Hamiltonian Cycles in Directed Toeplitz Graphs-Part 2

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Abstract. A directed Toeplitz graph is a digraph with a Toeplitz adjacency matrix. In this paper we contribute to [6]. The paper [6] investigates the hamiltonicity of the directed Toeplitz graphs $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$ with $s_2 = 2$ and in particular those with $s_3 = 3$. In this paper we extend this investigation to $s_2 = 3$ with $s_1 = t_1 = 1$.

Keywords: *Toeplitz graph; Hamiltonian graph.*

1 Introduction

We use [6] for terminology and notations not defined here, and consider finite directed graphs without multiple edges and without loops, because multiple edges and loops play no role in hamiltonicity investigations. Since all graphs will be directed, we shall omit mentioning it.

Properties of Toeplitz graphs, such as bipartiteness, planarity and colourability, have been studied in [2], [3], [4]. Hamiltonian properties of undirected Toeplitz graphs have been investigated in [1] and [5]. The paper [6] investigates the hamiltonicity of the directed Toeplitz graphs with $s_2 = 2$ and in particular those with $s_3 = 3$. In this paper we extend this investigation to the cases $(k = l = 1)$ and $(s_1 = t_1 = 1$ *and* $s_2 = 3)$.

Connectivity and hamiltonicity results obtained in the undirected case for a Toeplitz graph have a direct impact on the directed case. So connectedness of $T_n\langle s_1,\ldots,s_k;t_1,\ldots,t_l\rangle$ means precisely connectedness of $T_n\langle s_1,\ldots,s_k,t_1,\ldots,t_l\rangle$ (the first one is directed while the later is undirected). Hamiltonicity of $T_n\langle t_1,t_2,\ldots,t_i\rangle$ means hamiltonicity of $T_n\langle t_1,\ldots,t_i;t_1,\ldots,t_i\rangle$.

2 Toeplitz graphs with $k = l = 1$

It is known that, if $gcd(s_1, s_2) = 1$ and n is a multiple of $s_1 + s_2$ then $T_n\langle s_1, s_2; s_1, s_2\rangle$ is hamiltonian (Theorem 10 in [1]). For $k = l = 1$ we obtain a characterization of cycles among Toeplitz graphs.

Theorem 1. *$T_n\langle s; t\rangle$ is a cycle if and only if $gcd(s, t) = 1$ and $s + t = n$.*

Proof. Firstly, suppose $gcd(s, t) = 1$ and $s + t = n$.

If $s = t = 1$, then the statement is true. Otherwise, assume without loss of generality that $s < t$.

Let $s + t = n$. From [1] we know that $T_n\langle s; t\rangle$ is connected. We show that each vertex has indegree and outdegree one.

Indeed, let $v \in V(T_n\langle s; t\rangle)$.

(a) If $v \leq s$, then its incident edges are $(v + t, v)$, $(v, v + s)$.

(b) If $s + 1 \leq v \leq t$, then its incident edges are $(v - s, v)$, $(v, v + s)$.

(c) If $v \geq t + 1$, then its incident edges are $(v - s, v)$, $(v, v - t)$.

Thus, $T_n\langle s; t\rangle$ is a cycle. (see Figure 1 for the case $s = 6$ and $t = 11$).

Conversely suppose $T_n\langle s; t\rangle$ is a cycle, so is connected which shows that $gcd(s, t) = 1$ (see [1]). The number of edges in $T_n\langle s; t\rangle$ is $(n - s) + (n - t) = 2n - (s + t)$ (see [1]), which must be n since $T_n\langle s; t\rangle$ is a cycle. This implies $n = s + t$.□

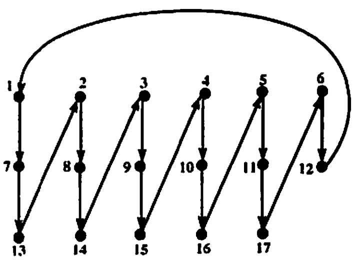

Fig. 1. The Toeplitz graph $T_{17}\langle 6; 11\rangle$.

3 Toeplitz graphs with $s_1 = t_1 = 1$ and $s_2 = 3$

In this section we will present a few results on Toeplitz graphs with $s_1 = t_1 = 1$ and $s_2 = 3$. They will sometimes depend upon the parity of n.

Theorem 2. *$T_n\langle 1,3;1,2\rangle$ is hamiltonian for all n.*

Proof.
Case 1. $n \equiv 1\,(mod\,3)$.
From Theorem 1 in [6], $T_n\langle 1,3;1,2\rangle$ is hamiltonian (with hamiltonian cycle containing the edge $(n-1,\ n)$).

