Defensive alliances in circulant graphs

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Abstract

In this paper we study defensive alliances in some specific regular graphs, the circulant graphs, i.e. Cayley graphs on a cyclic group. The critical defensive alliances of a circulant graph of degree at most 6 are completely determined. For the general case, we give tight lower and upper bounds on the alliance number of a circulant graph with d generators.

Keywords: Alliance, circulant graphs.

1 Introduction

Informally, an alliance is a perfect example of the well known slogan "United we stand, divided we fall." An alliance in a graph is a set of vertices with the property that the set is protected from attacks by other vertices (in the case of defensive alliances), or, for offensive alliances, is able to collaborate to attack other vertices.

Alliances were introduced in [14]. A *defensive alliance* is a set of vertices with the property that each vertex has at least as many neighbors in the alliance (including itself) as neighbors not belonging to the alliance. A defensive alliance is *strong* if each vertex has more neighbors in the alliance than not in it, and it is *critical* if it does not include other defensive alliances. An *offensive alliance* [7] is a set of vertices such that each vertex has at least as many neighbors on its boundary in the alliance as neighbors not belonging to the alliance (including itself). Strong and critical offensive alliances are defined similarly to strong and critical defensive alliances.

An alliance is called *global* if it is also a dominating set. Global defensive alliances and global offensive alliances were first studied in [12] and [23] respectively.

The concept of alliance is relatively new, but it is related to some other well known concepts and problems. Moreover it has given rise to new concepts and problems that are worth studying in different contexts. In complex networks, the definition of web community, as in [10], coincides with the definition of offensive alliance. Some works relate alliances to community detection and partitioning [10, 13]. Other related concepts are modules [18]; and in distributed computing, coalitions and monopolies [11, 17, 21]. From an algorithmic point of view, the clustering coefficient is defined in terms of small alliances in [5], and a study of algorithms for global alliances is given in [28]. Some of the work related to alliances in the context of graph theory include [9, 26], where the concept of k-alliance is defined and studied, and [22, 25], in which the authors focus on the spectral properties of alliances. Questions about complexity and alliances are studied in [8].

We focus on the study of critical defensive alliances in regular graphs. In a d-regular graph, a defensive alliance is a set of vertices that induces a subgraph with minimum degree at least $\lfloor \frac{d}{2} \rfloor$ and maximum degree at most d. As the description of graphs that induce critical defensive alliances in d-regular graphs is known for $d \leq 5$ (see [14, 24]), the authors of this paper solved, in [3], the problem of which graphs of cardinality $k \leq 8$ induce a critical defensive alliance in 6-regular graphs. Even in these restricted cases, there is no easy description of such alliances. Because of the complexity of the problem, in this paper we restrict the question to a family of very symmetric graphs, the well known circulant graphs.

Circulant digraphs were first defined as graphs whose adjacency matrix is a circulant matrix [6]. Circulant digraphs are, in fact, Cayley graphs on the cyclic group $\mathbb{Z}_n$. If the set of generators is closed under inversion, then the digraph is symmetric and it can be seen as a graph.

The regularity and the underlying algebraic structure of Cayley graphs and, particularly, circulant (di)graphs make them good candidates for interconnecting nodes of a network [15]. A problem that has been widely studied is the isomorphism of circulant graphs. The Ádám conjecture, proposed in 1967 in [1], gave rise to a large amount of literature. Alspach [2] gave a good overview of the state of the problem about ten years ago. The problem was recently closed in [20]. Other problems on circulant graphs under study include automorphism groups of circulant graphs [19], spanning trees [4], arboricity [27], and extremal problems [16].

The paper is organized as follows. Basic definitions and properties are given in Section 2. In Section 3 we enumerate specific results about defensive alliances in regular graphs. More of these results appear in [3] and will be used in Section 4. The main results of this paper on alliances in circulant (undirected) graphs are given in Section 4. Finally, in Section 5, we finish with some conclusions and open problems.

2 Definition and basic properties

First, we introduce some notation and basic definitions. Given a graph $G = (V, E)$ we denote by n and m its order and size respectively. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) := \{u \in V : u \sim v\}$, and the *closed neighborhood* of v is the set $N[v] := N(v) \cup \{v\}$. The degree of v is $d(v) := |N(v)|$. The *boundary* of S is the set $\partial(S) = \cup_{v \in S} N(v) - S$ and we denote by $\langle S \rangle$ the subgraph of G induced by S.

Given a non-empty set of vertices S, the neighborhood of v in S is $N_S(v) := \{u \in S : , u \sim v\} = N(v) \cap S$. Denoting by $\overline{S}$ the complement in V of S, we have $N(v) = N_S(v) \cup N_{\overline{S}}(v)$. We denote by $\langle S \rangle$ the subgraph of G induced by S.

2.1 Alliances

The following definitions are taken from [14].

Definition 2.1 (Defensive alliance) *A non-empty set $S \subseteq V$ is a* defensive alliance *of G if, for every $v \in S$,*

$$|N_S[v]| \geq |N_{\overline{S}}(v)|. \tag{1}$$

We say that the alliance is strong *if, for every $v \in S$, the inequality is strict.*

The inequality (1) is called the (defensive) boundary condition.

Definition 2.2 (Offensive alliance) *A non-empty set $S \subseteq V$ is an* offensive alliance *of G if, for every $v \in \partial(S)$,*

$$|N_S(v)| \geq |N_{\overline{S}}[v]|. \tag{2}$$

We say that the alliance is strong *if, for every $v \in \partial(S)$, the inequality is strict.*

The inequality (2) is called the (offensive) boundary condition.

An alliance (of any type) is said to be *global* if it is also a dominating set of the graph. (Recall that S is a *dominating set* if every vertex of G

is in S or has a neighbor in S, that is, $N[S] = V$.) An alliance (of either type) is said to be *critical* if none of its proper subsets is an alliance of the same type. A *dual (or powerful) alliance* is a set that is both a defensive and an offensive alliance.

In the remainder of the paper we will focus on defensive alliances. Note that the whole graph G is a defensive alliance in G. Moreover if S is a critical (strong) defensive alliance in G, then $\langle S \rangle$ is connected.

2.2 Alliance numbers

From the definition of alliance, some problems naturally arise. The first problem studied is to find the minimum cardinality of a defensive alliance of a given graph G. The problem we are interested in is which subsets of V, or the induced subgraphs of G, are critical defensive alliances and, among them, which are the minimal defensive alliances.

For a graph G, we can consider the following classes;

- $\mathcal{A}(G)$, the class of critical defensive alliances.
- $\hat{\mathcal{A}}(G)$, the class of critical strong defensive alliances.

