On potentially $K_6 - 3K_2$-graphic sequences*

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Abstract: Let $K_m - H$ be the graph obtained from the complete graph on m vertices K_m by removing the edges set $E(H)$ of H, where H is a subgraph of K_m. In this paper, we characterize the potentially $K_6 - 3K_2$-graphic sequences, where pK_2 is the matching consisted of p edges.

Keywords: graph; degree sequence; potentially $K_6 - 3K_2$-graphic sequences

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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative, non-increasing integers with $d_1 \leq n - 1$ is denoted by NS_n. If each term of a sequence $\pi \in NS_n$ is nonzero, then π is said to be *positive*. A sequence $\pi \in NS_n$ is called to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is refereed to as a *realization* of π. The set of all graphic sequences in NS_n is denote by GS_n. For a $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, define $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. In a degree sequence, r^t means r repeats t times, that is, in the realization of the sequence there are t vertices of degree r. For a given graph H, a graphic sequence π is *potentially H-graphic* if there exists a realization of π containing H as a subgraph. Let $G - H$ denote the graph obtained from G by removing the edges set $E(H)$, where H is a subgraph of G. Let K_k, C_k, P_k and $K_{r,s}$ denote a complete graph on k vertices, a cycle on k vertices, a path on $k+1$ vertices, and the $r \times s$ complete bipartite graph, respectively. Given any two graphs G and H, $G \cup H$ is the disjoin union of G and H. If $G = H$, we abbreviate $G \cup H$ as $2G$.

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Gould et al. [4] considered the following variation of the classical Turán-type extremal problems: determine the smallest even integer $\sigma(H, n)$ such that every n-term graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with $\sigma(\pi) \geq \sigma(H, n)$ has a realization G containing H as a subgraph. If $H = K_{r+1}$, this problem was considered by Erdős et al. [3] where they showed that $\sigma(K_3, n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_{r+1}, n) = (r-1)(2n-r)+2$ for sufficiently large n. Gould et al. [4] and Li and Song [18] proved independently that the conjecture holds for $r = 3$ and $n \geq 8$. Recently, Li et al. [19,20] showed that the conjecture is true for $r = 4$ and $n \geq 10$ and for $r \geq 5$ and $n \geq \binom{r}{2}+3$. Li and Yin [21] further determined $\sigma(K_{r+1}, n)$ for $r \geq 6$ and $n \geq 2r+3$. The problem about determining $\sigma(K_{r+1}, n)$ was completely solved. Yin, Li and Mao [27] determined $\sigma(K_{r+1} - K_2, n)$ for $r \geq 3$ and $r+1 \leq n \leq 2r$ and $\sigma(K_5 - K_2, n)$ for $n \geq 5$, and Yin and Li [25] further determined $\sigma(K_{r+1} - K_2, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$. Yin [29] determined $\sigma(K_{r+1} - K_3, n)$ for $r \geq 3$ and $n \geq 3r+5$. Lai [13-15] determined $\sigma(K_5 - C_4, n)$, $\sigma(K_5 - P_3, n)$, $\sigma(K_5 - P_4, n)$ and $\sigma(K_5 - K_3, n)$ for $n \geq 5$. Lai and Hu [16] determined $\sigma(K_{r+1} - H, n)$ for $n \geq 4r+10$, $r \geq 3$, $r+1 \geq k \geq 4$ and H be a graph on k vertices which containing a tree on 4 vertices but not contain a cycle on 3 vertices and $\sigma(K_{r+1} - P_2, n)$ for $n \geq 4r+8$, $r \geq 3$. Lai and Yan [17] determined the values of $\sigma(K_{r+1} - U, n)$ for $n \geq 5r+18$, $r+1 \geq k \geq 7$, $j \geq 6$, where U is a graph on k vertices and j edges which contains a graph $K_3 \cup P_3$ but not contains a cycle on 4 vertices and not contains $K_4 - P_2$.

In the research of degree sequence, another important question is to characterize the potentially H-graphic sequences without zero terms. Luo [23] characterized the potentially K_3-graphic sequences. Luo and Warner [24] characterized then potentially K_4-graphic sequences. Elaine and Niu [1] characterized the potentially $K_4 - K_2$-graphic sequences. In [23], Luo characterized the potentially $K_4 - 2K_2$-graphic sequences, where pK_2 is the matching consisted of p edges. Yin and Luo [28] characterized the potentially K_5-graphic sequences. Yin and Yin [31] characterized the potentially $K_5 - K_2$-graphic sequences. Luo [23] also characterized the potentially $K_5 - C_5$-graphic sequences. Hu and Lai [5,6] characterized the potentially $K_5 - C_4$, $K_5 - P_4$ and $K_5 - E_3$-graphic sequences, where E_3 denotes graphs with 5 vertices and 3 edges. Hu and Lai [7] characterized the potentially $K_5 - P_3$, $K_5 - A_3$, $K_5 - K_3$ and $K_5 - K_{1,3}$-graphic sequences, where A_3 is $P_2 \cup K_2$. Moreover, they also characterized the potentially $K_5 - 2K_2$-graphic sequences. In [31], Yin et al. also characterized the potentially K_6-graphic. For $K_6 - H$, Hu and Lai [8] characterized the potentially $K_6 - 2K_3$ and $K_6 - C_6$-graphic sequences. Latterly, Yin [30] characterized the potentially $K_6 - K_3$-graphic sequences. The purpose of this paper is to characterize $K_6 - 3K_2$-graphic sequences. As an application of this characterization, it is straightforward to find the values of $\sigma(K_6 - 3K_2, n)$.

2. Preparations

Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, $1 \le k \le n$. Denote

$$\pi_k'' = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n), \\ if\ d_k \ge k, \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n), \\ if\ d_k < k, \end{cases}$$

and $\pi_k' = (d_1', d_2', \ldots, d_{n-1}')$, where $d_1' \ge d_2' \ge \cdots \ge d_{n-1}'$ is a rearrangement of the $n-1$ terms of π_k''. Then π_k' is called the residual sequence obtained by laying off d_k from π. It is easy to see that if π_k' is graphic then so is π, since a realization G of π can be obtained from a realization G' of π_k' by adding a new vertex of degree d_1 to G' and joining it to the vertices whose degrees are reduced by one in going from π to π_k'. In fact more is true:

Theorem 2.1 [9] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and $1 \le k \le n$. Then π is graphic if and only if π_k' is graphic.

Theorem 2.2 [2] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then π is graphic if and only if for any t, $1 \le t \le n-1$,

$$\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{j=t+1}^{n} \min\{d_j, t\}.$$

Theorem 2.3 [4] If $\pi = (d_1, d_2, \ldots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Theorem 2.4 [22] Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$. If $\sigma(\pi)$ is even and $d_1 - d_n \le 1$, then π is graphic.

Theorem 2.5 [7] Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a positive sequence with $n \ge 5$. Then π is potentially $K_5 - 2K_2$-graphic if and only if the following conditions hold:

(1) $d_1 \ge 4$ and $d_5 \ge 3$.

(2)

$$\pi \ne \begin{cases} (n-i, n-j, 3^{n-i-j-2k}, 2^{2k}, 1^{i+j-2}), & if\ n-i-j\ is\ even, \\ (n-i, n-j, 3^{n-i-j-2k-1}, 2^{2k+1}, 1^{i+j-2}), & if\ n-i-j\ is\ odd, \end{cases}$$

where $1 \le j \le 5$ and $0 \le k \le \lfloor \frac{n-i-j-4}{2} \rfloor$.

(3) $\pi \ne (4^2, 3^4), (4, 3^4, 2), (5, 4, 3^5), (5, 3^5, 2), (4^7), (4^3, 3^4), (4^2, 3^4, 2), (4, 3^6), (4, 3^5, 1), (4, 3^4, 2^2), (5, 3^7), (5, 3^6, 1), (4^8), (4^2, 3^6), (4^2, 3^5, 1), (4, 3^6, 2), (4, 3^5, 2, 1), (4, 3^7, 1), (4, 3^6, 1^2), (n-1, 3^5, 1^{n-6})$ and $(n-1, 3^6, 1^{n-7})$.

Lemma 2.6 [30] Let $\pi = (4^x, 3^y, 2^z, 1^m)$ with even $\sigma(\pi)$, $x + y + z + m = n \ge 5$ and $x \ge 1$. Then $\pi \in GS_n$ if and only if $\pi \notin A$, where

$A = \{(4,3^2,1^2), (4,3,1^3), (4^2,2,1^2), (4^2,3,2,1), (4^3,1^2), (4^3,2^2), (4^3,3,1), (4^4,2), (4^2,3,1^3), (4^2,1^4), (4^3,2,1^2), (4^4,1^2), (4^3,1^4)\}$.

Lemma 2.7 [30] Let $\pi = (3^x, 2^y, 1^z)$, where $x+y+z \geq 1$ and $\sigma(\pi)$ is even. Then $\pi \in GS_n$ if and only if $\pi \notin S$, where $S = \{(2),(2^2),(3,1),(3^2),(3,2,1),(3^2,2),(3^3,1),(3^2,1^2)\}$.