Case 2. $n \equiv 0,\ 2\,(mod\,3)$.
We take first representatives from each residue class. For $n \in \{6,\ 5\}$, $T_n\langle$ 1,3;1,2$\rangle$ has a hamiltonian cycle containing the edge $(n-1,\ n)$.

Indeed, $T_6\langle$ 1,3;1,2$\rangle$ has a hamiltonian cycle (1, 2, 5, 6, 4, 3, 1) and $T_5\langle$ 1,3;1,2$\rangle$ has a hamiltonian cycle (1, 4, 5, 3, 2, 1) (see Figures 2-3).

Suppose $T_n\langle 1,3;1,2\rangle$ has a hamiltonian cycle containing the edge $(n-1,\ n)$. We prove that $T_{n+3}\langle 1,3;1,2\rangle$ has the same property. Indeed, since $(n-1,\ n)$ is an edge in a hamiltonian cycle of $T_n\langle 1,3;1,2\rangle$, we transform this cycle to a hamiltonian cycle in $T_{n+3}\langle 1,3;1,2\rangle$, by replacing the edge $(n-1,\ n)$ with the path $(n-1,\ \underline{n+2,\ n+3},\ n+1,\ n)$. Hence $T_n\langle 1,3;1,2\rangle$ is hamiltonian for all n. □

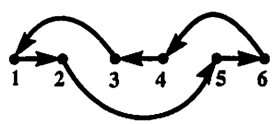

Fig. 2. $T_6\langle 1,3;1,2\rangle$.

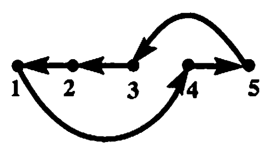

Fig. 3. $T_5\langle 1,3;1,2\rangle$.

Theorem 3. $T_n\langle 1,3;1,4\rangle$ *is hamiltonian for all* n.

Proof.
Claim 1. For $n \in \{5, 7, 12\}$, $T_n\langle 1,3;1,4\rangle$ has a hamiltonian cycle containing the edge $(n-2, n-1)$.

Indeed, $T_5\langle 1,3;1,4\rangle$ has the hamiltonian cycle $T_5\langle 1;4\rangle$, $T_7\langle 1,3;1,4\rangle$ has a hamiltonian cycle (1, 4, 5, 6, 7, 3, 2, 1), and $T_{12}\langle 1,3;1,4\rangle$ has a hamiltonian cycle (1, 4, 5, 6, 9, 10, 11, 12, 8, 7, 3, 2, 1) (see Figures 4-5).

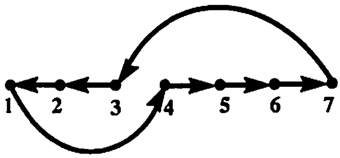

Fig. 4. $T_7\langle 1,3;1,4\rangle$.

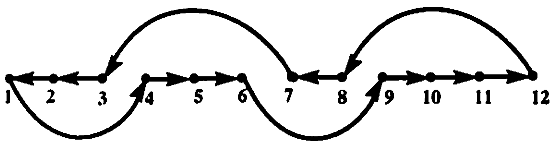

Fig. 5. $T_{12}\langle 1,3;1,4\rangle$.

Claim 2. For $n \in \{6, 9\}$, $T_n\langle 1,3;1,4\rangle$ is hamiltonian.

Indeed, $T_6\langle 1,3;1,4\rangle$ has a hamiltonian cycle (1, 4, 3, 6, 2, 5, 1) (see Figure 6), and $T_9\langle 1,3;1,4\rangle$ has a hamiltonian cycle (1, 4, 5, 6, 9, 8, 7, 3, 2, 1) (see Figure 7).

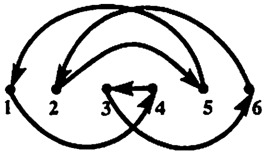

Fig. 6. $T_6\langle 1,3;1,4\rangle$.

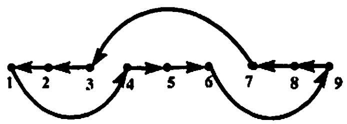

Fig. 7. $T_9\langle 1,3;1,4\rangle$.

Suppose $T_n\langle 1,3;1,4\rangle$; $n \notin \{6,\ 9\}$, has a hamiltonian cycle containing the edge $(n-2,\ n-1)$. We prove that $T_{n+3}\langle 1,3;1,4\rangle$ has the same property. Indeed, since $(n-2,\ n-1)$ is an edge in a hamiltonian cycle of $T_n\langle 1,3;1,4\rangle$, we transform this cycle to a hamiltonian cycle in $T_{n+3}\langle 1,3;1,4\rangle$, by replacing the edge $(n-2,\ n-1)$ with the path $(n-2,\ n+1,\ n+2,\ n+3,\ n-1)$.