Associated with these classes, the following invariants are defined:

- The defensive alliance number of G,

$$a(G) := \min\{|S| : S \in \mathcal{A}(G)\}.$$

  The upper defensive alliance number of G,

$$A(G) := \max\{|S| : S \in \mathcal{A}(G)\}.$$

- The strong defensive alliance number of G,

$$\hat{a}(G) := \min\{|S| : S \in \hat{\mathcal{A}}(G)\}.$$

  The upper strong defensive alliance number of G,

$$\hat{A}(G) := \max\{|S| : S \in \hat{\mathcal{A}}(G)\}.$$

For the defensive alliance number of a graph, called alliance number from here on, it is easy to find tight lower bounds in terms of the minimum degree of the graph, as well as tight upper bounds in terms of the order:

$$\left\lfloor \frac{\delta_G}{2} \right\rfloor + 1 \leq \quad a(G) \quad \leq \left\lceil \frac{n}{2} \right\rceil, \tag{3}$$

$$\left\lceil \frac{\delta_G}{2} \right\rceil + 1 \leq \quad \hat{a}(G) \quad \leq \left\lfloor \frac{n}{2} \right\rfloor + 1. \tag{4}$$

The alliance number of a graph G is also related to its girth $g(G)$, i.e., the length of the shortest cycle of the graph (if any): If $\delta_G \geq 4$, then

$$g(G) \leq a(G).$$

The classes of critical offensive alliances and critical strong offensive alliances, with their corresponding alliance numbers, can be defined analogously. Also, we can define the classes and alliance numbers for global alliances of any type.

It is worth mentioning that the decision problems associated with the different variations of alliances are all NP-complete (see [8] and the references therein). Therefore it makes sense to study both the properties of the different types of alliance numbers and the alliance number of restricted classes of graphs.

3 Defensive alliances in regular graphs

The alliance numbers of d-regular graphs are known only for $d \leq 5$ (see [14, 24]). In this section we enumerate these results and also give the principal results of [3] that will be used in the following section.

We denote by $g(G)$ the girth of G and by $lc(G)$ the maximum length of an induced cycle in G. If G is d-regular, then it is known that:

- $d = 1 \Rightarrow a(G) = A(G) = 1$, $\hat{a}(G) = \hat{A}(G) = 2$;
- $d = 2 \Rightarrow a(G) = A(G) = \hat{a}(G) = \hat{A}(G) = 2$;
- $d = 3 \Rightarrow a(G) = A(G) = 2$, $\hat{a}(G) = g(G)$, and $\hat{A}(G) = lc(G)$;
- $d = 4 \Rightarrow a(G) = \hat{a}(G) = g(G)$, $A(G) = \hat{A}(G) = lc(G)$; and
- $d = 5 \Rightarrow a(G) = g(G)$, $A(G) = lc(G)$.

If $G = (V, E)$ is a graph, we say that a vertex $v \in S \subset V$ is defended in S if and only if it satisfies the boundary condition with respect to S. Similarly, if v satisfies the strong boundary condition with respect to S we say that v is strongly defended in S. Let $G = (V, E)$ be a graph, and $v \in S \subset V$. The following properties are direct consequences of the definition of alliance and strong alliance.

Property 3.1 *If $d(v) = 2k$, v is defended in S if and only if $d_S(v) \geq k$. Moreover the strong boundary condition is equivalent to the boundary condition, i.e., v is defended in S if and only if it is strongly defended in S.*

Property 3.2 *If $d(v) = 2k+1$, v is defended in S if and only if $d_S(v) \geq k$; v is strongly defended in S if and only if $d_S(v) \geq k+1$.*

Property 3.3 *If G is d-regular, then S is an alliance in G if and only if S induces a subgraph of minimum degree $\delta_S \geq \lfloor \frac{d}{2} \rfloor$; S is a strong alliance in G if and only if it induces a subgraph of minimum degree $\delta_S \geq \lceil \frac{d}{2} \rceil$.*

In fact, the known results for regular graphs of degree $d \leq 5$ allow us to completely characterize critical alliances for these graphs:

- If G is 1-regular, the critical alliances are exactly the singletons.
- The strong critical alliances in a 1-regular or 2-regular graph and the critical alliances in a 2-regular or 3-regular graph are exactly the edges.
- The strong critical alliances in a 3-regular or 4-regular graph and the critical alliances in a 4-regular or 5-regular graph are exactly the induced cycles.
- The strong critical alliances in a 4-regular or 5-regular graph and the critical alliances in a 5-regular or 6-regular graph are studied by the authors of the present paper in [3].

In this section we give some definitions and results that appear in [3] and which we will use to prove the main results of this paper

Definition 3.4 (Induced alliance set) *The* (k, d)-induced alliance set *is the set of graphs H of order k, minimum degree $\delta_H \geq \lfloor \frac{d}{2} \rfloor$, and maximum degree $\Delta_H \leq d$, with no proper subgraph of minimum degree greater than $\lfloor \frac{d}{2} \rfloor$. We denote this set by $\mathcal{S}_{(k,d)}$.*

Similarly, the (k, d)-induced strong alliance set *is the set of graphs H of order k, minimum degree $\delta_H \geq \lceil \frac{d}{2} \rceil$, and maximum degree $\Delta_H \leq d$, with no proper subgraph of minimum degree greater than $\lceil \frac{d}{2} \rceil$. We denote this set by $\hat{\mathcal{S}}_{(k,d)}$.*

For instance, $\mathcal{S}_{(2,2)} = \mathcal{S}_{(2,3)} = \{K_2\}$, and $\mathcal{S}_{(k,2)} = \mathcal{S}_{(k,3)} = \emptyset$, if $k \geq 3$; $\mathcal{S}_{(5,4)} = \mathcal{S}_{(5,5)} = \{C_5\}$, and $\mathcal{S}_{(k,4)} = \mathcal{S}_{(k,5)} = \{C_k\}$, if $k \geq 6$.

The following result is a consequence of the definitions of defensive alliance and (k, d)-induced alliance set, or (k, d)-ias for short.

Proposition 3.5 *If G is d-regular, then S is a critical alliance of G of cardinality k if and only if $\langle S \rangle \in \mathcal{S}_{(k,d)}$.*

Notice that Proposition 3.5 says that alliances in regular graphs are defined by induced subgraphs of given minimum degree. The family of graphs that can be induced by a critical alliance can be described by its degree sequence.

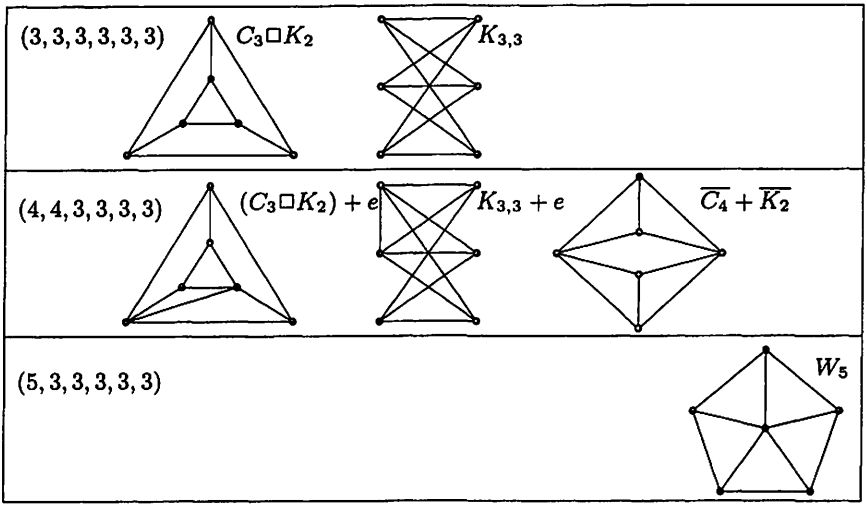

Figure 1: The (6, 6)-induced alliance set, with associated degree sequences.

Definition 3.6 (admissible sequence) *A sequence* $\mathbf{s} = (d_1, d_2, \ldots, d_k)$ *is a* (k, d)-admissible sequence, *or an admissible sequence, if there is a graph* $G_{\mathbf{s}}$ *in* $\mathcal{S}_{(k,d)}$ *with degree sequence* $\mathbf{s}$.