The set of all sequences $\pi = (4^x, 3^y, 2^z, 1^m)$ is denoted by B, where $\sigma(\pi)$ is even , $x \geq 1$ and $x+y+z+m \leq 4$. Then

$B = \{(4),(4,2),(4^2),(4,1^2),(4,3,1),(4,3^2),(4,2^2),(4^2,2),(4^3),(4,2,1^2),(4,2^3),(4,3,2,1),(4,3^2,2),(4^2,1^2),(4^2,2^2),(4^2,3,1),(4^2,3^2),(4^3,2),(4^4)\}$.

Obviously, if $\pi = (d_1, d_2, \ldots, d_n)$ satisfies $\sigma(\pi)$ is even, $4 \geq d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ and $\pi \notin A \cup B \cup S$, then π must be graphic.

Lemma 2.8 Let $\pi = (d_1, d_2, 3^{n-2})$ be non-negative integer sequence with $n \geq 5$ and $n-2 \geq d_1 \geq d_2$. If $\pi \neq (1,0,3^3)$ and $\sigma(\pi)$ is even, then π is graphic.

Proof. If $d_2 \geq 3$, by Theorem 2.2, then $\pi \in GS_n$. Assume $d_2 \leq 2$. Consider the residual sequence $\pi_1' = (3^{n-2-d_1}, 2^{d_1}, d_2)$ of π by laying off d_1, where $n-2 \geq d_1$ and $2 \geq d_2 \geq 0$. Since $n \geq 5$, $\sigma(\pi_1')$ is even and $\pi \neq (1,0,3^3)$, π_1' is not one of following sequences:

$$(2),(2^2),(3,1),(3^2),(3,2,1),(3^2,2),(3^3,1),(3^2,1^2).$$

By Theorem 2.4 and Lemma 2.7, π_1' is graphic, so is π. □

3. Main result

Theorem 3.1 Let $n \geq 6$ and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ be a positive sequence. Then π is potentially $K_6 - 3K_2$-graphic if and only if π satisfies the following conditions:

(1) $d_6 \geq 4$.

(2) $d_1 = n-1$ implies $d_2 \geq 5$, and the residual positive sequence obtained by laying off d_1 from π is not one of the following sequences:

$(4^2,3^4),(4,3^4,2),(5,4,3^5),(5,3^5,2),(4^7),(4^3,3^4),(4^2,3^4,2),(4,3^6),(4,3^5,1),(4,3^4,2^2),(5,3^7),(5,3^6,1),(4^8),(4^2,3^6),(4^2,3^5,1),(4,3^6,2),(4,3^5,2,1),(4,3^7,1),(4,3^6,1^2)$.

(3) $d_3 = n-1$ implies $d_4 \geq 5$.

(4) If $\pi = (d_1, d_2, d_3, 4^i, 3^j, 2^k, 1^{n-i-j-k-3})$ with $i \geq 3$, then

$$d_1 + d_2 + d_3 < n + 2(i+j) + k + 3.$$

(5) If $d_1 = 4$, i.e., $\pi = (4^6, d_7, \ldots, d_n)$, then sequence $(d_7, \ldots, d_n) \notin A \cup B \cup S$.

(6) π is not one of the following sequences:

$$(n-i, n-j, 4^5, 2^{n-i-j-5}, 1^{i+j-2}), (n-i, n-j, 4^6, 2^{n-i-j-6}, 1^{i+j-2});$$

$(n-2,4^5,3,1^{n-7}),(n-2,5,4^5,1^{n-7}),(n-3,4^6,1^{n-7}),(n-2,4^7,1^{n-8}),$
$(n-2,4^6,2,1^{n-8}),(n-2,4^5,3^2,1^{n-8}),(n-3,4^6,3,1^{n-8}),(n-3,5,4^6,1^{n-8}),$
$(n-2,5,4^5,3,1^{n-8}),(n-2,5^2,4^5,1^{n-8}),(n-2,6,4^6,1^{n-8}),(n-4,4^7,1^{n-8}),$
$(n-2,4^7,3,1^{n-9}),(n-2,4^6,3,2,1^{n-9}),(n-3,4^7,2,1^{n-9}),(n-3,4^8,1^{n-9}),$
$(n-2,5,4^7,1^{n-9}),(n-2,5,4^6,2,1^{n-9}),(n-2,4^8,2,1^{n-10}),(n-2,4^7,2^2,1^{n-10})$;

$n=8:$ $(5,4^5,3,2),(5^2,4^5,2),(5^2,4^4,3^2),(5^3,4^4,3),(5^4,4^4),(5^8),(6,5,4^5,1),$
$(6^2,4^5,2)$;

$n=9:$ $(5,4^7,3),(5,4^6,3,2),(5,4^6,2,1),(5,4^5,3^3),(5,4^5,3^2,1),(5^2,4^7),(5^2,$
$4^6,2),(5^2,4^5,3^2),(5^2,4^5,3,1),(5^3,4^5,3),(5^3,4^5,1),(5^4,4^5),(5^8,4),(6,4^6,3^2),$
$(6,5,4^6,3),(6,5^2,4^6)$,$(6,5^8)$,$(6^2,4^7)$;

$n=10:$ $(5,4^8,3),(5,4^8,1),(5,4^7,3,2),(5,4^7,2,1),(5,4^6,3^2,1),(5,4^6,3,1^2),$
$(5^2,4^8),(5^2,4^7,2),(5^2,4^6,3,1),(5^2,4^6,1^2),(5^2,4^5,3,2,1),(5^3,4^6,1),(5^{10}),$
$(5^9,1),(6,4^9),(6,4^8,2),$ $(6,4^7,3,1)$.

Proof. Suppose that π is potentially K_6-3K_2-graphic. Let G be a realization of π which contains K_6-3K_2 and $d(v_i)=d_i$ for $v_i \in V(G)$, $i=1,2,\ldots,n$. (1) is obvious. If $d_1=n-1$, then $G-v_1$ contains K_5-2K_2 as a subgraph. Thus, $G-v_1$ contains at least one vertex with degree at least 4, and the positive degree sequence of $G-v_1$ is not one of the following sequences:

$(4^2,3^4),(4,3^4,2),(5,4,3^5),(5,3^5,2),(4^7),(4^3,3^4),(4^2,3^4,2),(4,3^6),$
$(4,3^5,1),(4,3^4,2^2),(5,3^7),(5,3^6,1),(4^8),(4^2,3^6),(4^2,3^5,1),(4,3^6,2),$
$(4,3^5,2,1),(4,3^7,1),(4,3^6,1^2)$.

By Theorem 2.1 and Theorem 2.5, (2) holds. If $d_3=n-1$, then $G-v_1-v_2-v_3$ contains $K_{1,2}$ as a subgraph. Thus, $G-v_1-v_2-v_3$ contains at least one vertex with degree at least 2. Therefore, $d_3=n-1$ implies that $d_4\geq 5$. Hence, (3) holds.

If $\pi=(d_1,d_2,d_3,4^i,3^j,2^k,1^{n-i-j-k-3})$ is potentially K_6-3K_2-graphic, then according to Theorem 2.3, there exists a realization G of π containing K_6-3K_2 as subgraph so that the vertices of K_6-3K_2 have the largest degree of π. Therefore, the sequence $\pi_1=(d_1-4,d_2-4,d_3-4,0^3,4^{i-3},3^j,2^k,1^{n-i-j-k-3})$ obtained from $G-(K_6-3K_2)$ must be graphic. Then, there exist at most two vertices in $\{v_1,\ v_2,\ v_3\}$ which are adjacent in $G-(K_6-3K_2)$. Thus, $d_1-4+d_2-4+d_3-4\leq 2+3(i-3)+3j+2k+n-i-j-k-3$. If $d_1+d_2+d_3=n+2(i+j)+k+3$, then $n+2(i+j)+k+3-12\leq 2+3(i-3)+3j+2k+n-i-j-k-3$, i.e., $0\leq -1$, a contradiction. Hence, (4) holds.

Let $\pi = (4^6, d_7, \ldots, d_n)$ be potentially $K_6 - 3K_2$-graphic and G be a realization of π with $K_6 - 3K_2 \subseteq G$. Then $G - (K_6 - 3K_2)$ is a realization of the sequence $\pi_2 = (d_7, \ldots, d_n)$. So π_2 is graphic and $\pi_2 \notin A \cup B \cup S$. Hence, (5) holds.

Assume $\pi = (n-i, n-j, 4^5, 2^{n-i-j-5}, 1^{i+j-2})$ is potentially $K_6 - 3K_2$-graphic. Let G be a realization of π. If $v_1v_2 \notin E(G)$, then $d_1 + d_2 \leq 2 \times 5 + 2(n-i-j-5) + i + j - 2$, i.e., $n - i + n - j \leq 2n - i - j - 2$, a contradiction. Assume $v_1v_2 \in E(G)$. By Theorem 2.3, sequence $\pi_3 = (n-i-4, n-j-4, 0^4, 4^1, 2^{n-i-j-5}, 1^{i+j-2})$ obtained from $G - (K_6 - 3K_2)$ must be graphic. If $n - j - 4 \leq 1$, then $n - i - j - 5 < 0$, a contradiction. Suppose $n - j - 4 \geq 2$. By Theorem 2.2, then $(n-i-4) + (n-j-4) + 4 \leq 3 \times 2 + 2(n-i-j-5) + i + j - 2$, i.e., $-4 \leq -6$, a contradiction. Therefore $\pi = (n-i, n-j, 4^5, 2^{n-i-j-5}, 1^{i+j-2})$ is not potentially $K_6 - 3K_2$-graphic. If $\pi = (n-i, n-j, 4^6, 2^{n-i-j-6}, 1^{i+j-2})$, then we can prove π is not potentially $K_6 - 3K_2$-graphic similarly.

