Graphs with prescribed star complement for 1 as the second largest eigenvalue

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Abstract

Let G be a graph of order n and let μ be an eigenvalue of multiplicity m. A star complement for μ in G is an induced subgraph of G of order $n-m$ with no eigenvalue μ. In this paper, we study maximal and regular graphs which have $K_{r,s}+tK_1$ as a star complement for 1 as the second largest eigenvalue. It turns out that some well known strongly regular graphs are uniquely determined by such a star complement.

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1 Introduction

Let G be a simple graph of order n and the vertex set $V(G)$. Let μ be an eigenvalue of G of multiplicity m. An m-subset X of $V(G)$ is called a *star set* for μ in G if μ is not an eigenvalue of $G \setminus X$. The induced subgraph $H = G \setminus X$ is said to be a *star complement* for μ in G. Star sets exist for any eigenvalue in a graph and they are not necessarily unique. For the background and results on star sets and star complements, one may consult [8, 9, 11, 15].

The following theorem which establishes a relation between a graph and its substructures corresponding to an eigenvalue is the basis of the so called *star complement technique.*

Theorem 1 (The Reconstruction Theorem) *Let G be a graph with adjacency matrix*

$$\begin{pmatrix} A_X & B^t \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by a subset X of vertices. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and $\mu I - A_X = B^t(\mu I - C)^{-1}B$.

This theorem states that the triple (μ,B,C) determines A_X uniquely. In other words, given eigenvalue μ, a star complement H and H-neighborhoods of elements of X, G is uniquely determined. Here, by the H-neighborhood of $x \in X$, we mean the set of all neighbors of x in $V(H)$.

From the theorem, it is seen that for any two vertices u and v of X, we have

$$< \mathbf{b}_u, \mathbf{b}_v >= \mathbf{b}_u^t(\mu I - C)^{-1}\mathbf{b}_v = \begin{cases} \mu & \text{if } u = v, \\ -1 & \text{if } u \sim v, \\ 0 & \text{if } u \not\sim v, \end{cases} \tag{1}$$

where $\mathbf{b}_x$ is the column of B corresponding to a vertex x. It is well known that if $\mu \neq 0, -1$, then the H-neighborhoods of vertices of X are distinct and nonempty.

Let H be a graph of order t with no eigenvalue μ. The star complement technique is a method for determining all graphs G prescribing H as a star complement for eigenvalue μ. It is known that if $\mu \neq -1, 0$, then $|V(G)| \leq \binom{t+1}{2}$ (see [3]) and therefore there are only finitely many such graphs G. Now we briefly review the star complement technique. We use the notation of Theorem 1. Given C (the adjacency matrix of H) with no eigenvalue μ, we are interested in finding the solutions for B (note that by Theorem 1, A_X will then be determined uniquely). Hence, first of all we need to find $(0,1)$ column vectors of dimension t which are candidates for columns of B. In other words, we need to find all possible extensions $H+u$ of H by adding a new vertex u such that $H+u$ has μ as an eigenvalue. In order to do this, we identify all vectors $\mathbf{b}$ satisfying

$$< \mathbf{b}, \mathbf{b} >= \mu,$$

and let them be the vertices of the *compatibility* graph $\mathcal{G}(H,\mu)$. An edge is inserted between $\mathbf{b}$ and $\mathbf{b}'$ if and only if

$$< \mathbf{b}, \mathbf{b}' >= 0, -1.$$

Now by Theorem 1, any clique in $\mathcal{G}(H,\mu)$ determines the vertices of a star set X and therefore a graph G having H as a star complement for eigenvalue μ. To describe all the graphs with H as a star complement for μ, it suffices to determine the *maximal* graphs, i.e. those graphs for which the corresponding clique in $\mathcal{G}(H,\mu)$ is maximal, since any graph with H as a star complement for μ is an induced subgraph of such a graph.