By Claim 1, $T_n\langle 1,3;1,4\rangle$ enjoys the above property for $n \in \{5,\ 7,\ 12\}$. It follows that the property holds for $n = 5,\ 7,\ 8$ and all $n \geq 10$. This together with Claim 2 proves the theorem. □

Theorem 4. *$T_n\langle 1,3;1,6\rangle$ is hamiltonian for all n.*

Proof.

Claim 1. For $n \in \{7,\ 8,\ 9,\ 10,\ 16\}$, $T_n\langle 1,3;1,6\rangle$ has a hamiltonian cycle containing the edge $(n-2,\ n-1)$.

Indeed $T_7\langle 1,3;1,6\rangle$ has the hamiltonian cycle (1, 2, 3, 4, 5, 6, 7, 1), $T_8\langle$ 1, 3;1,6$\rangle$ has a hamiltonian cycle (1, 4, 5, 8, 2, 3, 6, 7, 1), $T_9\langle$ 1,3;1,6$\rangle$ has a hamiltonian cycle (1, 4, 5, 6, 7, 8, 9, 3, 2, 1), $T_{10}\langle$ 1,3;1,6$\rangle$ has a hamiltonian cycle (1, 2, 3, 6, 5, 8, 9, 10, 4, 7, 1), and $T_{16}\langle$ 1, 3; 1, 6$\rangle$ has a hamiltonian cycle (1, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, 10, 9, 3, 2, 1) (see Figures 8-12, respectively).

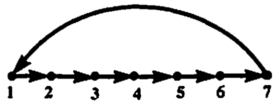

Fig. 8. $T_7\langle 1,3;1,6\rangle$.

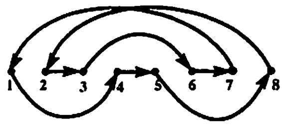

Fig. 9. $T_8\langle 1,3;1,6\rangle$.

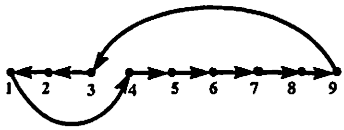

Fig. 10. $T_9\langle 1,3;1,6\rangle$.

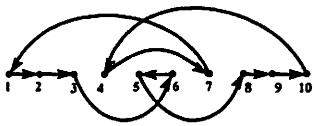

Fig. 11. $T_{10}\langle 1,3;1,6\rangle$.

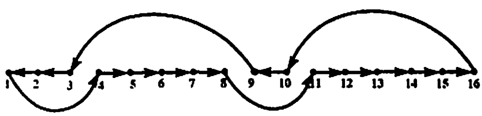

Fig. 12. $T_{16}\langle 1,3;1,6\rangle$.

Claim 2. $T_{11}\langle 1,3;1,6\rangle$ is hamiltonian.

Indeed, $T_{11}\langle 1,3;1,6\rangle$ has a hamiltonian cycle (1, 4, 5, 6, 7, 8, 11, 10, 9, 3, 2, 1) (see Figure 13).

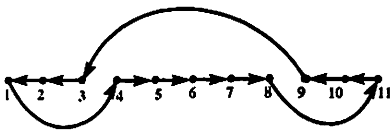

Fig. 13. $T_{11}\langle 1,3;1,6\rangle$.

Suppose $T_n\langle 1,3;1,6\rangle$; $n \neq 11$, has a hamiltonian cycle containing the edge $(n-2, n-1)$. We prove that $T_{n+5}\langle 1,3;1,6\rangle$ has the same property. Indeed, since $(n-2,\ n-1)$ is an edge in a hamiltonian cycle of $T_n\langle 1,3;1,6\rangle$, we transform this cycle to a hamiltonian cycle in $T_{n+5}\langle 1,3;1,6\rangle$, by replacing the edge $(n-2,\ n-1)$ with the path $(n-2,\ n+1,\ n+2,\ \underline{n+3,\ n+4},\ n+5,\ n-1)$.

By Claim 1, $T_n\langle 1,3;1,6\rangle$ enjoys the above property for $n \in \{7, 8, 9, 10, 16\}$. It follows that the property holds for $n = 7, 8, 9, 10$ and all $n \geq 12$. This together with Claim 2 proves the theorem. □

Theorem 5. *$T_n\langle 1,3;1,t_2\rangle$, where $t_2(\geq 8)$ is even, is hamiltonian if $n \equiv 0,2,4,6,5,7,9,\ldots,t_2-3$ (mod (t_2-1)).*

Proof. Put $t_2 = 2m$, for some integer $m \geq 4$.

Let

$$n \equiv n_0\,(mod\,(2m-1)),$$

where

$$n_0 = 0,\ 2,\ 4,\ 6,\ 5,\ 7,\ 9,\ \ldots .2m-3.$$

Since $n > 2m$, we take representatives of each class between $2m+1$ and $4m-2$.