All results in defensive alliances of d-regular graphs are based on determining all (k, d)-admissible sequences and then describing the corresponding (k, d)-induced alliance sets.

For a 6-regular graph G and a critical alliance of G, S, the cardinality of S determines the degree sequence associated to $\langle S \rangle$, as a graph.

- If $|S| = 4$ then $\langle S \rangle = K_4$ and its associated degree sequence is $(3, 3, 3, 3)$. That is, $\mathcal{S}_{(4,6)} = \{K_4\}$.
- If $|S| = 5$ then $\langle S \rangle = W_4$ and its associated degree sequence is $(4, 3, 3, 3, 3)$. That is, $\mathcal{S}_{(5,6)} = \{W_4\}$.

Proposition 3.7 *The* $(6, 6)$*-ias are:*

$$\mathcal{S}_{(6,6)} = \{C_3 \square K_2,\ K_{3,3},\ (C_3 \square K_2) + e,\ K_{3,3} + e,\ \overline{C_4} + \overline{K_2},\ W_5\}$$

This set contains exactly the six graphs in Figure 1.

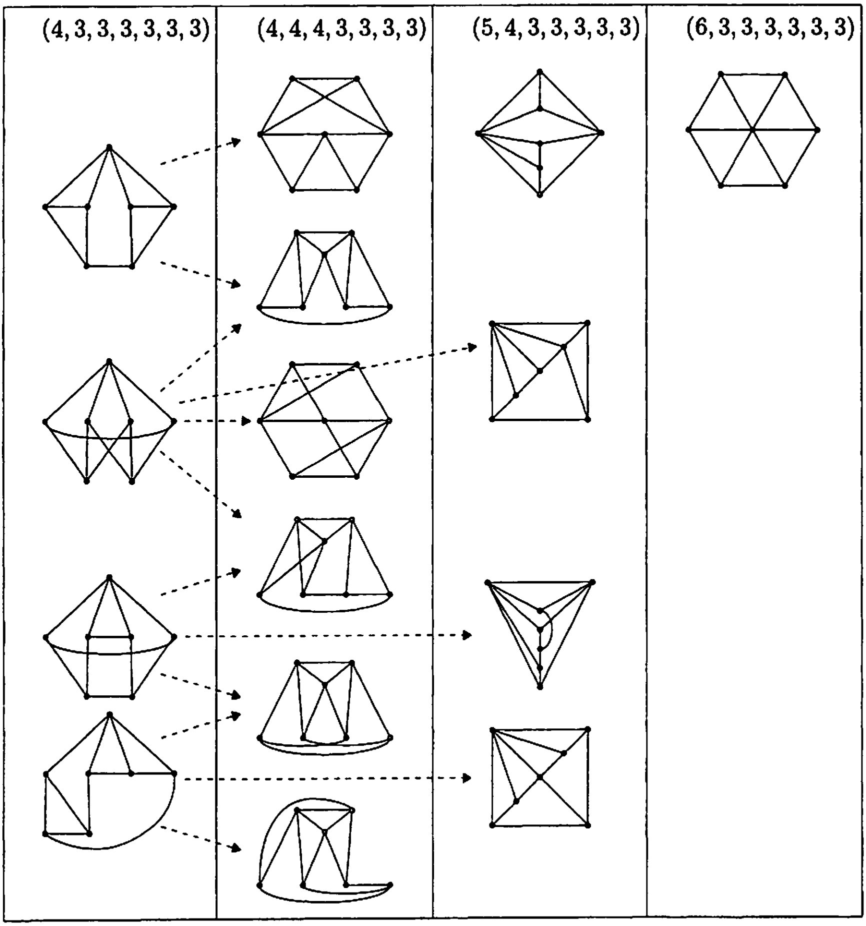

Figure 2: The (7, 6)-induced alliance set, with associated degree sequence. The arrows indicate the subgraph relation.

Proposition 3.8 *The set* $\mathcal{S}_{(7,6)}$ *of the* $(7,6)$*-ias contains exactly the* 15 *graphs in Figure 2.*

Corollary 3.9 *Let* $G = (V, E)$ *be a* 6*-regular graph.*

- $a(G) = 4 \Leftrightarrow K_4$ *is an induced subgraph of* G;
- $a(G) = 5 \Leftrightarrow W_4$ *is an induced subgraph of* G *and* K_4 *is not; and*
- $a(G) = 6 \Leftrightarrow$ *some graph in* $\mathcal{S}_{(6,6)}$ *is an induced subgraph of* G, *and neither* K_4 *nor* W_4 *are.*

- *$a(G) = 7 \Leftrightarrow$ some graph in $\mathcal{S}_{(7,6)}$ is an induced subgraph of G, and neither K_4 nor W_4, nor any of the graphs in $\mathcal{S}_{(6,6)}$ are.*

The number of graphs in $\mathcal{S}_{(m,6)}$ increases with the cardinality, m. A similar but longer chain of reasoning gives the following claim.

Claim 3.10 *The set $\mathcal{S}_{(8,6)}$ of the $(8,6)$-ias contains exactly the 65 graphs in Figure 3.*

We can easily extend the previous results to 7-regular graphs. The set of $(8,7)$-ias, $\mathcal{S}_{(8,7)}$, contains exactly the graphs in $\mathcal{S}_{(8,6)}$, plus W_7. To summarize, we have

$$\mathcal{S}_{(6,7)} = \mathcal{S}_{(6,6)}, \quad \mathcal{S}_{(7,7)} = \mathcal{S}_{(7,6)}, \quad \mathcal{S}_{(8,7)} = \mathcal{S}_{(8,6)} \cup \{W_7\}.$$

(See Figures 1, 2 and 3.)

3.1 Strong defensive alliances in regular graphs

Defensive alliances and strong defensive alliances coincide if G is d-regular, with d even. For d odd, a defensive alliance is a set of vertices that induces a subgraph with minimum degree at least $\frac{d-1}{2}$ and maximum degree at most d, while a strong defensive alliance is a set of vertices that induces a subgraph with minimum degree at least $\frac{d+1}{2}$ and maximum degree at most d. (See Property 3.3.)

5-regular graphs. We have that $\hat{\mathcal{S}}_{(m,5)}$, defined in Definition 3.4, is the set of graphs of minimum degree at least 3 and maximum degree at most 5. Thus if $m \leq 6$, $\hat{\mathcal{S}}_{(m,5)} = \mathcal{S}_{(m,6)}$. For $m = 7, 8$, we have to remove from $\mathcal{S}_{(m,6)}$ the graphs with maximum degree 6. To be precise, $\hat{\mathcal{S}}_{(7,5)} = \mathcal{S}_{(7,6)} \setminus \{W_6\}$ and $\hat{\mathcal{S}}_{(8,5)}$ contains the 59 graphs in $\mathcal{S}_{(8,6)}$ (see Figure 3).

6-regular graphs. We have that $\hat{\mathcal{S}}_{(m,6)} = \mathcal{S}_{(m,6)}$.

7-regular graphs. A graph is in $\hat{\mathcal{S}}_{(m,7)}$ if it has minimum degree 4 and maximum degree at most 7, and it contains no subgraph isomorphic to a graph in $\hat{\mathcal{S}}_{(m',7)}$, for any $4 \leq m' < m$.