Two main problems arise in the context of star complement. One of these is the *general problem* which is to find all maximal graphs having a given graph H as a star complement for some eigenvalue. In other words, by the notation of Theorem 1, given C, we want to find all solutions for μ, B, A_X. The other problem is the *restricted problem* which is about the determination of all maximal graphs prescribing a given graph H as a star complement for a given eigenvalue μ. This means that given C and μ, we need to find all solutions for B and A_X. These problems are interesting for some reasons as is described in the following. Sometimes there is only a unique maximal graph and hence that graph is characterized by a means of its star complement. Also the problems usually build unexpected links to other areas of combinatorics such as extremal set theory and t-designs. The general and restricted problems have been dealt with for some special families of graphs such as complete graphs, complete bipartite graphs, stars, paths, cycles and so on. A list of references includes [1, 2, 4, 5, 9, 10, 12, 13, 15, 18, 19].

In this paper, we consider the restricted problem for $H = K_{r,s} + tK_1$ and $\mu = 1$. Note that by the following lemma, in any extension G of H for

$\mu = 1$, one is the second largest eigenvalue. For a graph G of order n we denote the ith largest eigenvalue of G by $\lambda_i(G)$ and we also let $\lambda_0(G) = \infty$ and $\lambda_{n+1}(G) = -\infty$.

Lemma 1 *Given a graph G of order n with eigenvalue μ of multiplicity $m \geq 1$, let H be a star complement for μ in G. Let $\lambda_{r+1}(H) < \mu < \lambda_r(H)$ for some $0 \leq r \leq n - m$. Then $\lambda_{r+1}(G) = \cdots = \lambda_{r+m}(G) = \mu$.*

Proof. By interlacing, we have the inequalities

$$\lambda_{r+m}(G) \leq \lambda_r(H) \leq \lambda_r(G),$$

$$\lambda_{r+1+m}(G) \leq \lambda_{r+1}(H) \leq \lambda_{r+1}(G),$$

which yield $\lambda_{r+m+1}(G) < \mu < \lambda_r(G)$. Since G has eigenvalue μ of multiplicity m, the assertion follows. □

We determine the appropriate r, s, t and identify the maximal graphs in one of the obtained cases as well as regular graphs which have H as a star complement for the eigenvalue 1. Some special cases of this problem have already been investigated: In [12] (see also [16, 18]), it is shown that the complement of the Clebsch graph (srg(16, 5, 0, 2)) is the unique maximal graph which has $K_{1,5}$ as a star complement. For $H = K_{1,9}$, it is known that there are exactly 15 maximal graphs [18] and for $H = K_{1,10}$, there is a unique maximal graph [12]. We also know that the complement of the Schläfli graph is the unique maximal graph which admits $H = K_{2,5}$ as a star complement [12, 13]. Finally, for $H = K_{1,s} + 2K_1$, the maximal graphs are found in [10]. We note that in [12] (see also [13]) some general observations on the general problem for $H = K_{r,s} + tK_1$ and arbitrary μ are given.

We introduce some notation which will be used throughout the paper. We assume that $H = K_{r,s} + tK_1$ is a star complement for the eigenvalue $\mu = 1$ in G. With no loss of generality, we suppose that $1 \leq r \leq s$, $(r, s) \neq (1, 1)$. Let also $W = \{w_1, w_2, \ldots, w_t\}$ denote the set of isolated vertices in H and let $U = \{u_1, u_2, \ldots, u_r\}$ and $V = \{v_1, v_2, \ldots, v_s\}$ be the two subsets of vertices of H with all edges of H between U and V. The star set corresponding to H and μ is denoted by X. Let $H(a, b, c)$ be a graph obtained from H by introducing a new vertex and joining it to a vertices of U, b vertices of V and c vertices of W. The (0,1) column vector

$\mathbf{b}_u$ denotes the neighborhood of $u \in X$ in H. Let $H+u = H(a,b,c)$ and $H+v = H(\alpha,\beta,\gamma)$. Then it is an easy task to show that

$$(1-rs) < \mathbf{b}_u, \mathbf{b}_v >= (1-rs)\rho + a(\beta+\alpha s) + b(\alpha+\beta r), \tag{2}$$

where ρ is the number of common neighbors of u and v in H (see [12, Eq. (7.4)]).