Assume $\pi = (n-2, 4^8, 2, 1^{n-10})$ is potentially $K_6 - 3K_2$-graphic. By Theorem 2.3, then $\pi_4 = (n-2-4, 0^5, 4^3, 2, 1^{n-10})$ is graphic. But $(3^3, 1)$, the residual sequence by laying off the term $n-6$ from π_4, is not graphic, a contradiction. Hence, $\pi = (n-2, 4^8, 2, 1^{n-10})$ is not potentially $K_6 - 3K_2$-graphic. We can similarly show that the following sequences are not potentially $K_6 - 3K_2$-graphic:

$(n-2, 4^7, 3, 1^{n-9}), (n-2, 4^7, 2^2, 1^{n-10}), (n-2, 4^7, 1^{n-8}), (n-2, 4^6, 3, 2, 1^{n-9})$ $(n-2, 4^6, 2, 1^{n-8}), (n-2, 4^5, 3, 1^{n-7}), (n-2, 4^5, 3^2, 1^{n-8}), (n-3, 4^6, 1^{n-7})$, $(n-3, 4^6, 3, 1^{n-8}), (n-3, 4^7, 2, 1^{n-9}), (n-3, 4^8, 1^{n-9})$, $(n-4, 4^7, 1^{n-8})$.

Assume $\pi = (n-3, 5, 4^6, 1^{n-8})$ is potentially $K_6 - 3K_2$-graphic. Let G be a realization of π. According to Theorem 2.3 , there exists a realization G of π containing $K_6 - 3K_2$ as subgraph so that the vertices of $K_6 - 3K_2$ have the largest degree of π. Thus, $G - v_1$ contain $K_5 - 2K_2$. If $v_1v_2 \notin E(G)$, then $\pi_5 = (5, 4, 3^5, 0^{n-8})$ or $\pi_6 = (5, 3^6, 1, 0^{n-9})$ is degree sequence of $G - v_1$. By Theorem 2.5, π_5 and π_6 are not potentially $K_5 - 2K_2$-graphic, a contradiction. If $v_1v_2 \in E(G)$, then $\pi_7 = (4, 3^6, 1^2, 0^{n-10})$ or $\pi_8 = (4^2, 3^5, 1, 0^{n-9})$ or $\pi_9 = (4^3, 3^4, 0^{n-8})$ is degree sequence of $G - v_1$. By Theorem 2.5, π_7, π_8 and π_9 are not potentially $K_5 - 2K_2$-graphic, a contradiction. Thus, $\pi = (n-3, 5, 4^6, 1^{n-8})$ is not potentially $K_6 - 3K_2$-graphic. If π is one of following sequences:

$$(n-2, 5^2, 4^5, 1^{n-8}), (n-2, 6, 4^6, 1^{n-8}), (n-2, 5, 4^5, 3, 1^{n-8}),$$

$$(n-2, 5, 4^7, 1^{n-9}), (n-2, 5, 4^6, 2, 1^{n-9}), (n-2, 5, 4^5, 1^{n-7}),$$

then we can similarly prove that π is not potentially $K_6 - 3K_2$-graphic as above. Now it is easy to check that the others sequences in (6) are not potentially $K_6 - 3K_2$-graphic. Hence, (6) holds.

Next, we will prove the sufficient conditions. Suppose the graphic positive sequence $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ satisfies the conditions (1)-(6).

If $d_1 = n-1$, then the residual sequence $\pi'_1 = (d'_1, d'_2, \ldots, d'_{n-1})$ obtained by laying off d_1 from π such that $d'_1 = d_2 - 1 \geq 4$, $d'_5 = d_6 - 1 \geq 3$. Since π satisfies conditions (2),(4) and (6) , π'_1 is potentially $K_5 - 2K_2$-graphic by Theorem 2.5. So π is potentially $K_6 - 3K_2$-graphic by Theorem 2.1. Suppose $d_1 \leq n-2$.

Our proof is by induction on n. If $n = 6$, then $\pi = (4^6)$. It is easy to see that the sequence is the degree sequence of $K_6 - 3K_2$. Now suppose that the sufficiency holds for $n-1 (n \geq 7)$. We will prove that π is potentially $K_6 - 3K_2$-graphic in terms of the following cases:

Case 1. $d_n \geq 5$. Consider the residual sequence $\pi'_n = (d'_1, d'_2, \ldots, d'_{n-1})$. If $d_n \geq 6$, then the residual positive sequence π'_n satisfies $d'_{n-1} \geq 5$. Since $\pi'_n \neq (5^{10}), (5^8), (6, 5^8)$, it is easy to check that π'_n satisfies (1),(2),(3) and (6). Then by the induction hypothesis, π'_n is potentially $K_6 - 3K_2$-graphic. Hence, by Theorem 2.1, π is potentially $K_6 - 3K_2$-graphic.

Suppose $d_n = 5$. Then π'_n satisfies $d'_2 \geq 5$, $d'_{n-1} \geq 4$. Obviously, π'_n satisfies (1). If π'_n satisfies (2),(3),(4) and (6), then by the induction hypothesis, π'_n is potentially $K_6 - 3K_2$-graphic, and hence so is π.

If $d'_1 = n-2$, by $d_1 \leq n-2$, then $d_6 = d_1 = n-2$. Clearly, π'_n satisfies (2) and (3).

Assume π'_n does not satisfy (4). Since $d'_{n-1} \geq 4$, $\pi'_n = (d'_1, d'_2, d'_3, 4^{n-4})$ where $d'_1 + d'_2 + d'_3 = (n-1) + 2(n-4) + 3 = 3(n-2)$. By $d'_1 \leq n-2$, $d'_1 = d'_3 = n-2$. If $n \geq 10$, by $d_n = 5$, then it is impossible. If $n \leq 9$, then π'_n is just $(6^3, 4^4)$, a contradiction. Hence, π'_n satisfies (4).

If π'_n does not satisfy (6), by $\pi \neq (6, 5^8), (5^8)$, then π'_n is one of the following sequences:

$$(5^8), (5^8, 4), (6, 5^8), (5^{10}).$$

Then π is one of the following sequences:

$$(6^5, 5^4), (6^4, 5^6), (6^6, 5^4), (7, 6^4, 5^5), (6^5, 5^6).$$

It is easy to check that all of those are potentially $K_6 - 3K_2$-graphic.

Case 2. $d_n = 4$. Consider the residual sequence $\pi'_n = (d'_1, d'_2, \ldots, d'_{n-1})$, where $d'_{n-5} \geq 4$ and $d'_{n-1} \geq 3$. If π'_n satisfies (1)-(6), then by the induction hypothesis, π'_n is potentially $K_6 - 3K_2$-graphic, and hence so is π.

If π'_n does not satisfy (1), i.e., $d'_6 \leq 3$, then $d_4 = 4$. Since $d'_{n-5} \geq 4$, we have $7 \leq n \leq 10$. If $n = 7$, then $\pi = (d_1, d_2, d_3, 4^4)$, where $4 \leq d_3 \leq d_2 \leq d_1 \leq 5$. Since $\sigma(\pi)$ is even, $\pi = (5^2, 4^5)$ or $\pi = (4^7)$, which is impossible by (5) and (6). If $n = 8$, then $\pi = (d_1, d_2, 4^6)$, where $4 \leq d_2 \leq d_1 \leq 6$. Hence, π is one of following sequences:

$$(4^8), (5^2, 4^6), (6, 4^7), (6^2, 4^6),$$

which is also impossible by (5) and (6). If $n = 9$, then $\pi = (d_1, 4^8)$, where

$4 \le d_1 \le 7$ and d_1 is even. Hence, $\pi = (4^9)$ or $\pi = (6, 4^8)$, which is impossible by (5) and (6). If $n = 10$, then $\pi = (4^{10})$, a contradiction. Thus, π'_n satisfies (1).

If $d'_1 = n-2$, by $d_1 \le n-2$, then $d_5 = d_1 = n-2$. Clearly, π'_n satisfies (2) and (3).

If π'_n does not satisfy (4), by $d'_{n-5} \ge 4$ and $d'_{n-1} \ge 3$, then $\pi'_n = (d'_1, d'_2, d'_3, 4^i, 3^{n-i-4})$, where $i \ge n-8$ and $d'_1+d'_2+d'_3 = 3(n-2)$. Then $d'_1 = d'_2 = d'_3 = n-2$. By $d_n = 4$ and Theorem 2.1, then $\pi = ((n-1)^3, 5, 4^{n-4})$ or $\pi = ((n-1)^3, 4^{n-3})$, a contradiction. Hence, π'_n satisfies (4).

If π'_n does not satisfy (5), by $\pi \ne (5^4, 4^4)$, $(5^4, 4^5)$, $(5^2, 4^7)$, $(5^2, 4^8)$, then π'_n is one of the following sequences: $(4^9), (4^{10}), (4^8, 3^2)$. Thus, π is one of the following sequences: $(5^4, 4^6), (5^4, 4^7), (5^2, 4^9)$. It is easy to check that three of above sequences are potentially $K_6 - 3K_2$-graphic.