In this case, we cannot derive any result about the $(m,7)$-induced strong alliance set from the $(m,7)$-induced alliance set.

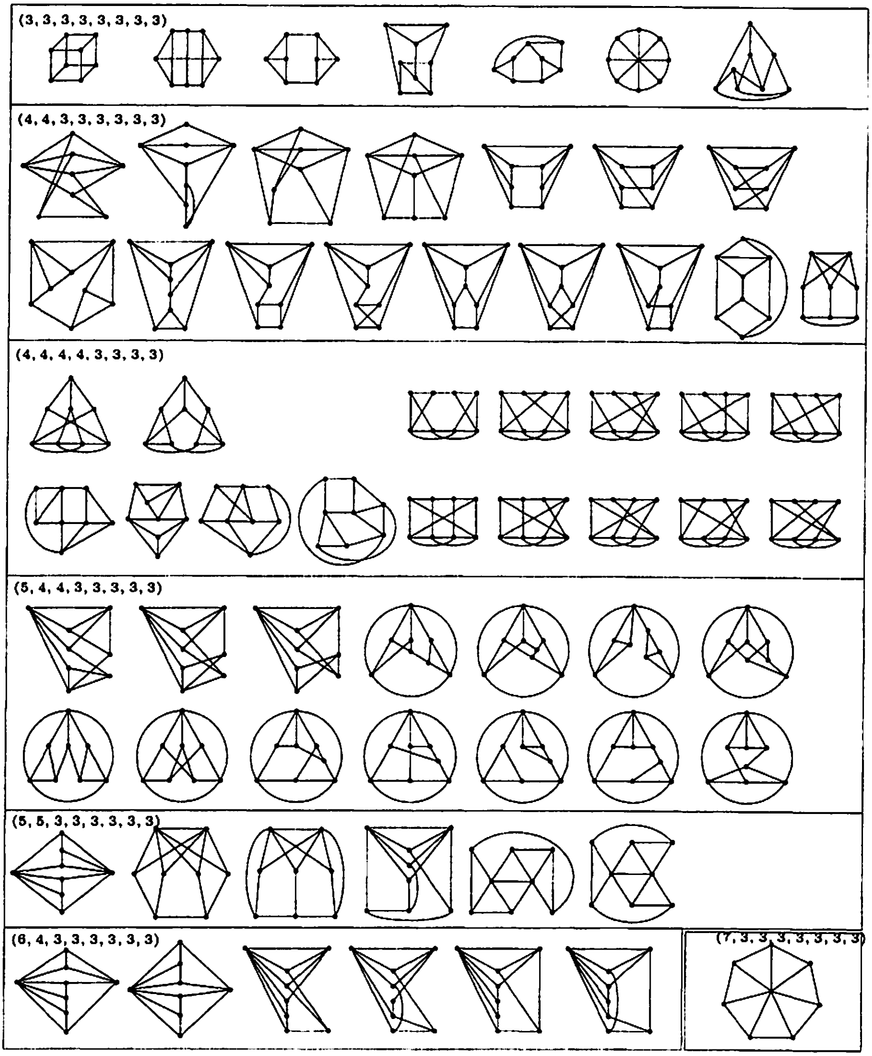

Figure 3: The (8, 6)- and the (8, 7)-induced alliance set, with associated degree sequence. The first 65 graphs are the graphs in $\mathcal{S}_{(8,6)}$, which are also in $\mathcal{S}_{(8,7)}$. The set $\mathcal{S}_{(8,7)} \setminus \mathcal{S}_{(8,6)}$ contains only the graph W_7.

4 Defensive alliances in circulant graphs

In this section we give the main results of this paper. As we have seen, the study of defensive alliances in regular graphs becomes more and more complex as the degree increases. Therefore it makes sense to restrict the study of alliances to more symmetric graphs.

In this paper we begin our study of defensive alliances on a family of highly symmetric graphs, the well known (undirected) circulant graphs. Circulant graphs are Cayley graphs on the cyclic group $\mathbb{Z}_n$. Since we are studying undirected graphs, the set of generators must be closed under additive inversion. An undirected circulant graph can be defined as follows.

Definition 4.1 (Circulant graph of order n with generators $c_1, c_2, \ldots, c_d$) *The* circulant graph *of order n with generators $c_1, c_2, \ldots, c_d$ is the graph $G = C_n(c_1, c_2, \ldots, c_d)$ with vertex set $\mathbb{Z}_n$ and adjacencies defined by*

$$v \sim v \pm c_i$$

for every $v \in \mathbb{Z}_n$ and $i = 1, \ldots, d$.

Remark. The usual notation for a symmetric or undirected circulant graph with generators $c_1, c_2, \ldots, c_d$ is $C_n(\pm c_1, \pm c_2, \ldots, \pm c_d)$. Since we are only dealing with undirected graphs, we use the simpler notation $C_n(c_1, c_2, \ldots, c_d)$, assuming that both c_i and $-c_i$ are in the set of generators.

According to this notation, a permutation of the set of generators, which gives an isomorphic circulant graph, is given by a permutation of the set $\{c_1, c_2, \ldots, c_d\}$, but also by the change of the signs of an arbitrary subset of $\{c_1, c_2, \ldots, c_d\}$.

4.1 Properties of circulant graphs

We first recall some well known properties of circulant graphs.

- The circulant graph $C_n(c_1, c_2, \ldots, c_d)$ is connected if and only if $\gcd(c_1, c_2, \ldots, c_d, n) = 1$.

  If $\gcd(c_1, \ldots, c_d, n) = m$ then $C_n(c_1, c_2, \ldots, c_d)$ is isomorphic to m copies of the connected circulant graph $C_{\frac{n}{m}}(\frac{c_1}{m}, \frac{c_2}{m}, \ldots, \frac{c_d}{m})$.

  Thus we can restrict our study to the case of connected circulant graphs and therefore we always assume the connectedness condition $\gcd(c_1, \ldots, c_d, n) = 1$.

- If $\frac{n}{2} \notin \{c_1, \ldots, c_d\}$, then $C_n(c_1, \ldots, c_d)$ is $2d$-regular.

  If $\frac{n}{2} \in \{c_1, c_2, \ldots, c_d\}$, then $C_n(c_1, \ldots, c_d)$ is $(2d-1)$-regular.

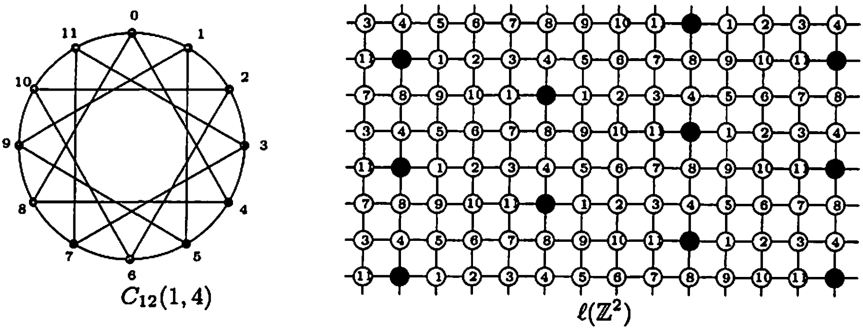

Figure 4: The circulant graph $C_{12}(1,4)$ with its lattice representation. The vertices of the sublattice $\ell^{-1}(0)$ are solid grey.