2 Extension by a vertex

The first step in the star complement technique is to find all possible extensions of a star complement by adding a new vertex. We proceed to determine all possible graphs $H+u$ by adding a vertex u such that $H+u$ has μ as an eigenvalue.

Let $H+u = H(a,b,c)$. Using (1) and (2), we find that

$$1-rs = (a+b+c)(1-rs) + 2ab + a^2s + b^2r. \tag{3}$$

Assuming $r = a+x$ and $s = b+y$, (3) is converted to

$$ab(c-3) + (b+c-1)ay + (a+c-1)bx + (a+b+c-1)(xy-1) = 0. \tag{4}$$

We make use of (4) to obtain the solutions of (3). The proofs of the next two lemmas are straightforward.

Lemma 2 *Let $m \geq n \geq q \geq 1$ be integers. Then the solutions of $mnq = m+n+q+2$ are $(m,n,q) \in \{(2,2,2),(3,3,1),(5,2,1)\}$.*

Lemma 3 *Let $c \geq 3$. Then the solutions of* (3) *are as follows.*

r	3	2	2	1	1	1	1	1
s	3	5	2	5	3	2	2	2
a	3	2	2	1	1	1	1	0
b	3	5	2	5	3	2	1	2
c	4	4	5	5	6	8	4	3

Proof. First suppose that $xy \neq 0$. Since $c \geq 3$, all sentences in the left hand side of (4) are nonnegative and thus they all must be 0. Consequently, we obtain that $a = b = 0$ and $x = y = 1$, which is not acceptable. So $xy = 0$.

Assume that $x \neq 0$ and $y = 0$. Then (4) yields $ab(c-3) + (a+c-1)bx = a+b+c-1$. It is easily obtained that $b = 1, 2$. If $b = 2$, then clearly, $x = 1, c = 3, a = 0$ and we have the solution $(r, s, a, b, c) = (1, 2, 0, 2, 3)$. If $b = 1$, then $x = 1, c = 4, a = 1$ and the solution $(r, s, a, b, c) = (2, 1, 1, 1, 4)$ is obtained which is not acceptable (since $r \leq s$). The case $x = 0$ and $y \neq 0$ gives the same solutions with the roles of r and s interchanged. Therefore in this case we have the solution $(r, s, a, b, c) = (1, 2, 1, 1, 4)$. Finally, let $x = y = 0$. Then we have $ab(c-3) = a+b+c-1$ and hence by Lemma 2, $(a, b, c) = (2, 2, 5), (1, 3, 6), (3, 3, 4), (1, 2, 8), (1, 5, 5), (2, 5, 4)$. □

In the next lemmas we consider the remaining cases $c = 0, 1, 2$.

Lemma 4 *Let $c = 0$. Then the solutions of* (3) *are as follows.*

r	5	3	3	2	2	2	1	1	1	1
s	10	11	3	5	5	13	5	9	10	10
a	5	3	1	0	1	2	0	1	1	1
b	6	7	1	3	1	9	2	3	2	5

Proof. For $c = 0$, the equation (4) becomes

$$abx + aby + (a+b-1)(xy-1) = 3ab + ay + bx. \tag{5}$$

With no loss of generality we assume that $x \leq y$ (if we find a solution such that $r > s$, we should interchange the roles of r and s). Note that $y \neq 0$. First suppose that $x = 0$. Then $(b-1)ay = a+b-1+3ab$. Since $r \geq 1$, we have $a \geq 1$. Also $b \geq 2$, since otherwise we get $y = 0$ or $a = 0$, a contradiction. We now conclude that $4a$ is congruent to 0 modulo $b-1$ and $b-1$ is congruent to 0 modulo a. Therefore, $b-1 = a, 2a$ or $4a$. First let $b-1 = a$. Then $ay = 5+3a$ which gives the solutions $(r, s, a, b) = (1, 10, 1, 2), (5, 10, 5, 6)$. Next let $b-1 = 2a$. Then $ay = 3+3a$ which gives the solutions $(r, s, a, b) = (1, 9, 1, 3), (3, 11, 3, 7)$. Finally, let $b-1 = 4a$. Then from $ay = 3a+2$ we find the solutions $(r, s, a, b) = (1, 10, 1, 5), (2, 13, 2, 9)$.