If π'_n does not satisfy (6), by $d'_{n-1} \ge 3$ and $\pi \ne (6, 5^2, 4^5), (6, 5^2, 4^6)$, $(6^2, 4^7)$, $(5^8, 4)$, $(7, 5, 4^7)$,$(6, 4^9)$, then π'_n is one of the following sequences:

$n-1 = 7:$ $(5^2, 4^5)$;

$n-1 = 8:$ $(6, 4^7), (5^2, 4^6), (6, 5, 4^5, 3), (6, 5^2, 4^5), (6^2, 4^6), (5^3, 4^4, 3), (5^4, 4^4)$, (5^8);

$n-1 = 9:$ $(7, 4^7, 3), (6, 4^8), (7, 5, 4^7), (5, 4^7, 3), (5^2, 4^7), (5^2, 4^5, 3^2), (5^3, 4^5, 3)$, $(5^4, 4^5), (5^8, 4)$,$(6, 4^6, 3^2), (6, 5, 4^6, 3), (6, 5^2, 4^6), (6, 5^8), (6^2, 4^7)$;

$n-1 = 10:$ $(5, 4^8, 3)$, $(5^2, 4^8)$, (5^{10}), $(6, 4^9)$.

Hence π is one of the following sequences:

$n = 8:$ $(6^2, 5^2, 4^4), (6, 5^4, 4^3)$;

$n = 9:$ $(7, 5^3, 4^5)$, $(6^2, 5^2, 4^5)$,$(6, 5^4, 4^4)$, $(5^6, 4^3)$, $(7, 6, 5, 4^6)$, $(7, 6^2, 5, 4^5)$, $(7^2, 5^2, 4^5)$, $(6^3, 4^6)$, $(6^4, 4^5)$, $(6^2, 5^4, 4^3)$, $(6, 5^6, 4^2)$,$(6^3, 5^2, 4^4)$,$(6^4, 5^4, 4)$;

$n = 10$: $(8, 5^2, 4^7)$, $(7, 5^3, 4^6)$, $(8, 6, 5^2, 4^6)$, $(8, 5^4, 4^4, 3)$, $(6, 5^2, 4^7)$, $(6^2, 5^2, 4^6)$, $(6, 5^4, 4^5)$, $(5^6, 4^4)$, $(6^2, 4^8)$, $(6^3, 4^7)$, $(6^4, 4^6)$, $(6^3, 5^2, 4^4, 5)$, $(6^2, 5^4, 4^4)$, $(6, 5^6, 4^3)$, $(5^8, 4^2)$,$(6^4, 5^4, 4^2)$,$(6^3, 5^6, 4)$, $(7, 5, 4^8)$, $(7, 6, 5, 4^7)$, $(7, 6^2, 5, 4^6)$, $(7, 6, 5^3, 4^5)$, $(7, 5^5, 4^4)$, $(7, 6^3, 5^5, 4)$, $(6^5, 5^4, 4)$, $(7^2, 5^2, 4^6)$;

$n = 11$: $(6, 5^2, 4^8)$, $(6^2, 5^2, 4^7)$, $(6, 5^4, 4^6)$, $(5^6, 4^5)$, $(6^4, 5^6, 4)$, $(7, 5^3, 4^7)$.

It is easy to check that all of those are potentially $K_6 - 3K_2$-graphic.

Case 3. $d_n = 3$. Consider the residual sequence $\pi'_n = (d'_1, d'_2, \ldots, d'_{n-1})$, where $d'_3 \ge 4$ and $d'_{n-1} \ge 3$. If π'_n satisfies (1)-(6), then by the induction hypothesis, π'_n is potentially $K_6 - 3K_2$-graphic, and hence so is π.

If π'_n does not satisfy (1), i.e., $d'_6 \le 3$, then $d_3 = 4$. Since $d'_3 \ge 4$, we have $d_9 = 3$ or $7 \le n \le 8$. If $n = 7$, then $\pi = (d_1, d_2, 4^4, 3)$, where $4 \le d_2 \le d_1 \le 5$. Since $\sigma(\pi)$ is even, $\pi = (5, 4^5, 3)$, which is impossible by (6). If $n = 8$, then $\pi = (d_1, d_2, 4^4, d_7, 3)$, where $3 \le d_7 \le d_2 \le d_1 \le 6$. Hence, π is one of following sequences:

$$(5^2, 4^4, 3^2), (4^6, 3^2), (5, 4^6, 3), (6, 4^5, 3^2), (6^2, 4^4, 3^2).$$

It is easy to see that $\pi = (6^2, 4^4, 3^2)$ is potentially $K_6 - 3K_2$-graphic and others are in contradiction with (5) and (6). Suppose $n \ge 9$. If $d_2 = 4$, by

$d_9 = 3$, then $\pi = (d_1, 4, 4^4, d_7, d_8, 3^{n-8})$, where $4 \geq d_7 \geq d_8 \geq 3$. Consider the sequence $\rho_1 = (d_1 - 4, d_7, d_8, 3^{n-8})$. Clearly, $\sigma(\rho_1) = \sigma(\pi) - 24$ is even. Let $\rho_1' = (s_1, s_2, \ldots, s_{n-6})$ be the residual sequence of ρ_1 by laying off $d_1 - 4$. Clearly, $h \geq s_1 \geq s_2 \geq \cdots \geq s_{n-6} \geq h-1$, where $h \in \{3, 4\}$. By $\pi \neq (4^7, 3^2), (4^8, 3^2), (5, 4^5, 3^3), (6, 4^6, 3^2)$, then ρ_1' is not one of the following sequence:

$$(4, 3^2), (3^2, 2), (4^2, 3^2).$$

By Theorem 2.4, ρ_1' is graphic. Hence, by Theorem 2.1, ρ_1 is graphic. Let G_1^* be a realization of ρ_1 such that $d_{G_1^*}(v_1) = d_1 - 4$, $d_{G_1^*}(v_7) = d_7$, $d_{G_1^*}(v_8) = d_8$ and $d_{G_1^*}(v_i) = 3$ where $i = 9, 10, \ldots, n$. In G_1^*, we add new vertices $v_2, v_3, \ldots, v_6$, and new edges

$$v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_2v_3, v_2v_4, v_2v_5, v_2v_6, v_3v_5, v_3v_6, v_4v_5, v_4v_6,$$

and denote
$G = G_1^* + \{v_2, v_3, \ldots, v_6\} +$

$$\{v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_2v_3, v_2v_4, v_2v_5, v_2v_6, v_3v_5, v_3v_6, v_4v_5, v_4v_6\}.$$

It is easy to see that G is a realization of π and $K_6 - 3K_2 \subseteq G[\{v_1, v_2, \ldots, v_6\}]$. Thus, π is potentially $K_6 - 3K_2$-graphic. If $d_2 \geq 5$, then $d_7 = 3$, i.e., $\pi = (d_1, d_2, 4^4, 3^{n-6})$. Consider the sequence $\rho_2 = (d_1 - 4, d_2 - 4, 3^{n-6})$, where $n - 6 \geq d_1 - 4 \geq d_2 - 4 \geq 0$. Since $\pi \neq (5, 4^5, 3^3)$ and Lemma 2.8, ρ_2 is graphic. Let G_2^* be a realization of ρ_2 such that $d_{G_2^*}(v_1) = d_1 - 4$, $d_{G_2^*}(v_2) = d_2 - 4$, and $d_{G_2^*}(v_i) = 3$, $i = 7, 8, \ldots, n$. In G_2^*, we add new vertices v_3, v_4, v_5, v_6, and new edges

$$v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_2v_3, v_2v_4, v_2v_5, v_2v_6, v_3v_5, v_3v_6, v_4v_5, v_4v_6,$$

and denote
$G = G_2^* + \{v_3, \ldots, v_6\} +$

$$\{v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_2v_3, v_2v_4, v_2v_5, v_2v_6, v_3v_5, v_3v_6, v_4v_5, v_4v_6\}.$$

It is easy to see that G is a realization of π and $K_6 - 3K_2 \subseteq G[\{v_1, v_2, \ldots, v_6\}]$. Thus, π is potentially $K_6 - 3K_2$-graphic.

If $d_1' = n - 2$, by $d_1 \leq n - 2$, then $d_4 = d_1 = n - 2$. Clearly, π_n' satisfies (2) and (3).

If π_n' does not satisfy (4), by $d_{n-1}' \geq 3$, then $\pi_n' = (d_1', d_2', d_3', 4^i, 3^{n-i-4})$, where $d_3' \leq d_2' \leq d_1' \leq n - 2$ and $d_1' + d_2' + d_3' = 3(n - 2)$. Then $d_1' = d_2' = d_3' = n - 2$. If $n - 2 \geq 6$, then $\pi = ((n-1)^3, 4^i, 3^{n-i-3})$, a contradiction. If $n - 2 = 5$, then $\pi_n' = (5^3, 3^3)$ or $\pi_n' = (5^3, 4^2, 3)$. Thus π is one of the following sequence: $(6^3, 3^4)$, $(6^3, 4^2, 3^2)$, $(6^2, 5^2, 4, 3^2)$, $(6, 5^4, 3^2)$, a contradiction. Hence, π_n' satisfies (4).