- If $\lambda \in \mathbb{Z}_n^*$ then $C_n(c_1, c_2, \ldots, c_d) \cong C_n(\lambda \cdot c_1, \lambda \cdot c_2, \ldots, \lambda \cdot c_d)$. This kind of isomorphism is called Ádám-isomorphism [1]. In particular, if one of the generators of a circulant graph, say c_1, is invertible, then we can always assume that it is equal to 1. Indeed,

$$C_n(c_1, c_2, \ldots, c_d) \cong C_n(1, c_2', \ldots, c_d'),$$

  where $c_i' = c_1^{-1} \cdot c_i$, for $i = 2, \ldots, c_d$.

- Every circulant graph is vertex-symmetric. For every $v \in \mathbb{Z}_n$, the mapping $f_v \colon \mathbb{Z}_n \to \mathbb{Z}_n$ defined by $f_v(u) = u + v$ is an automorphism of $C_n(c_1, c_2, \ldots, c_d)$, which applies the edge $\{u, u + c\}$ to the edge $\{u + v, u + v + c\}$, for any $u \in \mathbb{Z}_n$ and c in the set of generators.

- A circulant graph $C_n(c_1, c_2, \ldots, c_d)$ contains triangles if, for some i, j and k pairwise distinct, $c_i \pm c_j \pm c_k = 0$, or $2c_i \pm c_j = 0$, or $3c_i = 0$, with the additions and products modulo n.

Lattice representation. Let $C_n(c_1, \ldots, c_d)$ be a circulant graph, and consider the infinite integer lattice $\mathbb{Z}^d$, with the usual adjacencies $(x_1, \ldots, x_i, \ldots, x_d) \sim (x_1, \ldots, x_i \pm 1, \ldots, x_d)$, for $i = 1, \ldots, d$.

The vertices of this lattice can be labeled in $\mathbb{Z}_n$ by

$$\ell(x_1, \ldots, x_d) = x_1 \cdot c_1 + x_2 \cdot c_2 + \cdots + x_d \cdot c_d \pmod{n}.$$

Note that for every $v \in \mathbb{Z}_n$, the set $\ell^{-1}(v)$ is an infinite set of vertices of $\mathbb{Z}^d$. It can be easily seen that since the map ℓ is linear, $\ell^{-1}(v)$ is a sublattice of $\mathbb{Z}^d$. If $v, w \in \mathbb{Z}_n$, the lattices $\ell^{-1}(v)$ and $\ell^{-1}(w)$ are isomorphic.

This surjective map provides a useful geometric representation of circulant graphs which, in fact, can be easily generalized to Cayley graphs on Abelian groups. See Figure 4 for an example of a circulant graph and its lattice representation.

4.2 Alliances in circulant graphs

Using the geometric representation of a circulant graph $G = C_n(c_1, \ldots, c_d)$, we can give a first bound on its alliance number.

Proposition 4.2 *Let $G = C_n(c_1, \ldots, c_d)$, a circulant graph with d generators.*

1. *If $\delta_G = 2d$, i.e., $\frac{n}{2} \notin \{c_1, \ldots, c_d\}$, then the alliance number of G satisfies $d + 1 \leq a(G) \leq 2^d$.*

2. *If $\delta_G = 2d - 1$, i.e., $\frac{n}{2} \in \{c_1, \ldots, c_d\}$, then the alliance number of G satisfies $d \leq a(G) \leq 2^{d-1}$.*

Proof. Assume that $\frac{n}{2} \notin \{c_1, \ldots, c_d\}$. Since $\delta_G = 2d$, we know that $a(G) \geq d + 1$.

To show that $a(G) \leq 2^d$, we show that the set $S = \ell(\{0,1\}^d)$ is a defensive alliance of G of cardinality $|S| \leq 2^d$. Indeed, every vertex in S has at least d neighbors in S. Moreover, S contains 2^d d-tuples. However, since the labeling ℓ is not injective, some of the d-tuples might be assigned by ℓ to the same vertex. Thus $|S| \leq 2^d$.

In the second case, that is if $\frac{n}{2} \notin \{c_1, \ldots, c_d\}$, we can assume, w.l.o.g., that $\frac{n}{2} = c_d$. Then the set $S = \ell(\{0,1\}^{d-1} \times \{0\})$, i.e., the d-tuples with coordinates in $\{0, 1\}$ with 0 in the last one, is a defensive alliance of G of cardinality $|S| \leq 2^{d-1}$. As in the previous case, every vertex in S has at least $d - 1$ neighbors in S, and S contains 2^{d-1} d-tuples. Again, some of the d-tuples might be assigned by ℓ to the same vertex. Thus $|S| \leq 2^{d-1}$. ■

For $d = 3$ and $\frac{n}{2} \notin \{a, b, c\}$, we have that $G = C_n(a, b, c)$ is 6-regular and its alliance number satisfies

$$4 \leq a(G) \leq 8.$$

We have characterized all the alliances of cardinality at most 8 of 6-regular graphs. In what follows, we study the alliances in circulant graphs, up to 3 generators.

4.2.1 Alliances in circulant graphs of small degrees

For the sake of completeness we give a short review of circulant graphs with 1 and 2 generators.

- $G = C_n(c_1) \cong C_n$ is the n-cycle. In this case, the alliance numbers are $a(G) = \hat{a}(G) = 2$. The critical alliances are the edges.

- $G = C_n(c_1, c_2)$ is a 3-regular graph if and only if $n = 2m$ and $c_2 = m$. In this case, $G \cong C_{2m}(1, m)$ or $G \cong C_{2m}(2, m)$. The alliance numbers are $a(G) = 2$ and $\hat{a}(G) = g(G)$. G contains triangles (and thus $\hat{a}(G) = 3$) if and only if $n = 4$, which implies $G \cong K_4$, and if $n = 6$ and $G \cong C_6(2, 3)$. In the remaining cases, $\hat{a}(G) = g(G) = 4$.

  The critical alliances are the edges, and the strong critical alliances are the induced cycles.

- $G = C_n(c_1, c_2)$, with $c_2 \neq \frac{n}{2}$, is a 4-regular graph. Therefore $a(G) = \hat{a}(G) = g(G)$. In this case, G contains triangles if and only if $c_2 = 2c_1$, that is, $G \cong C_n(1, 2)$, or $n = 3m$ and $c_2 = \pm m$, that is, $G \cong C_{3m}(1, m)$ or $G \cong C_{3m}(3, m)$.

  If $G \cong C_n(1, 2)$, $G \cong C_{3m}(1, m)$, or $G \cong C_{3m}(3, m)$, then $a(G) = \hat{a}(G) = 3$. Otherwise $a(G) = \hat{a}(G) = 4$.

Circulant graphs of degree 5. Let $G = C_{2m}(a, b, m)$ be a circulant graph of degree 5. Note that since $\delta = 5$, $2m \geq 6$. But if $2m = 6$ we have that $G = K_6$ and then Equations (3) and (4) imply that $a(K_6) = 3$ and $\hat{a}(K_6) = 4$. Thus we can assume that $2m \geq 8$.

The known bounds give $a(G) = g(G) \leq 4$.

- G contains triangles (and thus $g(G) = a(G) = 3$) if and only if $a + b = m$, or $m = 2m'$ is even and $b = \pm m'$, or $m = 3m'$ and $b = \pm 2m'$. The connectedness condition, $\gcd(a, b, m, 2m) = 1$, implies that these three cases correspond to $G \cong C_{2m}(1, 2, m)$, $G \cong C_{4m'}(1, m', 2m')$ and $G \cong C_{6m'}(1, 2m', 3m')$ respectively.

