherwise} \end{cases}$$

Proof. First note that for any given f, n, t, and x, with $w = n - x$,

$$t - xf \geq 0 \text{ and } t - xw - \lfloor x(f-w)/2 \rfloor \geq 0$$

so that each of the bounds given in Theorem 4.1(I) is bounded by $\binom{w}{2}$. Second, for any given j, Lemma 2.3 yields, $x \geq 2j$ when $n(f-1)$ is even and $x \geq 2j-1$, otherwise. Therefore, when $n(f-1)$ is even we have $\binom{w}{2} = \binom{n-x}{2} \leq \binom{n-2j}{2}$; and when $n(f-1)$ is odd $\binom{w}{2} = \binom{n-x}{2} \leq \binom{n-2j+1}{2}$. ■

For the case $f = n-2$, we have the following theorem.

Theorem 4.3. If $f = n-2$, then the maximum outdegree of f-graphs of size $t = n(f-1)/2 + j$ with $1 \leq j \leq \lfloor n/2 \rfloor$ is given by

$$M(t,n,f) = \begin{cases} B(t,n,f) \text{ when } B(t,n,f) = 0 \text{ or } 1 \\ B(t,n,f) - 2, \text{otherwise.} \end{cases}$$

Proof.
(i) $B(t, n, f) = 0$ or 1
$M(t, n, f) = B(t, n, f)$ is a direct consequence of Theorem 3.1.

(ii) $B(t, n, f) > 1$
If $f = n-2$, then $n(f-1)$ is even and by Lemma 2.3, $x \geq 2j$. Consider the complement of the union of j copies of a K_2 and an $(n-2j)$-cycle. For these graphs the number of admissible edges is equal to $n-2j$. There are no other graphs with the number of admissible edges greater than $n-2j$. Thus, from Theorem 3.1, $M(t, n, f) = n - 2j - 4 = B(t, n, f) - 2$. ■

5. Concluding remarks

We have studied $M(t, n, f)$, the maximum number of non-isomorphic one-edge f-graph extensions of an f-graph G among all f-graphs of size t and order n. For the case $f = n-2$ an exact solution is given in Theorem 3.1. $B(t, n, f)$, an upper bound for $M(t, n, f)$ is given in Theorem 4.1. In general, $B(t, n, f)$ is not equal $M(t, n, f)$. This is due to the fact that equivalent admissible edges may yield isomorphic one-edge extensions of a given graph. Even using the fact that almost all graphs are identity graphs does not completely resolve the problem because there are identity graphs that have pairs of admissible edges that give rise to isomorphic one-edge extensions (see Figure 1.1).

We propose the following open problems.

1. For what values of t, n, and f with $\lfloor n(f-1)/2 \rfloor < t < \lfloor nf/2 \rfloor$ and $3 \leq f \leq n-3$ is $M(t, n, f) = B(t, n, f)$?
   In particular, are there any infinite classes, for which this occurs?
2. Determine $M(t, n, f)$ for $f = n - i$ with $i > 2$.

The work in this paper relates to the study of digraphs whose vertices are f-graphs in the following way. Let $D(n, f)$ denote the digraph whose *node set* consists of all unlabeled f-graphs of order n. An arc (G, H) in $D(n, f)$ exists if and only if a graph isomorphic to H can be obtained from G by the insertion of an edge. $D(n, f)$ is the underlying digraph of the *transition digraph* of the Random f-Graph Process and $M(t, n, f)$ is the *maximum outdegree* of a node in $D(n, f)$ [4][5]. A probabilistic study of $M(t, n, f)$ remains open.

Graphs whose vertices are graphs with bounded degree and their adjacency relation, one-edge transitions between f-graphs, are of interest in many contexts. For example, in chemistry and physics, f-graphs can be thought of as molecules or system of molecules (polymers) and the insertion or deletion of an edge can be interpreted as the creation or breaking of a bond between atoms or a link between molecules. Since the valence of an atom is bounded, the bounded degree condition is obvious in applications in chemistry. For other applications other conditions naturally bound vertex degrees. Examples of both deterministic and probabilistic applications of graphs with bounded degree can be found in Chapter 9 of [5] and in [6].

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Appendix

Table A. $B(t, n, f)$ and $M(t, n, f)$,
$6 \le n \le 12,\ 3 \le f \le n-2,\ t = \lfloor n(f-1)/2 \rfloor + j,\ 1 \le j \le \lfloor n/2 \rfloor$

$n=6$	$B(t,6,f)$		$M(t,6,f)$	
j	$f=3$	4	$f=3$	4
1	5	4	3	2
2	1	1	1	1
3	0	0	0	0

$n=7$	$B(t,7,f)$			$M(t,7,f)$		
j	$f=3$	4	5	$f=3$	4	5
1	8	8	5	5	6	3
2	3	6	1	2	4	1
3	0	1	0	0	1	0
4	-	0	-	-	0	-

$n=8$	$B(t,8,f)$				$M(t,8,f)$			
j	$f=3$	4	5	6	$f=3$	4	5	6
1	12	10	8	6	9	9	7	4
2	6	6	6	4	5	5	4	2
3	1	1	1	1	1	1	1	1
4	0	0	0	0	0	0	0	0

$n=9$	$B(t,9,f)$					$M(t,9,f)$				
j	$f=3$	4	5	6	7	$f=3$	4	5	6	7
1	17	18	12	11	7	13	18	12	11	5
2	10	12	10	9	5	9	12	7	7	3
3	3	6	3	6	3	3	6	3	4	1
4	0	1	0	1	0	0	1	0	1	0
5	-	0	-	0	-	-	0	-	0	-

$n=10$	$B(t,10,f)$						$M(t,10,f)$					
j	$f=3$	4	5	6	7	8	$f=3$	4	5	6	7	8
1	23	20	17	14	11	8	19	20	17	14	11	6
2	15	14	13	12	9	6	14	13	12	10	7	4
3	6	6	6	6	6	4	6	6	6	5	4	2
4	1	1	1	1	1	1	1	1	1	1	1	1
5	0	0	0	0	0	0	0	0	0	0	0	0

$n=11$	$B(t,11,f)$							$M(t,11,f)$						
j	$f=3$	4	5	6	7	8	9	$f=3$	4	5	6	7	8	9
1	30	32	23	23	16	14	9	25	32	23	23	16	14	7
2	20	22	17	17	14	12	7	19	22	17	17	12	11	5
3	10	15	10	15	10	9	5	10	14	10	12	8	7	3
4	3	6	3	6	3	6	3	3	6	3	6	3	4	1
5	0	1	0	1	0	1	0	0	1	0	1	0	1	0
6	-	0	-	0	-	0	-	-	0	-	0	-	0	-

$n=12$	$B(t,12,f)$								$M(t,12,f)$							
j	$f=3$	4	5	6	7	8	9	10	$f=3$	4	5	6	7	8	9	10
1	38	34	30	26	22	18	14	10	32	34	30	26	22	18	14	8
2	26	24	22	20	18	16	12	8	25	24	22	19	17	15	11	6
3	15	15	15	15	15	12	9	6	15	15	14	13	12	10	7	4
4	6	6	6	6	6	6	6	4	6	6	6	6	6	6	4	2
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0