If π'_n does not satisfy (5), by $\pi \neq (5,4^7,3),(5^2,4^5,3^2),(5^3,4^4,3),(5^3,4^5,3),(5,4^8,3)$, then π'_n is one of the following sequences:

$$(4^6,3^2),(4^7,3^2),(4^9),(4^8,3^2),(4^{10}).$$

So π is one of the following sequences:

$$(5^3,4^3,3^3),(5^2,4^6,3^2),(5^3,4^4,3^3),(5^3,4^6,3),$$
$$(5,4^9,3),(5^2,4^7,3^2),(5^3,4^5,3^3),(5^3,4^7,3).$$

It is easy to check that the eight sequences above are all potentially K_6-3K_2-graphic.

If π'_n does not satisfy (6), by $d'_{n-1}\geq 3$ and $\pi \neq (6,5,4^5,3)$, $(6,5,4^6,3)$, $(7,4^7,3)$, then π'_n is one of the following sequences:

$n-1=7:$ $(5,4^5,3),(5^2,4^5)$;

$n-1=8:$ $(6,4^7),(5,4^6,3),(5^2,4^6),(6,5,4^5,3),(6,5^2,4^5),(6^2,4^6),(5^2,4^4,3^2),(5^3,4^4,3),(5^4,4^4),(5^8)$;

$n-1=9:$ $(7,4^7,3),(6,4^8),(7,5,4^7),(5,4^7,3),(5,4^5,3^3),(5^2,4^7),(5^2,4^5,3^2),(5^3,4^5,3),(5^4,4^5),(5^8,4),(6,4^6,3^2),(6,5,4^6,3),(6,5^2,4^6),(6,5^8),(6^2,4^7)$;

$n-1=10:$ $(5,4^8,3),(5^2,4^8),(5^{10}),(6,4^9)$;

Then π is one of the following sequences:

$n=8:$ $(5^4,4^2,3^2),(6,5^2,4^3,3^2),(6^2,5,4^4,3),(6,5^3,4^3,3),(5^5,4^2,3)$;

$n=9:$ $(7,5^2,4^5,3),(7,5^2,4^3,3^3)$ $(7,5,4^5,3^2),(7,4^7,3),(6,5^2,4^4,3^2),(6,5,4^6,3),(5^4,4^3,3^2),(6^2,5,4^5,3),(6,5^3,4^4,3),(5^5,4^3,3),(7,6,5,4^4,3^3),(7,5^3,4^3,3^2),(7,6,4^6,3),(7,6^2,4^5,3),(7,5^4,4^3,3),(7,6,5^2,4^4,3),(7^2,5,4^5,3),(6^2,5,4^3,3^3),(5^5,4{\cdot}3^3),(6^2,4^5,3^2),(6,5^3,4^2,3^3),(6^3,4^4,3^2),(6^2,5^2,4^3,3^2),(6,5^4,4^2,3^2),(5^6,4,3^2),(6^3,5,4^4,3),(6^2,5^3,4^3,3),(6,5^5,4^2,3),(5^7,4,3),(6^3,5^5,3)$;

$n=10$: $(8,5^2,4^5,3^2),(8,5,4^7,3),(7,5^2,4^6,3),(8,6,5,4^6,3),(8,5^3,4^5,3),(6,5^2,4^5,3^2),(6,5,4^7,3),(5^4,4^4,3^2),(6,5^2,4^3,3^4),(6,5,4^5,3^3),(6,4^7,3^2),(5^4,4^2,3^4),(6^2,5,4^6,3),(6,5^3,4^5,3),(5^5,4^4,3),(6^2,5,4^4,3^3),(6,5^3,4^3,3^3),(5^5,4^2,3^3),(6^2,4^6,3^2),(6^3,4^5,3^2),(6^2,5^2,4^4,3^2),(6,5^4,4^3,3^2),(5^6,4^2,3^2),(6^3,5,4^5,3),(6^2,5^3,4^4,3),(6,5^5,4^3,3),(5^7,4^2,3),(6^3,5^5,4,3),(6^2,5^7,3),(7,5^2,4^4,3^3),(7,4^8,3),(7,5,4^6,3^2),(7,6,5,4^5,3^2),(7,5^3,4^4,3^2),(7,6,4^7,3),(7,6^2,4^6,3),(7,5^2,5^2,4^4,3),(7,6,5^2,4^5,3),(7,6^2,5^6,3),(6^4,5^5,3),(7^2,5,4^6,3)$;

$n=11:$ $(6,5^2,4^6,3^2),(5^4,4^5,3^2),(6,5,4^7,4,3),(6^2,5,4^7,3),(6,5^3,4^6,3),(5^2,5^3,4^5,3),(6^3,5^7,3),(7,5^2,4^7,3)$.

It is easy to check that all of those are potentially K_6-3K_2-graphic.

Case 4. $d_n=2$ Consider the residual sequence $\pi'_n=(d'_1,d'_2,\ldots,d'_{n-1})$, where $d'_4\geq 4$ and $d'_{n-1}\geq 2$. If π'_n satisfies (1)-(6), then by the induction hypothesis, π'_n is potentially K_6-3K_2-graphic, and hence so is π.

If π'_n does not satisfy (1), then $d_2=4$. Let $\pi=(d_1,4^5,d_7,d_8\ldots,d_n)$, where $n-2\geq d_1\geq 4\geq d_7\geq d_8\geq\cdots\geq d_n=2$. If $d_1=4$, then $d_8\leq 3$.

If $d_1 \geq 5$, then $d_7 \leq 3$. Consider the sequence $\rho_3 = (d_1 - 4, d_7, \ldots, d_n)$. Clearly, $\sigma(\rho_3) = \sigma(\pi) - 24$ is even. Let $\rho_3' = (s_1, s_2, \ldots, s_{n-6})$ be the residual sequence of ρ_3 by laying off $d_1 - 4$. If $d_1 = 4$, then $4 \geq s_1$ and $3 \geq s_2 \geq \cdots \geq s_{n-6} \geq 2$; If $d_1 \geq 5$, then $h \geq s_1 \geq s_2 \geq \cdots \geq s_{n-6} \geq h-1$, where $h \in \{2, 3\}$. Since π is not one of the following sequences:

$$(4^6, 2), (4^6, 2^2), (4^6, 3^2, 2), (4^6, 4, 2),$$

$$(4^6, 4, 2^2), (4^6, 4, 2^3), (4^6, 4, 3^2, 2), (5, 4^6, 3, 2).$$

ρ_3' is not one of the following sequences:

$$(2), (2^2), (3^2, 2), (4, 2), (4, 2^2), (4, 2^3), (4, 3^2, 2).$$

By $\rho_3' \notin A \cup B \cup S$ and Theorem 2.1, ρ_3 is graphic. Let G_3^* be a realization of ρ_3 such that $d_{G_3^*}(v_1) = d_1 - 4$, $d_{G_3^*}(v_i) = d_i$, where $i = 7, 8, \ldots, n$. In G_3^*, we add new vertices $v_2, v_3, \ldots, v_6$, and new edges

$$v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_2v_3, v_2v_4, v_2v_5, v_2v_6v_3v_5, v_3v_6, v_4v_5, v_4v_6,$$

and denote
$G = G_3^* + \{v_2, v_3, \ldots, v_6\} +$

$$\{v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_2v_3, v_2v_4, v_2v_5, v_2v_6v_3v_5, v_3v_6, v_4v_5, v_4v_6\}.$$

It is easy to see that G is a realization of π and $K_6 - 3K_2 \subseteq G[\{v_1, v_2, \ldots, v_6\}]$. Thus, π is potentially $K_6 - 3K_2$-graphic.

If $d_1' = n-2$, by $d_1 \leq n-2$, then $d_1 = d_3 = n-2$. If $d_1' = n-2$ and $d_2' = 4$, by $d_6 \geq 4$, then $n = 7$ and $\pi_n' = (5, 4^5)$, a contradiction. If π_n' does not satisfy (2), then π_n' is just $(6, 5^2, 4^4)$. Then π is one of the following sequence: $(7, 6, 5, 4^4, 2), (7, 5^3, 4^3, 2), (6^3, 4^4, 2)$. It is easy to check that π is potentially $K_6 - 3K_2$-graphic.

If $d_1' = d_3' = n-2$, by $d_1 \leq n-2$, then $d_1 = d_2 = \cdots = d_5 = n-2$. If $n \geq 8$, then $d_4' \geq 5$. If $n = 7$, by $d_6 \geq 4$, then $\pi = (5^5, 5, 2)$. It is easy to see that $\pi_n' = (5^4, 4^2)$ satisfies (3). Therefore, π_n' satisfies (3).

If π_n' does not satisfy (4), then $\pi_n' = (d_1', d_2', d_3', 4^i, 3^j, 2^{n-1-i-j-3})$, where $i \geq 1$ and $d_1' + d_2' + d_3' = (n-1) + 2(i+j) + (n-1-i-j-3) + 3 = 2(n-1) + (i+j)$.

If $d_2' \geq 6$, then $\pi = (d_1, d_2, d_3, 4^i, 3^j, 2^{n-i-j-3})$, where $d_1 + d_2 + d_3 = d_1' + d_2' + d_3' + 2 = n + 2(i+j) + (n-i-j-3) + 3$, a contradiction.