- Otherwise, $g(G) = a(G) = 4$.

4.2.2 Alliances in circulant graphs of degree 6

Let us concentrate now on circulant graphs with 3 generators, $C_n(a, b, c)$, with $\frac{n}{2} \notin \{a, b, c\}$. Note that since $\delta = 6$, $n \geq 7$. But if $n = 7$ then $C_n(a, b, c) = K_7$ and Equations (3) and (4) imply that $a(K_7) = \hat{a}(K_7) = 4$. Thus we can assume that $n \geq 8$.

Recall that $4 \le a(G) \le 8$ (see Proposition 4.2), and also that G contains triangles if and only if, up to a permutation of the generators,

$$a+b+c=0\,(\mathrm{mod}\ n),\ 2a+b=0\,(\mathrm{mod}\ n),\ \text{ or } 3c=0\,(\mathrm{mod}\ n).$$

In the previous section we found that $\mathcal{S}_{(4,6)} = \{K_4\}$, $\mathcal{S}_{(5,6)} = \{W_4\}$, $\mathcal{S}_{(6,6)} = \{C_3\Box K_2,\ K_{3,3}, (C_3\Box K_2)+e,\ K_{3,3}+e,\ \overline{C_4}+\overline{K_2},\ W_5\}$, and $\mathcal{S}_{(7,6)}$ contains exactly the 15 graphs in Figure 2. To apply these results to 6-regular circulant graphs, it should be noted that every graph in $\mathcal{S}_{(7,6)}$ contains triangles, while the only triangle-free graph in $\mathcal{S}_{(6,6)}$ is $K_{3,3}$.

In the remainder of the section, we give a classification of 6-regular circulant graphs according to their alliance number, showing which alliance is a minimal alliance in these graphs. In summary, we show that either $G = C_n(a,b,c)$ contains triangles and then $a(G)$ ranges from 4 to 7, or G is triangle-free. In this latter case, either G contains $K_{3,3}$ and $a(G) = 6$, or $a(G) = 8$ and the minimal alliance is the cube Q_3. Before giving the complete classification theorem we prove some technical lemmas.

In all of the following lemmas we use the vertex-symmetry of circulant graphs, and also that a circulant graph is isomorphic to any circulant graph obtained by a permutation of its generators.

Lemma 4.3 *If $G = C_n(a,b,c)$ is 6-regular then $a(G) = 4$ if and only if $G \cong C_n(1,2,3)$.*

Proof. Assume that $G = C_n(a,b,c)$ contains a subgraph isomorphic to K_4. Because of the vertex-symmetry of G, we can fix 0 to be any of the vertices of this induced K_4. Now the set of vertices that induce K_4 is, up to a permutation of generators, either $\{0,a,b,-a\}$ or $\{0,a,b,c\}$.

- If $\langle\{0,a,b,-a\}\rangle \cong K_4$, then all the integers $\pm 2a, \pm(b-a), \pm(b+a)$ are generators of G.

  Let us first study which of the generators equals $2a$. The case $2a = a$ is impossible. The case $2a = -b$ is, up to a permutation of the generators, the same as $2a = b$. The case $2a = -c$ is, up to a permutation of the generators, the same as $2a = c$. Thus $2a \in \{-a,b,c\}$.

  - $2a = -a$ gives $3a = 0$; that is $n = 3m$ and $a = m$. Now, $b-a$ can only be equal to $-b$ or c. If $b-a = -b$ then $n = 6m'$, $a = 2m'$, $b = m'$, which implies $a+b = 3m' = \pm c$. But then the connectedness condition is not fulfilled. If, otherwise, $b-a = c$, then $b+a = -b$, which implies $2b = -a = 2a$. In both cases we get a contradiction.
  - $2a = b$ gives $b-a = a$ and then $a+b = 3a = \pm c$, which implies $G \cong C_n(a,2a,3a) \cong C_n(1,2,3)$.

- $2a = c$ implies that $b - a = -b$, that is, $2b = a$, and thus $a + b = 3b$. This implies b, $2b$, $3b$, and $4b$ are all generators of G and this is a contradiction if $n \geq 7$.

- If $\langle\{0, a, b, c\}\rangle \cong K_4$, then all the integers $\pm(a-b), \pm(b-c), \pm(c-a)$ are generators of G.

  Let us first study which of the generators equals $c - a$. The cases $c-a = c$ and $c-a = -a$ are both impossible. Thus, $c-a \in \{a, \pm b, c\}$.

  - $c-a = a$ gives $c = 2a$. Now, $c-b$ can only be equal to $-a$ or $-c$. If $c - b = -a$ we get $3a = b$, which implies $G \cong C_n(a, 2a, 3a) \cong C_n(1, 2, 3)$. The case $c - b = -c$ implies $2c = 4a = b$ and then $b - a = 3a$. So a, $2a$, $3a$, and $4a$ are all generators of G and this is a contradiction if $n \geq 7$.
  - $c - a = b$ gives $a + b = c$ and $b - c = a$. Now, $a - b$ can only be equal to $-a$ or b. Both cases imply $G \cong C_n(1, 2, 3)$.
  - $c-a = -b$ implies that $a-b = c$ and $c-b$ can only be equal to b or $-c$. As in the previous case, both cases imply $G \cong C_n(1, 2, 3)$.
  - $c - a = -c$ gives $2c = a$. Now, $c - b$ can only be equal to $-a$ or b. If $c - b = -a$ then $3c = b$, which implies $G \cong (2c, 3c, c) \cong C_n(1, 2, 3)$. If, otherwise, $c - b = b$, we get $2b = c$ and thus $a = 4b$. This implies b, $2b$, $3b$, and $4b$ are all generators of G and this is a contradiction if $n \geq 7$.

  All possible cases give $G \cong C_n(1, 2, 3)$.

■

Lemma 4.4 *If $G = C_n(a, b, c)$ is 6-regular, then $a(G) = 5$ if and only if $G \cong C_{3m}(a, m - a, m)$ or $G \cong C_n(1, 2, 4)$.*

Proof. Assume that $G = C_n(a, b, c)$ contains an induced subgraph isomorphic to W_4, and G does not contain any subgraph isomorphic to K_4.

By symmetry, we can assume w.l.o.g. that either $\langle\{0, \pm a, \pm b\}\rangle \cong W_4$ or $\langle\{0, \pm a, b, c\}\rangle \cong W_4$. That is, either $\langle\{\pm a, \pm b\}\rangle \cong C_4$ or $\langle\{\pm a, b, c\}\rangle \cong C_4$.

For each case there are two possibilities:

- $\langle\{0, \pm a, \pm b\}\rangle \cong W_4$ and $(a, b, -a, -b, a)$ is an induced 4-cycle.

  This implies that $\pm(b - a)$ and $\pm(b + a)$ are generators of G. It is easy to see that in this case, $b - a \in \{a, -b, c\}$ and in all three of the cases, $G \cong C_n(1, 2, 3)$, which satisfies $a(G) = 4$.

- $\langle\{0, \pm a, \pm b\}\rangle \cong W_4$ and $(a, b, -b, -a, a)$ is an induced 4-cycle.