Now we assume that $x > 0$. We claim that $x > 2$ is impossible. On the contrary, suppose that $x > 2$. From (5), it can easily been seen that $ab \neq 0$. Since we have assumed $y \geq x$, (5) yields $3(ab-b)+3(ab-a)+9(a+b-1)+1 \leq a+b+3ab$ which in turn gives $3ab+5a+5b \leq 8$, a contradiction. Therefore, $x \leq 2$. First let $x = 1$. Then (5) yields $(ab+b-1)(y-2) = a+1$. This gives

$y \geq 3$. Now $a+1$ is congruent to 0 modulo $y-2$ and $y-2$ is congruent to 0 modulo $a+1$. Therefore, $y-2=a+1$ and so $ab+b=2$ which gives the solutions $(r,s,a,b)=(1,5,0,2),(2,5,1,1)$. Next let $x=2$. From (5) we find $(ab+a)y+(2b-2)y+1=a+ab+3b$. If $b=0$, then $(a-2)y=a-1$ and we find the solution $(r,s,a,b)=(5,2,3,0)$. If $a=0$, then $(2b-2)y=3b-1$ and we have the solution $(r,s,a,b)=(2,5,0,3)$. Hence, let $a,b\neq 0$. Then we have $ab+a+b\leq 3$ and so the solution $(r,s,a,b)=(3,3,1,1)$ is obtained. □

Lemma 5 *Let $c=1$. Then the solutions of* (3) *are as follows:*

(i) *r,s arbitrary and $a=b=0$.*

(ii) $(r,s,a,b)=(2,5,2,2),(1,5,1,1)$.

(iii) *r,s arbitrary and $a=r-1, b=s-1$.*

Proof. With $c=1$ the equation (4) becomes

$$ab(x+y)+(a+b)xy=a+b+2ab. \tag{6}$$

If $a,b=0$, then obviously x,y are arbitrary and hence (i) holds. So let $a+b\neq 0$. If $x=0$, then $ab(y-2)=a+b$ which means that $a=b$ and so we find the solutions $(r,s,a,b)=(2,5,2,2),(1,5,1,1)$. For $y=0$, the same solutions are found with the roles of r and s interchanged. Now let $xy\neq 0$. We have $x,y<2$, since otherwise from (6) we have $ab(2+y)+2(a+b)y\leq a+b+2ab$ which is a contradiction. Therefore, $x=y=1$ and (iii) holds. □

Lemma 6 *Let $c=2$. Then in* (3) *we have $r=a=1$ and $b=s-2$.*

Proof. Letting $c=2$ in (4) we have

$$ab(x+y)+(a+b+1)xy+ay+bx=a+b+1+ab. \tag{7}$$

If $a,b=0$, then we find $x=y=1$ and hence $r=s=1$, a contradiction. So let $a+b\neq 0$. It is seen from (7) that $xy=0$. If $x=0$, then (7) yields $a(b+1)(y-1)=b+1$ and thus $a=1$ and $y=2$. The case $y=0$ is similar with the roles of r and s interchanged. □

We summarize the results of the previous lemmas in the following Theorem.

Theorem 2 *Let $1 \le r \le s$, $(r,s) \ne (1,1)$ and let $H = K_{r,s} + tK_1$ be a star complement for the eigenvalue 1 in G. Suppose that $H + u = H(a,b,c)$ is an extension by a vertex of H in G. Then the graph $H(a,b,c)$ has one of the forms presented in Table 1.*

Table 1 All possible extensions by a vertex of $H = K_{r,s} + tK_1$ for the eigenvalue 1.