Case 1. $n \equiv 0$ (*mod* $(2m-1)$).

For $n = 4m-2$, a hamiltonian cycle in $T_n\langle 1,3;1,2m\rangle$ is $(1,\ 2,\ 3,\ldots,n-2m-1,\ n-2m+2,\ n-2m+1,\ n-2m+4,\ n-2m+5,\ \ldots,\ \underline{n-2,\ n-1},\ n,\ n-2m,\ n-2m+3,\ 1)$.

Case 2. $n \equiv 2\ (mod\ (2m-1))$.

For $n = 2m+1$, a hamiltonian cycle in $T_n\langle 1,3;1,2m\rangle$ is $(1,\ 2,\ 3, \ldots, n-2,\ n-1,\ n,\ 1)$.

Case 3. $n \equiv 4\ (mod\ (2m-1))$.

For $n = 2m+3$, a hamiltonian cycle in $T_n\langle 1,3;1,2m\rangle$ is $(1,\ 4,\ 5,\ 6, \ldots, n-2,\ n-1,\ n,\ 3,\ 2,\ 1)$.

Case 4. $n \equiv 6\ (mod\ (2m-1))$.

For $n = 2m+5$, a hamiltonian cycle in $T_n\langle 1,3;1,2m\rangle$ is $(1,\ 4,\ 5,\ 6, \ldots, n-3,\ n,\ n-1,\ n-2,\ 3,\ 2,\ 1)$.

Case 5. $n \equiv (2m-5)\ (mod\ (2m-1))$.

For $n = 4m-6$, a hamiltonian cycle in $T_n\langle 1,3;1,2m\rangle$ is $(1,\ 2,\ 3, \ldots, n-2m-1,\ n-2m+2,\ n-2m+1,\ n-2m+4,\ n-2m+5,\ n-2m+8,\ ,n-2m+9, \ldots, n-2,\ n-1,\ n,\ n-2m,\ n-2m+3,\ n-2m+6,\ n-2m+7,\ 1)$

Case 6. $n \equiv (2m-3)\ (mod\ (2m-1))$.

For $n = 4m-4$, a hamiltonian cycle in $T_n\langle 1,3;1,2m\rangle$ is $(1,\ 2,\ 3, \ldots, n-2m-1,\ n-2m+2,\ n-2m+1,\ n-2m+4,\ n-2m+7,\ n-2m+8, \ldots, n-2,\ n-1,\ n,\ n-2m,\ n-2m+3,\ n-2m+6,\ n-2m+5,\ 1)$. (for Cases 1-6, see Figures 14-19, respectively).

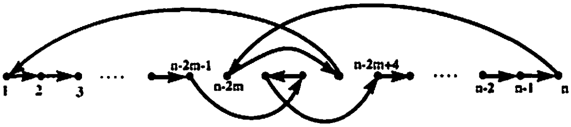

Fig. 14. $T_n\langle 1,3;1,2m\rangle$; $n = 4m-2$.

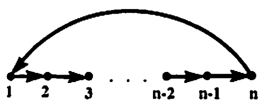

Fig. 15. $T_n\langle 1,3;1,2m\rangle$; $n = 2m+1$.

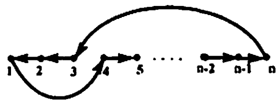

Fig. 16. $T_n\langle 1,3;1,2m\rangle$; $n = 2m + 3$.

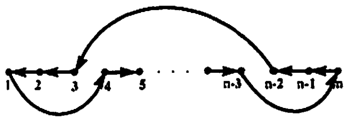

Fig. 17. $T_n\langle 1,3;1,2m\rangle$; $n = 2m + 5$.

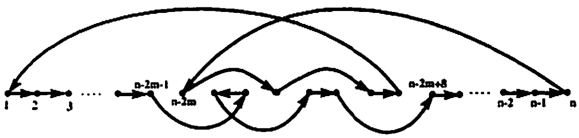

Fig. 18. $T_n\langle 1,3;1,2m\rangle$; $n = 4m - 6$.

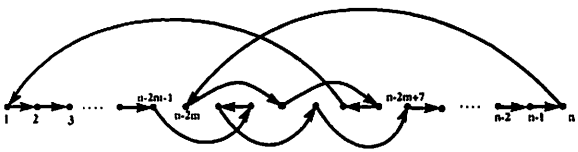

Fig. 19. $T_n\langle 1,3;1,2m\rangle$; $n = 4m - 4$.

Case 7. $n \equiv s \ (mod\ (2m-1))$, where $s = 5, 7, 9, \dots, 2m-7$.

We have three subcases.