If $d_2' = 5$, then $\pi_n' = (d_1', 5, d_3', 4^i, 3^j, 2^{n-4-i-j})$, where $4 \leq d_3' \leq d_1' \leq n-2$ and $d_1' + 5 + d_3' = 2(n-1) + i + j$. Then $d_1' + d_3' = 2n + i + j - 7 \leq (n-2) + 5$, i.e., $n \leq 10 - i - j$. Since $i \geq 1$ and $j \geq 0$, $n \leq 9$. If $n = 9$, then $i = 1$ and $j = 0$. Then $d_1' + d_3' = 2n + i + j - 7$ is even. It is easy to see that $\sigma(\pi_n')$ is odd, a contradiction. If $n = 8$, then $i = 1, j = 1$ or $i = 2, j = 0$. $\sigma(\pi_n')$

is odd for $n=8$ and $i=j=1$, a contradiction. If $n=8$ and $i=2, j=0$, by $d_1' \le n-2$, then $\pi_n' = (6,5^2,4^2,2^2)$. Hence π is one of the following sequences:

$$(7,6,5,4^2,2^3),(6^3,4^2,2^2),(7,5^3,4,2^3),$$

which is a contradiction. If $n=7$, by even $\sigma(\pi_n')$, then $i=2, j=1$,i.e., $\pi_n' = (5^3,4^2,3)$. Then π is one of the following sequences:

$$(6^2,5,4^2,3,2),(5^5,3,2),(6,5^3,4,3,2),$$

a contradiction.

If $d_2' = 4$, then $d_1' + 8 = 2(n-1)+i+j$. By $d_1' \le n-2$, $n \le 8-i-j$. Since $n \ge 7$ and $i \ge 1$, $n=7$, $i=1$, $j=0$. Thus, $\pi_n' = (5,4^3,2^2)$, a contradiction. Therefore, π_n' satisfies (4).

If π_n' does not satisfy (5), by $\pi \ne (4^8,2),(5,4^6,3,2),(4^8,2^2),(5^2,4^5,2)$, $(5^2,4^6,2)$, $(4^9,2)$, $(5,4^7,3,2)$, $(5^2,4^7,2)$, $(4^9,2^2)$, $(4^{10},2)$, $(5^2,4^8,2)$, then π_n' is one of the following sequences:

$$(4^6,2),(4^6,2^2),(4^6,3^2),(4^6,3^2,2),(4^9,2^2),(4^{10},2),(4^7,2),(4^7,3^2),$$

$$(4^7,2^2),(4^8,2),(4^7,2^3),(4^7,3^2,2),(4^8,2^2),(4^8,3^2),(4^9,2).$$

Then π is one of the following sequences:

$$(5^2,4^4,2^2),(5^2,4^4,2^3),(5^2,4^4,3^2,2),(5,4^6,3,2^2),(5^2,4^4,3^2,2^2),(5^2,4^7,2^3),$$

$$(5^2,4^8,2^2),(5^2,4^5,2^2),(5^2,4^5,3^2,2),(5^2,4^5,2^3),(5^2,4^6,2^2),(5^2,4^5,2^4),$$

$$(5,4^7,3,2^2),(5^2,4^5,3^2,2^2),(5^2,4^6,2^2,2),(5,4^8,3,2),(5^2,4^6,3^2,2),(5^2,4^7,2^2).$$

It is easy to check that the above sequences are potentially $K_6 - 3K_2$-graphic.

If π_n' does not satisfy (6), by $d_{n-1}' \ge 2$ and $\pi \ne (6^2,4^5,2),(6,4^6,2),(6,4^7,2)$ $(6,4^6,2^2)$, $(7,5,4^6,2)$, $(7,4^6,3,2)$, $(6,4^8,2)$, $(8,4^8,2)$, $(8,4^7,2^2)$, then π_n' is one of the following sequences:

$n-1=7$: $(5,4^5,3),(5^2,4^5)$;

$n-1=8$: $(6,4^6,2),(6,4^5,3^2),(5,4^6,3),(5^2,4^6),(6,5,4^5,3),(6,5^2,4^5),(6^2,4^6),$
$(5,4^5,3,2),(5^2,4^5,2),(5^2,4^4,3^2),(5^3,4^4,3),(5^4,4^4),(5^8),(6^2,4^5,2)$;

$n-1=9$: $(7,4^7,3),(7,4^6,3,2),(6,4^7,2),(6,4^8),(7,5,4^7),(7,5,4^6,2),(5,4^7,3),$
$(5,4^6,3,2),(5,4^5,3^3),(5^2,4^7),(5^2,4^6,2),(5^2,4^5,3^2),(5^3,4^5,3),$
$(5^4,4^5),(5^8,4),(6,4^6,3^2),(6,5,4^6,3),(6,5^2,4^6),(6,5^8),(6^2,4^7)$;

$n-1=10$: $(8,4^8,2),(8,4^7,2^2),(5,4^8,3),(5,4^7,3,2),(5^2,4^8),(5^2,4^7,2),$
$(5^{10}),(6,4^9),(6,4^8,2)$.

Then π is one of the following sequences:

$n=8$: $(6,5,4^4,3,2)$, $(5^3,4^3,3,2)$, $(6,5^2,4^4,2)$, $(5^4,4^3,2)$;

$n=9$: $(7,5,4^5,2^2)$, $(7,5,4^4,3^2,2)$, $(6,5,4^5,3,2)$, $(5^3,4^4,3,2)$, $(6^2,4^6,2)$,

$(6,5^2,4^5,2)$, $(7,6,4^5,3,2)$, $(7,5^2,4^4,3,2)$, $(7,6,5,4^5,2)$, $(6^3,4^5,2)$, $(7,5^3,4^4,2)$, $(7^2,4^6,2)$, $(6,5,4^4,3,2^2)$, $(5^3,4^3,3,2^2)$, $(6^2,4^5,2^2)$, $(6,5^2,4^4,2^2)$, $(5^4,4^3,2^2)$, $(6^2,4^4,3^2,2)$, $(6,5^2,4^3,3^2,2)$, $(5^4,4^2,3^2,2)$, $(6^2,5,4^4,3,2)$, $(6,5^3,4^3,3,2)$, $(5^5,4^2,3,2)$, $(6^2,5^2,4^4,2)$, $(6,5^4,4^3,2)$, $(5^6,4^2,2)$, $(6^2,5^6,2)$, $(7^2,4^5,2^2)$;

$n=10$: $(8,5,4^6,3,2)$, $(8,5,4^5,3,2^2)$, $(7,5,4^6,2^2)$, $(7,5,4^7,2)$, $(8,6,4^7,2)$, $(8,5^2,4^6,2)$, $(8,6,4^6,2^2)$, $(8,5^2,4^5,2^2)$, $(6,5,4^6,3,2)$, $(5^3,4^5,3,2)$, $(6,5,4^5,3,2^2)$, $(5^3,4^4,3,2^2)$, $(6,4^7,2^2)$, $(6,5,4^4,3^3,2)$, $(5^3,4^3,3^3,2)$, $(6,4^6,3^2,2)$, $(6^2,4^7,2)$, $(6,5^2,4^6,2)$, $(5^4,4^5,2)$, $(6^2,4^6,2^2)$, $(6,5^2,4^5,2^2)$, $(5^4,4^4,2^2)$, $(6^2,4^5,3^2,2)$, $(6,5^2,4^4,3^2,2)$, $(5^4,4^3,3^2,2)$, $(6^2,5,4^5,3,2)$, $(6,5^3,4^4,3,2)$, $(5^5,4^3,3,2)$, $(6^2,5^2,4^5,2)$, $(6,5^4,4^4,2)$, $(5^6,4^3,2)$, $(6^2,5^6,4,2)$, $(6,5^8,2)$, $(7,5,4^5,3^2,2)$, $(7,4^7,3,2)$, $(7,6,4^6,3,2)$, $(7,5^2,4^5,3,2)$, $(7,6,5,4^6,2)$, $(6^3,4^6,2)$, $(7,5^3,4^5,2)$, $(7,6,5^7,2)$, $(6^3,5^6,2)$, $(7^2,4^7,2)$;

$n=11$: $(9,5,4^7,2^2)$, $(9,5,4^6,2^3)$, $(6,5,4^7,3,2)$, $(6,4^9,2)$, $(5^3,4^6,3,2)$, $(6,5,4^6,3,2^2)$, $(6,4^8,2^2)$, $(5^3,4^5,3,2^2)$, $(6^2,4^8,2)$, $(6,5^2,4^7,2)$, $(5^4,4^6,2)$, $(6^2,4^7,2^2)$, $(6,5^2,4^6,2^2)$, $(5^4,4^5,2^2)$, $(6^2,5^8,2)$, $(7,5,4^8,2)$, $(7,5,4^7,2^2)$.

It is easy to check that all of those are potentially K_6-3K_2-graphic.

Case 5. $d_n=1$.

Consider the residual sequence $\pi'_n=(d'_1,d'_2,\ldots,d'_{n-1})$, where $d'_5\geq 4$ and $d'_{n-1}\geq 1$. If π'_n satisfies (1)-(6), then by the induction hypothesis, π'_n is potentially K_6-3K_2-graphic, and hence so is π.