  This implies that $\pm(b-a)$, $\pm 2a$, and $\pm 2b$ are generators of G. It is easy to see that in this case, $2a \in \{-a, \pm b, c\}$. By easy computations, we obtain:

  - $2a = -a$ implies $n = 9m$ and $a = 3m$, $b = m$, $c = 2m$, which gives that either $G \cong C_9(1,2,3)$, which satisfies $a(G) = 4$, or G is disconnected.
  - $2a = b$ implies $G \cong C_n(1,2,4)$.
  - $2a = -b$ implies $G \cong C_n(1,2,3)$, which satisfies $a(G) = 4$.
  - $2a = c$ is only possible if $b - a = -b$, and this implies $G \cong C_n(1,2,4)$.

  Summarizing, $G \cong C_n(1,2,4)$.

- $\langle\{0, \pm a, b, c\}\rangle \cong W_4$ and $(a, b, -a, c, a)$ is an induced 4-cycle.

  This implies that $\pm(b-a)$, $\pm(b+a)$, $\pm(c-a)$, and $\pm(c+a)$ are generators of G. It is easy to see that in this case, $b - a \in \{a, -b, \pm c\}$. Reasoning and computing as in the previous case we obtain:

  (1) $b - a = a$ implies $2a = b$ and $b + a = 3a$;
  (2) $b - a = -b$ implies $2b = a$ and $b + a = 3b$;
  (3) $b - a = c$ gives that $b + a$ can only be equal to $-b$ and thus $3b = c$; and
  (4) $b - a = -c$ gives that $b + a$ can only be equal to $-b$ and thus $3b = -c$.

  Any of these four cases implies that $G \cong C_n(1,2,3)$, which satisfies $a(G) = 4$.

- $\langle\{0, \pm a, b, c\}\rangle \cong W_4$ and $(a, b, c, -a, a)$ is an induced 4-cycle.

  This implies that $\pm(b-a)$, $\pm 2a$, and $\pm(c+a)$ are generators of G. It is easy to see that in this case, $2a \in \{-a, \pm b, \pm c\}$, and we obtain:

  (1) $2a = -a$ and $b - c = \pm a$ implies $G \cong C_{3m}(a, m-a, m)$;
  (2) $2a = -a$ and $b - c = -b$ implies $G \cong C_n(1,2,4)$;
  (3) $2a = b$ and $b - c = -b$ implies $G \cong C_n(1,2,4)$; and
  (4) $2a = -c$ and $b - c = c$ implies $G \cong C_n(1,2,4)$.

  All the remaining cases either give impossible values for the generators or imply that $G \cong C_n(1,2,3)$, which satisfies $a(G) = 4$.

This completes the proof. ■

The following two lemmas characterize the critical defensive alliances for a circulant graph G of degree 6 not containing either K_4 or W_4. These graphs satisfy $6 \leq a(G) \leq 8$. Lemma 4.5 deals with graphs containing triangles, and Lemma 4.6 deals with triangle-free graphs.

Lemma 4.5 *If $G = C_n(a, b, c)$ contains a triangle and $a(G) > 5$, then one of the following conditions holds.*

1. *$3c = 0 \,(mod\, n)$. Then $n = 3m$ and $G \cong C_{3m}(a', b', m)$, with $a' + b' \neq m$, but $G \not\cong C_n(1, 2, 4)$. In this case, $C_3 \Box K_2$ is an induced subgraph of G. Thus $a(G) = 6$.*

2. *$2a + b = 0 \,(mod\, n)$. Then $G = C_n(a, -2a, c)$, but $G \not\cong C_n(1, 2, 3)$ and $G \not\cong C_n(1, 2, 4)$. In this case, $C_3 \Box K_2$ is an induced subgraph of G. Thus $a(G) = 6$.*

3. *$a + b + c = 0 \,(mod\, n)$. Then $G \not\cong C_n(1, 2, 3)$ and there are two possibilities:*

   - *$G \cong C_{2m}(a, m - a, m - 2a)$ and it has an induced $\overline{C_4} + \overline{K_2}$. Thus $a(G) = 6$.*

   - *Otherwise, none of the graphs in $\mathcal{S}_{(6,6)}$ is an induced subgraph of G, but G contains W_6. Thus $a(G) = 7$.*

Proof. The three cases in the statement of this lemma correspond to the three cases for which G contains triangles.

Case 1: $3c = 0$. It is clear that $n = 3m$ and $G \cong C_{3m}(a', b', m)$. Moreover if $a' + b' = m$, we have $G \cong C_{3m}(a', m - a', m)$. Lemma 4.4 says that both $C_{3m}(a', m - a', m)$ and $C_n(1, 2, 4)$ contain W_4. On the other hand, $\langle\{0, m, -m, a', m + a', -m + a'\}\rangle \cong C_3 \Box K_2$, if $a' + b' \neq m$.

Case 2: $2a + b = 0$. It is clear that $G = C_n(a, -2a, c)$. Since $a(G) > 5$, we know that $G \not\cong C_n(1, 2, 3)$ and $G \not\cong C_n(1, 2, 4)$. One can also see that if $n = 3m$, $C_{3m}(a, 2a, c)$ cannot be isomorphic to $C_{3m}(a', m - a', m)$. Moreover $\langle\{0, a, 2a, c, a + c, 2a + c\}\rangle \cong C_3 \Box K_2$.

Case 3: $a + b + c = 0$. The condition $a(G) > 5$ implies that $G \not\cong C_n(1, 2, 3)$. We have that if $S = \{0, \pm a, \pm b, \pm c\}$, then $\langle S \rangle \cong W_6$ and thus $6 \leq a(G) \leq 7$. The adjacencies in $\langle S \rangle$ are given by $d_S(0) = 6$, and $(a, -b, c, -a, b, -c, a)$ is a 6-cycle.

Now $a(G) = 6$ if one of the graphs in

$$\mathcal{S}_{(6,6)} = \{C_3 \Box K_2,\ K_{3,3},\ (C_3 \Box K_2) + e,\ K_{3,3} + e,\ \overline{C_4} + \overline{K_2},\ W_5\}$$

is an induced subgraph of G (see Proposition 3.7). First, we prove that none of the graphs W_5, $K_{3,3}$, $K_{3,3}+e$, $C_3\Box K_2$ or $(C_3\Box K_2)+e$ can be an induced subgraph of G.

- If there is an induced subgraph isomorphic to W_5, by symmetry we can assume that its vertex of degree 5 is 0. Now, up to a permutation of the generators, we can assume that $W_5 \cong \langle\{0,a,-b,c,-a,b\}\rangle$, which clearly implies that a and b are adjacent. But then $K_4 \cong \langle\{0,a,b,-c\}\rangle$ is an induced subgraph of G, a contradiction.

- If either $K_{3,3}$ or $K_{3,3}+e$ is isomorphic to an induced subgraph of G, we can assume w.l.o.g. that this subgraph is induced by a set $\{0,a,b,c,x,y\}$, where x and y are at distance 2 from 0 and are both adjacent to a, b and c. If $x \not\sim y$, then the induced subgraph is $K_{3,3}$, and if $x \sim y$, then the induced subgraph is $K_{3,3}+e$.

  Easy computations give that up to symmetries and permutations of the generators, either $x=a-b \not\sim y=a-c$ or $x=a-b \sim y=2a$. There are three possible values for $a-b$:

  - $a-b=c-a$ implies $3a=0$, which corresponds to case 1,
  - $a-b=c-b$ implies $a=c$, which is impossible, and
  - $a-b=2c$ implies $2a=c$ and $3a=-b$, which corresponds to $G \cong C_n(1,2,3)$.