(*i*) If $4m - n \equiv 1 \,(mod\, 3)$, then a hamiltonian cycle in $T_n\langle 1,3;1,2m\rangle$ is $(1,\ 2,\ 3, \dots, n-2m-1,\ n-2m+2,\ n-2m+1,\ n-2m+4,\ n-2m+5, \dots, n-2m+3p-1,\ n-2m+3p-2,\ n-2m+3p+1,\ n-2m+3p+2, \dots, 2m,\ 2m+3,\ 2m+4, \dots, \underline{n-2,\ n-1},\ n,\ n-2m,\ n-2m+3,\ n-2m+6,\ \dots, 2m+2,\ 2m+1,\ 1)$, where p is a non-negative odd integer.

(*ii*) If $4m - n \equiv 0 \,(mod\, 3)$, then a hamiltonian cycle in $T_n\langle 1,3;1,2m\rangle$ is $(1,\ 2,\ 3, \dots, n-2m-1,\ n-2m+2,\ n-2m+1,\ n-2m+4,\ n-2m+5, \dots, n-2m+3p-1,\ n-2m+3p-2,\ n-2m+3p+1,\ n-2m+3p+2, \dots, 2m+2,\ 2m+3,\ 2m+4, \dots, \underline{n-2,\ n-1},\ n,\ n-2m,\ n-2m+3,\ n-2m+6,\ \dots, 2m,\ 2m+1,\ 1)$

(*iii*) If $4m - n \equiv 2 \,(mod\, 3)$, then a hamiltonian cycle in $T_n\langle 1,3;1,2m\rangle$ is $(1,\ 2,\ 3, \dots, n-2m-1,\ n-2m+2,\ n-2m+1,\ n-2m+4,\ n-2m+5, \dots, n-2m+3p-1,\ n-2m+3p-2,\ n-2m+3p+1,\ n-2m+3p+2, \dots, 2m+2,\ 2m+3, \dots, \underline{n-2,\ n-1},\ n,\ n-2m,\ n-2m+3,\ n-2m+6,\ \dots, 2m-2,\ 2m+1,\ 1)$ (for subcases (*i*)-(*iii*) see Figures 20-22, respectively).

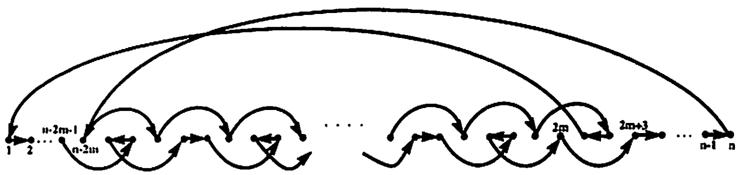

Fig. 20.

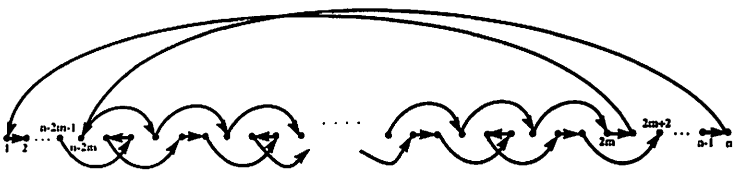

Fig. 21.

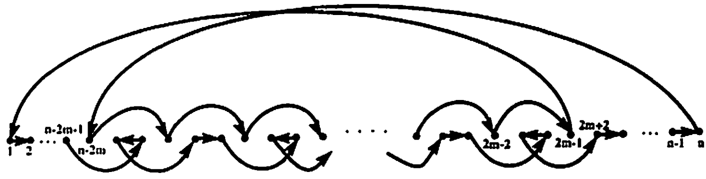

Fig. 22.

Note that for $n \neq 2m+5$, $(n-2,\ n-1)$ is an edge in each of the above hamiltonian cycles of $T_n\langle 1,3;1,2m\rangle$.

For $n = 2m+5$, since $(n,\ n-1)$ is an edge in the shown hamiltonian cycle of $T_n\langle 1,3;1,2m\rangle$, we transform this hamiltonian cycle to a hamiltonian cycle in $T_{n+(2m-1)}\langle 1,3;1,2m\rangle$, by replacing the edge $(n,\ n-1)$ with the path $(n,\ n+1,\ n+2,\dots,\underline{n+2m-3,\ n+2m-2},\ n+2m-1, n-1)$. Now $T_{n+(2m-1)}\langle\ 1,\ 3;\ 1,\ 6\rangle$ contains the edge $(n+2m-3,\ n+2m-2)$ (see Figure 23).