If π'_n does not satisfy (1), i.e., $d'_6\leq 3$, then $d_1=4$ and $d_7\leq 3$. Then $\pi=(4^6,d_7,d_8,\ldots,d_n)$, where $3\geq d_7\geq d_8\geq\cdots\geq d_n=1$. Consider the sequence $\rho_4=(d_7,\ldots,d_n)$. Clearly, $\sigma(\rho_4)=\sigma(\pi)-24$ is even. Since π is not one of the following sequences:

$$(4^6,3,1),(4^6,3,2,1),(4^6,3^3,1),(4^6,3^2,1^2),$$

$\rho_4\neq(3,1),(3,2,1),(3^3,1),(3^2,1^2)$. By Lemma 2.7, ρ_4 is graphic. Let G_4^* be a realization of ρ_4. Denote

$$G=G_4^*\cup(K_6-3K_2).$$

It is easy to see that G is a realization of π and K_6-3K_2 is subgraph of G. Thus, π is potentially K_6-3K_2-graphic.

If $d'_1=n-2$, by $d_1\leq n-2$, then $d_2=n-2$. If $d'_2\leq 4$, then $n=7$ and $d_3\leq 4$, i.e., $\pi=(5,5,4^4,1)$, a contradiction. Suppose that $n\geq 8$ or $d_3\geq 5$. If π'_n does not satisfy (2), then π'_n is one of the following sequences:

$$(6,5^2,4^4),(6,5,4^4,3),(7,6,5,4^5),(7,6,4^5,3).$$

By (2), $\pi \neq (7,5^2,4^4,1),(7,5,4^4,3,1),(8,6,5,4^5,1)$, $(8,6,4^5,3,1)$, thereby π is one of the following sequences:

$$(6^2,5,4^4,1),(6^2,4^4,3,1),(7^2,5,4^5,1),(7^2,4^5,3,1).$$

It is easy to see that all of those are potentially K_6-3K_2-graphic.

If π'_n does not satisfy (3), then $d'_3=n-2$ and $d'_4\leq 4$. By $d_1=d_4=n-2$, $n-3\leq d'_4\leq 4$ and $n\leq 7$, then $\pi=(5^4,4^2,1)$, a contradiction. Hence, π'_n satisfies (3).

If π'_n does not satisfy (4), then $\pi'_n=(d'_1,d'_2,d'_3,4^i,3^j,2^k,1^{n-1-i-j-k-3})$, where $i\geq 2$ and $d'_1+d'_2+d'_3=(n-1)+2(i+j)+k+3$.

If $d'_1\geq 6$, then $\pi=(d_1,d_2,d_3,4^i,3^j,2^k,1^{n-i-j-k-3})$, where $d_1+d_2+d_3=d'_1+d'_2+d'_3+1=n+2(i+j)+k+3$, a contradiction. If $d'_1=4$, then $\pi'_n=(4^3,4^i,3^j,2^k,1^{n-1-i-j-k-3})$, where $i\geq 2$, $j\geq 0$ and $k\geq 0$. Then $12=n-1+2(i+j)+k+3$, i.e., $n=10-2(i+j)-k$. Since $n\geq 7$, it is impossible. If $d'_1=5$, then $\pi'_n=(5,d'_2,d'_3,4^i,3^j,2^k,1^{n-1-i-j-k-3})$, where $4\leq d'_3\leq d'_2\leq n-2$ and $5+d'_2+d'_3=n-1+2(i+j)+k+3$. Then $d'_2+d'_3=n+2(i+j)+k-3\leq 10$, i.e., $n\leq 13-2(i+j)-k$. If $i=2$, by $n\geq 7$, $j\geq 0$ and $k\geq 0$, then $j=k=0, n\leq 9$ or $j=0,k=1,n\leq 8$ or $j=1,k=0,n=7$. Then π'_n is one of the following sequence:

$$(5^3,4^2,1^3),(5^2,4^3,1^2),(5,4^4,1),(5^3,4^2,2,1),(5^2,4^3,2),(5^3,4^2,3).$$

Thus, π is one of the following sequences:

$(6,5^2,4^2,1^4),(5^4,4,1^4),(6,5,4^3,1^3),(5^3,4^2,1^3),$
$(6,4^4,1^2),(5^2,4^3,1^2),(6,5^2,4^2,2,1^2),(5^4,4,2,1^2),$
$(6,5,4^3,2,1),(5^3,4^2,2,1),(6,5^2,4^2,3,1)(5^4,4,3,1),$

which is a contradiction. If $i\geq 3$, by $n\geq 7$, then $i=3$, $j=k=0$, $n=7$. Thus, $\pi'_n=(5^3,4^3)$, a contradiction. Therefore, π'_n satisfies (4).

If π'_n does not satisfy (5), by $\pi\neq(4^7,1^2),(5,4^5,3^2,1),(4^7,3,1),(4^7,2,1^2),(4^7,3,2,1),(4^7,3^2,1^2)$, $(4^7,3,1^3)$, $(4^8,3,1^3)$,$(4^8,1^4)$, $(4^9,2,1^2)$,$(5,4^8,2,1^3)$, $(4^{10},1^2)$,$(4^9,1^4)$,$(5,4^6,1)$, $(5,4^6,2,1)$,$(5,4^7,1)$, $(5,4^6,3,1^2)$, $(4^8,1^2)$,$(5,4^6,3^2,1)$,$(4^8,3,2,1)$,$(5,4^7,1^3)$, $(5,4^7,3,1^2)$,$(4^9,1^2)$,$(4^9,3,1)$,$(5,4^8,2,1)$, $(5,4^9,1)$, then the sequence π'_n is one of the following sequences:
$(4^6,2),(4^6,2^2),(4^6,3,1),(4^6,3,2,1),(4^6,3^2,2),(4^6,3^3,1),(4^6,3^2,1^2),(4^7,3^2,1^2),(4^7,3,1^3),(4^8,2,1^2),(4^8,3,2,1),(4^9,1^2),(4^9,2^2),(4^9,3,1),(4^{10},2),(4^8,3,1^3),(4^8,1^4),(4^9,2,1^2),(4^{10},1^2),(4^9,1^4),(4^7,1^2),(4^7,3^2),(4^7,2^2),(4^8,2),(4^9),(4^7,2,1^2),(4^7,2^3),(4^7,3,2,1),(4^7,3^2,2),(4^8,2^2),(4^8,3^2),(4^9,2),(4^{10})$.

Then π is one of the following sequences:
$(5,4^5,2,1),(5,4^5,2^2,1),(5,4^5,3,1^2),(5,4^5,3,2,1^2)$,$(5,4^5,3^2,2,1)$,$(5,4^5,3^3,1^2)$ $(5,4^5,3^2,1^3)$, $(4^7,3,1^3)$,$(5,4^6,3^2,1^3)$,$(5,4^6,3,1^4)$,$(4^8,1^4)$,$(5,4^7,2,1^3)$, $(5,4^7,3,2,1^2)$,$(5,4^8,1^3)$,$(5,4^8,2^2,1)$, $(5,4^8,3,1^2)$,$(5,4^9,2,1)$,$(5,4^7,3,1^4)$,$(4^9,1^4)$, $(5,4^7,1^5)$,$(5,4^8,2,1^3)$,$(5,4^9,1^3)$, $(5,4^8,1^5)$,$(5,4^6,1^3)$, $(4^8,3,1)$, $(5,4^6,2^2,1)$, $(5,4^7,2,1)$, $(5,4^8,1)$, $(5,4^6,2,1^3)$, $(5,4^6,2^3,1)$, $(5,4^6,3,2,1^2)$, $(4^8,2,1^2)$, $(5,4^6,$

$3^2,2,1),(4^8,3,2,1),(5,4^7,2^2,1),(5,4^7,3^2,1)(4^9,3,1)$.

It is easy to check that the above sequences are potentially K_6-3K_2-graphic.

Suppose that π'_n does not satisfy (6).

Assume $\pi'_n=(n-1-i,n-1-j,4^5,2^{n-1-i-j-5},1^{i+j-2})$. If $n-1-i\geq 6$, then $\pi=(n-i,n-(j+1),4^5,2^{n-i-(j+1)-5},1^{i+(j+1)-2})$ or $\pi=(n-(i+1),n-j,4^5,2^{n-(i+1)-j-5},1^{(i+1)+j-2})$, a contradiction. If $n-1-i\leq 5$, then $n-1-i-j-5<0$, a contradiction. Thus, $\pi'_n\neq(n-1-i,n-1-j,4^5,2^{n-1-i-j-5},1^{i+j-2})$. Similarly, $\pi'_n\neq(n-1-i,n-1-j,4^6,2^{n-1-i-j-6},1^{i+j-2})$.