- If $C_3\Box K_2$ is isomorphic to an induced subgraph of G, we can assume w.l.o.g. that the disjoint triangles in this graph are $T_1=\{0,a,-c\}$ and $T_2=\{2a,x,y\}$, with either $x\sim 0$ and $y\sim -c$, or $x\sim -c$ and $y\sim 0$. One can see that, up to symmetries, $x=3a$, and $y=2a-c$ or $y=2a-b$. A careful analysis shows that neither case is possible.

- If $(C_3\Box K_2)+e$ is isomorphic to an induced subgraph of G, we can assume w.l.o.g. that this subgraph is induced by a set $\{0,a,-b,c,-a,y\}$, with y a common neighbor of a, c and $-a$. One can see that this implies $y\in\{2c,c-a\}\cap\{2a,a-c\}\cap\{c-a,-2a,b-a\}$, which is impossible.

Let us now assume that G contains an induced subgraph isomorphic to $\overline{C_4}+\overline{K_2}$. Let $\{0,x,y,z,t,w\}$ be its set of vertices and $x\sim y$, $z\sim t$ the neighbors of 0 in this induced subgraph. We can assume w.l.o.g. that $\{x,y,z,t\}=\{a,\pm b,-c\}$ or $\{x,y,z,t\}=\{\pm a,\pm b\}$. In the first case, we get $w=a-b=b-c$ and $3b=0$, which corresponds to Case 1. Otherwise if $w=a-b=b-a$, then $2(b-a)=0$, which implies $n=2m$, $b=m+a$ and $c=-m-2a$. Thus $G\cong C_{2m}(a,m-a,m-2a)$.

This completes the proof. ∎

Lemma 4.6 *Let $G = C_n(a, b, c)$ be 6-regular and triangle-free. $K_{3,3}$ is an induced subgraph of G if and only if $G \cong C_n(1, 3, 5)$. In this case, $a(G) = 6$. Otherwise the minimal alliance of G is the cube Q_3 and thus $a(G) = 8$.*

Proof. First, recall that the only triangle-free graph in $\mathcal{S}_{(4,6)} \cup \mathcal{S}_{(5,6)} \cup \mathcal{S}_{(6,6)} \cup \mathcal{S}_{(7,6)}$ is $K_{3,3}$. Moreover if $G = C_n(a, b, c)$ is 6-regular, by Proposition 4.2, the set of vertices $\{0, a, b, c, a + b, a + c, b + c, a + b + c\}$ induce an alliance of cardinality at most 8 that, if G is triangle-free, is isomorphic to the cube Q_3. To prove the lemma, we only need to show that G, triangle-free, contains an induced $K_{3,3}$ if and only if $G \cong C_n(1, 3, 5)$. Notice that in $C_n(1, 3, 5)$, the set $\{0, 1, 2, 3, 4, 5\}$ induces a subgraph isomorphic to $K_{3,3}$.

Let us assume that G has an induced $K_{3,3}$. We can assume w.l.o.g. that $K_{3,3}$ is induced by $\{0, a, b, c, x, y\}$, with x and y common neighbors of a, b and c, at distance 2 from 0, or $\{0, a, b, -a, x, y\}$, with x and y common neighbors of a, b and $-a$, at distance 2 from 0. But easy computations show that only the first possibility can hold, provided x and y are conveniently chosen. Again by symmetry, $x \in \{2a, a + b, a - b\}$. For every possible value of x, we have to see which values are possible for y and how they determine the graph G.

- If $x = 2a$, since b and c are adjacent to x, we have that $2a - b$ and $2a - c$ are both in $\{\pm a, \pm b, \pm c\}$. In this case, $2a - b$ can only be equal to $-a$, and $\pm c$. We can see that $2a - b = -a$ implies $3a = b$ and $2a - b = -c$, which gives $G \cong C_n(1, 3, 5)$. If $2a - b = -c$, then $2a - c$ can only be equal to a, which implies $b = -3a$, $c = -5a$, and again, $G \cong C_n(1, 3, 5)$. On the other hand, for $2a - b = c$, and thus $2a - c = b$, we have to distinguish cases according to the values of y. It can be seen that $y \in \{a \pm b, a \pm c\}$. Some of these cases are impossible; the possible ones again give $G \cong C_n(1, 3, 5)$.

- If $x = a + b$, then we need only to consider the cases $y \in \{a - b, a \pm c\}$. One can see that the only valid possibilities give $G \cong C_n(1, 3, 5)$.

- Finally, if $x = a - b$, then we only need to consider the case $y = a - c$. This case can be reduced by symmetry to the previous cases.

Thus, we have shown that the only triangle-free circulant graph of degree 6 with an induced $K_{3,3}$ is, up to isomorphism, $G \cong C_n(1, 3, 5)$. This completes the proof. ■

Theorem 4.7 *The alliance number of $G = C_n(a, b, c)$, the circulant graph of degree 6 and order $n \geq 8$ is*

- $a(G) = 4 \Leftrightarrow G \cong C_n(1, 2, 3)$ *or* $G \cong C_n(1, 2, 4)$.

- $a(G) = 5 \Leftrightarrow G \cong C_{3m}(a, m-a, m)$.
- $a(G) = 6 \Leftrightarrow G$ *contains triangles and it is isomorphic to one of the graphs* $C_{3m}(a, b, m)$, *with* $a + b \neq m$, $C_{2m}(m-2c, m+c, c)$, $C_{4m}(m, b, -m-b)$, $C_n(1, 3, 4)$, $C_n(a, -2a, c)$, *but* G *is not isomorphic to* $C_n(1, 2, 3)$ *nor* $C_n(1, 2, 4)$; *or* G *is triangle-free and is isomorphic to* $C_n(1, 3, 5)$ *or* $C_n(1, 5, 7)$.
- $a(G) = 7 \Leftrightarrow G \cong C_n(a, b, -(a+b))$, *but* G *is not isomorphic to any of the graphs* $C_{2m}(m-2c, m+c, c)$, $C_{4m}(m, b, -m-b)$, $C_n(1, 3, 4)$.
- $a(G) = 8 \Leftrightarrow G$ *is triangle-free and is not isomorphic to* $C_n(1, 3, 5)$ *nor to* $C_n(1, 5, 7)$.

Proof. Follows straightforwardly from Lemmas 4.3, 4.4, 4.5, and 4.6 above. ■

5 Conclusions and open problems

In a previous paper ([3]), we studied defensive alliances of cardinality $k \leq 8$ in regular graphs of degree 6. In this paper, using these results, we have restricted the problem to circulant graphs.

Open problems. We leave some open problems about defensive alliances in circulant graphs.

- We have shown that if $G = C_n(a, b, c)$ is 6-regular, then $4 \leq a(G) \leq 8$. Moreover in the family of circulant graphs of degree 6, there are graphs with alliance number ranging from 4 to 8. Is this the same for circulant graphs of higher degree?
- On the other hand, assuming that the bounds in Proposition 4.2 are tight, an interesting problem is: Find a lower bound for n, the order of a circulant graph $G = C_n(c_1, \ldots, c_d)$, such that $a(G) = 2^d$ (or $a(G) = 2^{d-1}$, if $\frac{n}{2} \in \{c_1, \ldots, c_d\}$).

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