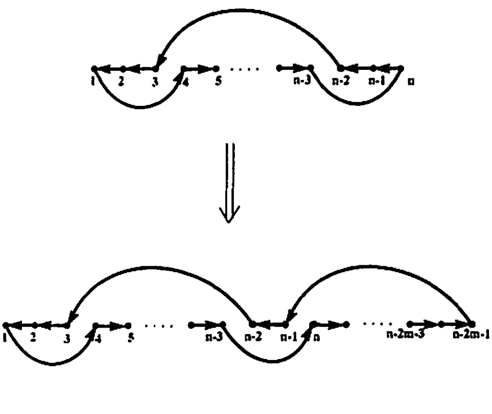

Fig. 23.

Suppose $T_n\langle 1,3;1,2m\rangle$, with $n = 4m+4+q(2m-1)$, $k+q(2m-1)$; $k = 2, 4, 5, 7, \ldots, 2m-5, 2m-3$, has a hamiltonian cycle containing the edge $(n-2,\ n-1)$, for some $q \in \mathbb{N}$. We prove that $T_{n+(2m-1)}\langle 1,3;1,2m\rangle$ has the same property. Since $(n-2,\ n-1)$ is an edge in a hamiltonian cycle of $T_n\langle 1,3;1,2m\rangle$, we transform this hamiltonian cycle to a hamiltonian cycle in
$T_{n+(2m-1)}\langle 1,3;1,2m\rangle$, by replacing the edge $(n-2,\ n-1)$ with the path $(n-2,\ n+1,\ n+2, \ldots, \underline{n+2m-3,\ n+2m-2},\ n+2m-1, n-1)$. This shows that $T_{n+(2m-1)}\langle 1,3;1,2m\rangle$ has the same property. This together with Case 3 proves the theorem.□

In Theorem 5, if $n \equiv 3\ (mod\ (t_2-1))$, then the hamiltonicity of $T_n\langle 1,3;1,t_2\rangle$ depends upon t_2 as described in Theorem 6.

Theorem 6. *$T_n\langle 1,3;1,t_2\rangle$ is hamiltonian if $t_2 \equiv 0,2\,(mod\ 3)$, $t_2(\geq 8)$ is even, and $n \equiv 3\ (mod\ (t_2-1))$.*

Proof. Put $t_2 = 2m$. Since $n \equiv 3\,(mod\,(2m-1))$, the smallest possible value for n is $2m+2$.

Case 1. If $2m \equiv 0\,(mod\ 3)$, then a hamiltonian cycle in $T_n\langle\ 1,\ 3;\ 1,\ 2m\rangle$ is $(1, 4, 5, 8, 7, \ldots, 3p+1, 3p+2, 3p+5, 3p+4, \ldots, n-3, n, 2, 3, 6, 9, \ldots, \underline{n-2, n-1}, 1)$, where p is a non-negative odd integer (see Figure 24).

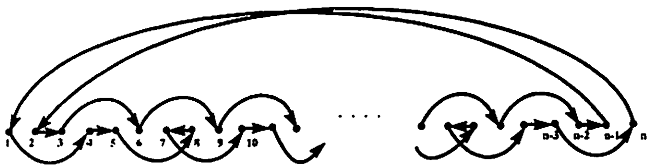

Fig. 24. $T_n\langle 1,3;1,2m\rangle$; $n = 2m+2$ where$2m \equiv 0\,(mod\ 3)$.

Case 2. If $2m \equiv 2\,(mod\ 3)$, then a hamiltonian cycle in $T_n\langle\ 1,\ 3;\ 1,\ 2m\rangle$ is $(1, 4, 3, 6, 7, \ldots, 3p+1, 3p, 3p+3, 3p+4, \ldots, n-3, n, 2, 5, 8, 11, \ldots, \underline{n-2, n-}$ (see Figure 25).

Note that $(n-2,\ n-1)$ is an edge in both of the above hamiltonian cycles. Suppose $T_n\langle 1,3;1,2m\rangle$, with $n = (2m+2)+q(2m-1)$, has a

Fig. 25. $T_n\langle 1,3;1,2m\rangle$; $n = 2m+2$ where$2m \equiv 2\,(mod\ 3)$.

hamiltonian cycle containing the edge $(n-2,\ n-1)$, for some non-negative integer q. We prove that $T_{n+(2m-1)}\langle 1,3;1,2m\rangle$ has the same property.

Since $(n-2,\ n-1)$ is an edge in a hamiltonian cycle of $T_n\langle 1,3;1,2m\rangle$, we transform this hamiltonian cycle to a hamiltonian cycle in $T_{n+(2m-1)}\langle$ 1, 3; 1, $2m\rangle$, by replacing the edge $(n-2,\ n-1)$ with the path $(n-2,\ n+1,$ $n+2,\ \ldots,\ \underline{n+2m-3},\ \underline{n+2m-2},\ n+2m-1,\ n-1)$. This shows that $T_{n+(2m-1)}\langle$ 1, 3; 1, $2m\rangle$ enjoys the same property. This finishes the proof.□

Theorem 7. *$T_n\langle 1,3;1,t_2\rangle$, where $t_2(\geq 3)$ is odd, is hamiltonian if and only if n is even.*

Proof. For $t_2 = 3$ it is done in ([1], Theorem 5).