If $\pi'_n=(n-3,4^8,2,1^{n-11})$, then $\pi=(n-2,4^8,2,1^{n-10})$, a contradiction. Hence, we can prove π'_n is not one of the following sequences similarly:

$$(n-3,4^7,3,1^{n-10}),(n-3,4^7,2^2,1^{n-11}),(n-3,4^7,1^{n-9}),(n-3,4^6,3,2,1^{n-10}),$$

$$(n-3,4^6,2,1^{n-9}),(n-3,4^5,3^2,1^{n-9}),(n-4,4^7,2,1^{n-10}),(n-4,4^8,1^{n-10}),$$

$$(n-3,5,4^7,1^{n-10}),(n-3,5,4^6,2,1^{n-10}).$$

Assume $\pi'_n=(n-4,5,4^6,1^{n-1-8})$. If $n\geq 10$, then $\pi=(n-3,5,4^6,1^{n-8})$, a contradiction. If $n\leq 9$, i.e., $\pi'_n=(5^2,4^6)$, then $\pi=(5^3,4^5,1)$ or $\pi=(6,5,4^6,1)$, a contradiction. Hence, we can show π'_n is not one of the following sequences similarly:

$$(n-4,4^6,3,1^{n-9}),(n-5,4^7,1^{n-9}),(n-3,6,4^6,1^{n-1-8}).$$

If $\pi'_n=(n-3,5,4^5,1^{n-8})$, then $\pi=(n-2,5,4^5,1^{n-7})$ for $n\geq 9$ or $\pi=(5^3,4^4,1)$ for $n=8$. By (6), $\pi=(n-2,5,4^5,1^{n-7})$, which is a contradiction. It is easy to check $\pi=(5^3,4^4,1)$ is potentially K_6-3K_2-graphic. Similarly, if π'_n is one of the following sequences:

$$(n-2,5,4^5,1^{n-7}),(n-3,4^5,3,1^{n-8}),(n-4,4^6,1^{n-8}),$$

$$(n-3,5^2,4^5,1^{n-1-8}),(n-3,5,4^5,3,1^{n-1-8}),$$

then π is potentially K_6-3K_2-graphic, where π satisfies (6).

Then π'_n is one of the following sequences:

$n-1=8$: $(5,4^5,3,2)(5^2,4^5,2),(5^2,4^4,3^2),(5^3,4^4,3),(5^4,4^4),(5^8),(6^2,4^5,2)$;

$n-1=9$: $(5,4^7,3),(5,4^6,3,2),(5,4^6,2,1),(5,4^5,3^3),(5,4^5,3^2,1),(5^2,4^7),$
$(5^2,4^6,2),(5^2,4^5,3^2),(5^2,4^5,3,1),(5^3,4^5,3),(5^3,4^5,1),(5^4,4^5),$
$(5^8,4),(6,4^6,3^2),(6,5,4^6,3),(6,5^2,4^6),(6,5^8),(6^2,4^7)$

$n-1=10$: $(5,4^8,3),(5,4^8,1),(5,4^7,3,2),(5,4^7,2,1),(5,4^6,3^2,1),(5,4^6,3,1^2),$
$(5^2,4^8),(5^2,4^7,2),(5^2,4^6,3,1),(5^2,4^6,1^2),(5^2,4^5,3,2,1),(5^3,4^6,1),$
$(5^{10}),(5^9,1),(6,4^9),(6,4^8,2),(6,4^7,3,1)$.

Since $\pi \neq (6,5,4^5,1)$, $(6,5,4^7,1)$,$(5^3,4^6,1)$, $(5^9,1)$,$(6,4^8,1^2)$, $(5^2,4^7,1^2)$, π is one of the following sequences:

$n=9$: $(6,4^5,3,2,1)$, $(5^2,4^4,3,2,1)$, $(6,5,4^5,2,1)$, $(5^3,4^4,2,1)$, $(6,5,4^4,3^2,1)$, $(5^3,4^3,3^2,1)$,$(6,5^2,4^4,3,1)$, $(5^4,4^3,3,1)$, $(6,5^3,4^4,1)$, $(5^5,4^3,1)$, $(6,5^7,1)$,$(6^2,4^5,1^2)$, $(7,6,4^5,2,1)$;

$n=10$: $(6,4^7,3,1)$, $(5^2,4^6,3,1)$, $(6,4^6,3,2,1)$, $(5^2,4^5,3,2,1)$, $(6,4^6,2,1^2)$, $(5^2,4^5,2,1^2)$,$(6,4^5,3^3,1)$, $(5^2,4^4,3^3,1)$, $(6,4^5,3^2,1^2)$, $(5^2,4^4,3^2,1^2)$, $(6,5,4^6,2,1)$, $(5^3,4^5,2,1)$, $(6,5,4^5,3^2,1)$, $(5^3,4^4,3^2,1)$, $(6,5,4^5,3,1^2$ $(5^3,4^4,3,1^2)$, $(6,5^2,4^5,3,1)$, $(5^4,4^4,3,1)$, $(6,5^2,4^5,1^2)$, $(5^4,4^4,1^2)$, $(6,5^3,4^5,1)$, $(5^5,4^4,1)$, $(6,5^7,4,1)$, $(7,4^6,3^2,1)$, $(7,5,4^6,3,1)$, $(6^2,4^6,3,1)$, $(7,5^2,4^6,1)$, $(6^2,5,4^6,1)$, $(7,5^8,1)$, $(6^2,5^7,1)$, $(7,6,4^7,1$

$n=11$: $(6,4^8,3,1)$, $(5^2,4^7,3,1)$,$(6,4^7,3,2,1)$, $(5^2,4^6,3,2,1)$, $(6,4^7,2,1^2)$, $(5^2,4^6,2,1^2)$, $(6,4^6,3^2,1^2)$,$(5^2,4^5,3^2,1^2)$,$(6,4^6,3,1^3)$, $(5^2,4^5,3,1^3)$ $(6,5,4^8,1)$, $(5^3,4^7,1)$, $(6,5,4^7,2,1)$, $(5^3,4^6,2,1)$,$(6,5,4^6,3,1^2)$,$(5^3,4^5,3,1^2)$, $(6,5,4^6,1^3)$, $(5^3,4^5,1^3)$, $(6,5,4^5,3,2,1^2)$, $(5^3,4^4,3,2,1^2)$, $(6,5^2,4^6,1^2)$, $(5^4,4^5,1^2)$, $(6,5^9,1)$, $(6,5^8,1^2)$, $(7,4^9,1)$, $(7,4^8,2,1)$, $(7,4^7,3,1^2)$.

It is easy to check that the above sequences are potentially K_6-3K_2-graphic. □

4. Application

In the remaining of this section, we will use the above theorem to find exact values of $\sigma(K_6-3K_2,n)$. Note that the value of $\sigma(K_6-3K_2,n)$ was determined by Yin and Li in [26] so another proof is given here.

Theorem 4.1 [26] Let $n \geq 6$. Then

$$\sigma(K_6-3K_2,n)=\begin{cases} 2\lfloor\frac{7n-13}{2}\rfloor, & if\ n\neq 8;\\ 44, & if\ n=8.\end{cases}$$

Proof. For $n=8$, since $\pi=(7,5^7)$ is not potentially K_6-3K_2-graphic, $\sigma(K_6-3K_2,n)\geq 7+5\times 7+2=44$. In order to prove $\sigma(K_6-3K_2,8)\leq 44$, it is enough to prove that if $\pi=(d_1,d_2,\ldots,d_8)\in GS_8$ with $\sigma(\pi)\geq 44$, then π is potentially K_6-3K_2-graphic. If $d_4=4$, then $\sigma(\pi)\leq 7\times 3+4\times 5<44$, a contradiction. Thus, $d_4\geq 5$. If $d_6=3$, then $d_4\leq 6$ and $\sigma(\pi)\leq 7\times 3+6\times 2+3\times 3=42<44$, a contradiction. Thus, $d_6\geq 4$. It is easy to check that $\pi\neq(5^4,4^4),(5^8)$. According to Theorem 3.1, $\pi\in GS_8$ with $\sigma(\pi)\geq 44$ is potentially K_6-3K_2-graphic.

Suppose $n\neq 8$. First, we prove that $\sigma(K_6-3K_2,n)\geq 2\lfloor\frac{7n-13}{2}\rfloor$. For n is odd, consider $\pi=((n-1)^3,4^{n-3})$. It is easy to see that $K_3+\frac{n-3}{2}K_2$ is unique realization of π. So π is graphic. By Theorem 3.1, π is not potentially K_6-3K_2-graphic. Thus, $\sigma(K_6-3K_2,n)\geq\sigma(\pi)+2=7n-13$, where n is odd. If n is even, then it is easy to see that $\pi=((n-1)^3,4^{n-4},3)$

is not potentially $K_6 - 3K_2$-graphic. Thus, $\sigma(K_6 - 3K_2, n) \geq \sigma(\pi) + 2 = 7n - 14$, where n is even.

Next, in order to prove $\sigma(K_6 - 3K_2, n) \leq 2\lfloor\frac{7n-13}{2}\rfloor$, it is enough to prove that if $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq 2\lfloor\frac{7n-13}{2}\rfloor$, then π is potentially $K_6 - 3K_2$-graphic.

We claim that $d_6 \geq 4$. By way of contradiction, we assume that $d_6 \leq 3$. By Theorem 2.2, then $\sum_{i=1}^{5} d_i \leq 20 + \sum_{j=6}^{n} \min\{5, d_j\} = 20 + 3(n-5)$. Thus, $\sigma(\pi) = \sum_{i=1}^{5} d_i + \sum_{j=6}^{n} d_j \leq 20 + 3(n-5) + 3(n-5) < 7n - 14$, a contradiction.

Clearly, $\sigma(K_6 - 3K_2, n) \geq 2\lfloor\frac{7n-13}{2}\rfloor$ implies $d_4 \geq 5$. It is easy to check that π is not one of the following sequences:

$$(n-1, 5^3, 4^4, 1^{n-8}), (n-1, 5^8, 1^{n-9}), (5^4, 4^5), (5^8, 4), (6, 5^8), (5^{10}), (5^9, 1).$$

Thus, π is potentially $K_6 - 3K_2$-graphic by Theorem 3.1. □

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