For $t_2 \geq 5$. First suppose n is even. We have $t_2 = 2m-1$ for some integer $m \geq 3$, and write

$$n \equiv n_0\,(mod\,(2m-2)),$$

where

$$2m \leq n_0 \leq 4m-4.$$

Clearly $n_0 - (2m-1)$ is odd. First, assume $n = n_0$. We show the existence of a hamiltonian cycle containing the edge $(n-2,\ n-1)$.

Case 1. If $n - (2m-1) = 1$, then a hamiltonian cycle in $T_n\langle$ 1, 3; 1, $2m-1\rangle$ is $(1,\ 2,\ 3, \ldots, \underline{n-2,\ n-1},\ n,\ 1)$ (see Figure 26).

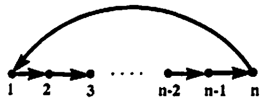

Fig. 26. $T_n\langle 1,3;1,2m-1\rangle$; $n-(2m-1)=1$.

Case 2. If $n-(2m-1)=3$, then a hamiltonian cycle in $T_n\langle 1, 3; 1, 2m-1\rangle$ is (1, 4, 5, 6, …, $\underline{n-2, n-1}$, n, 3, 2, 1) (see Figure 27).

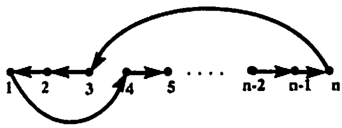

Fig. 27. $T_n\langle 1,3;1,2m-1\rangle$; $n-(2m-1)=3$.

Case 3. If $n-(2m-1)>3$, we have the following subcases.

(a) If $n-(2m-1)=2m-3$, then a hamiltonian cycle in $T_n\langle 1,3;1,2m-1\rangle$ is (1, 2, 3, …, $n-2m$, $n-2m+3$, $n-2m+2$, $n-2m+5$, $n-2m+6$, …, $\underline{n-2, n-1}$, n, $n-2m+1$, $n-2m+4$, 1) (see Figure 28).

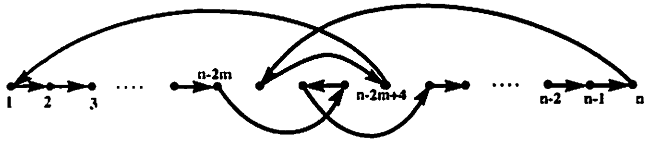

Fig. 28. $T_n\langle 1,3;1,2m-1\rangle$; $n-(2m-1)=2m-3$.

(b) If $n-(2m-1)=2m-5$, then a hamiltonian cycle in $T_n\langle 1,3;1,2m-1\rangle$ is (1, 2, 3, …, $n-2m$, $n-2m+3$, $n-2m+2$, $n-2m+5$, $n-2m+8$, $n-2m+9$, …, $\underline{n-2, n-1}$, n, $n-2m+1$, $n-2m+4$, $n-2m+7$, $n-2m+6$, 1)

(see Figure 29).

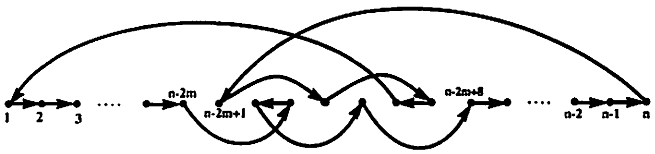

Fig. 29. $T_n\langle 1,3;1,2m-1\rangle$; $n-(2m-1)=2m-5$.

(c) If $n-(2m-1)\neq 2m-3$, $2m-5$, we have the following three subcases.

(*i*) If $4m-n\equiv 0\,(mod\,3)$, then a hamiltonian cycle in $T_n\langle 1,3;1,2m-1\rangle$ is $(1,\ 2,\ 3,\ldots,n-2m,\ n-2m+3,\ n-2m+2,\ n-2m+5,\ n-2m+6,\ldots,n-2m+3p,\ n-2m+3p-1,\ n-2m+3p+2,\ n-2m+3p+3,\ldots,2m-1,\ 2m+2,\ 2m+3,\ldots,\underline{n-2,\ n-1},\ n,\ n-2m+1,\ n-2m+4,\ n-2m+7,\ \ldots,2m-2,\ 2m+1,\ \overline{2m},\ 1)$, where p is a non-negative odd integer (see Figure 30).

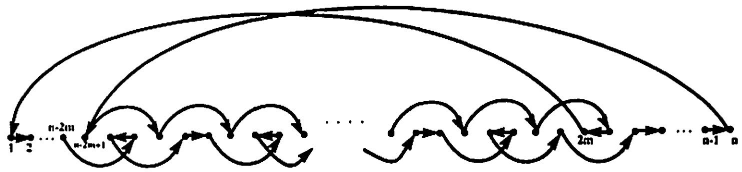

Fig. 30.

(*ii*) If $4m-n\equiv 1\,(mod\,3)$, then a hamiltonian cycle in $T_n\langle 1,3;1,2m-1\rangle$ is $(1,\ 2,\ 3,\ldots,n-2m,\ n-2m+3,\ n-2m+2,\ n-2m+5,\ n-2m+6,\ldots,n-2m+3p,\ n-2m+3p-1,\ n-2m+3p+2,\ n-2m+3p+3,\ldots,2m-1,\ 2m-2,\ 2m+1,\ 2m+2,\ldots,\underline{n-2,\ n-1},\ n,\ n-2m+1,\ n-2m+4,\ n-2m+7,\ldots,2m-3,\ 2m,\ 1)$ (see Figure 31).

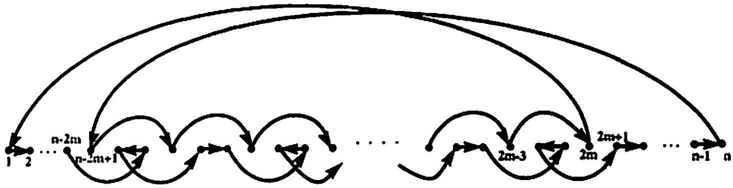

Fig. 31.

(*iii*) If $4m - n \equiv 2\,(mod\,3)$, then a hamiltonian cycle in $T_n\langle 1, 3; 1, 2m-1\rangle$ is $(1,\ 2,\ 3, \ldots, n-2m,\ n-2m+3,\ n-2m+2,\ n-2m+5,\ n-2m+6, \ldots, n-2m+3p,\ n-2m+3p-1,\ n-2m+3p+2,\ n-2m+3p+3, \ldots, 2m-3,\ \ 2m-2,\ 2m+1,\ 2m+2, \ldots, n-2,\ n-1,\ n,\ n-2m+1,\ n-2m+4,\ n-2m+7, \ldots, 2m-1,\ 2m,\ 1)$ (see Figure 32).

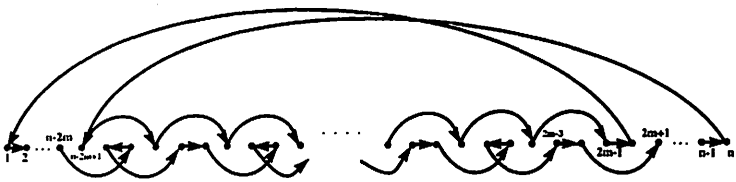

Fig. 32.

Suppose $T_n\langle 1, 3; 1, 2m-1\rangle$ with $n = n_0 + q(2m-2)$ has a hamiltonian cycle containing the edge $(n-2, n-1)$, for some non-negative integer q. We shall prove that $T_{n+(2m-2)}\langle 1, 3; 1, 2m-1\rangle$ enjoys the same property. Since $(n-2,\ n-1)$ is an edge in a hamiltonian cycle of $T_n\langle\ 1,\ 3;\ 1,\ 2m-1\rangle$, we transform this hamiltonian cycle to a hamiltonian cycle in $T_{n+(2m-2)}\langle 1, 3; 1, 2m-1\rangle$, by replacing the edge $(n-2,\ n-1)$ with the path $(n-2,\ n+1,\ n+2, \ldots,\ n+2m-4,\ n+2m-3,\ n+2m-2,\ n-1)$. This shows that $T_{n+(2m-2)}\langle 1, 3; 1, 2m-1\rangle$ enjoys the same property.

Conversely, since t_2 is odd, $T_n\langle 1, 3; 1, t_2\rangle$ is bipartite and, being hamiltonian, n must be even. □

4 Concluding Remarks

The investigation of the hamiltonicity of Toeplitz graphs, directed or not, is far from being achieved. The cases of small numbers k, l and s_i, t_j were studied in [6]. Of course, these cases are most relevant to this study, but there is still much left to do. In [6] it is shown that the investigation is complete for $s_2 = 2$ and $s_3 = 3$. In this paper we tried to enlarge s_i (in particular s_2) a little bit, so we extended this investigation here to $s_2 = 3$. The next task is, in our opinion, the investigation of the case of k, l still small, but larger s_i, t_j. Also, other characterizations of hamiltonian graphs inside subfamilies of Toeplitz graphs would be most welcome.

In this paper again we provided no negative results, except for those implied by the characterizations of hamiltonian graphs inside classes of Toeplitz graphs. Such results, besides those in [1] yielding disconnectedness, would also be of interest.

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