#	H	(a,b,c)
1	$K_{1,2} + tK_1$	$(0,2,3),(1,1,4),(1,2,8),(0,0,1),(0,1,1),(1,0,2)$
2	$K_{1,3} + tK_1$	$(1,3,6),(0,0,1),(0,2,1),(1,1,2)$
3	$K_{1,5} + tK_1$	$(0,2,0),(0,0,1),(1,1,1),(0,4,1),(1,3,2),(1,5,5)$
4	$K_{1,9} + tK_1$	$(1,3,0),(0,0,1),(0,8,1),(1,7,2)$
5	$K_{1,10} + tK_1$	$(1,2,0),(1,5,0),(0,0,1),(0,9,1),(1,8,2)$
6	$K_{2,2} + tK_1$	$(2,2,5),(0,0,1),(1,1,1)$
7	$K_{2,5} + tK_1$	$(1,1,0),(0,3,0),(0,0,1),(2,2,1),(1,4,1),(2,5,4)$
8	$K_{2,13} + tK_1$	$(2,9,0),(0,0,1),(1,12,1)$
9	$K_{3,3} + tK_1$	$(1,1,0),(0,0,1),(2,2,1),(3,3,4)$
10	$K_{3,11} + tK_1$	$(3,7,0),(0,0,1),(2,10,1)$
11	$K_{5,10} + tK_1$	$(5,6,0),(0,0,1),(4,9,1)$
12	$K_{1,s} + tK_1$	$(0,0,1),(0,s-1,1),(1,s-2,2)$
13	$K_{r,s} + tK_1$	$(0,0,1),(r-1,s-1,1)$

3 Maximal graphs

When H is one of the cases #1 to #12 in Theorem 2, there are different types of vertices in the star set which makes it a tedious task to find all maximal graphs with H as a star complement. However, in the case #13 there are only two types of vertices and it seems tractable. Hence, in this section we investigate the maximal extensions G of $H = K_{r,s} + tK_1$ when H is the case #13 in Theorem 2. Note that $t \ge 1$ and there are two types of vertices in the star set X. Let $u \in X$ and $H + u = H(a,b,c)$. We say that u is of type 1 (2) if $(a,b,c) = (0,0,1)$ ($(a,b,c) = (r-1,s-1,1)$). Then (1) and (2) show that any vertex of type 1 lies in a component of G which is K_2. We can ignore such vertices for the following reason: If G is a maximal graph for $H = K_{r,s} + tK_1$ containing r vertices of type 1, then G' obtained from G by removing r components K_2 is a maximal graph for $H' = K_{r,s} + (t-r)K_1$. Therefore, we may assume that G has no vertices of type 1. We index a vertex of type 2 by (i, j, k) if it is not adjacent (not adjacent, adjacent) to u_i (v_j, w_k) in U (V, W). Therefore, the vertices of the compatibility graph are indexed by the triples (i, j, k), where $1 \leq i \leq r, 1 \leq j \leq s$ and $1 \leq k \leq t$.

Let $u, v \in X$ be two distinct vertices of type 2. Then by (2),

$$< \mathbf{b}_u, \mathbf{b}_v >= \rho - r - s + 2, \tag{8}$$

where ρ denotes the number of common neighbors of u and v in H. Since $< \mathbf{b}_u, \mathbf{b}_v >= -1, 0$, we conclude that at least one and at most two of U-, V- and W-neighborhoods of u and v must coincide. Moreover, u is joined to v in G if and only they coincide for exactly one of these neighborhoods. In the compatibility graph, (i_1, j_1, k_1) is joined to (i_2, j_2, k_2) if and only if they coincide in at least one coordinate and at most two. We now find the maximal cliques in the compatibility graph.

Theorem 3 *The vertex set of a maximal clique in the compatibility graph, up to isomorphism, is one of the following forms:*

(i) $M_l = \{(i_1, i_2, i_3) \mid i_l = 1\}$, $1 \leq l \leq 3$.

(ii) $\{(i_1, i_2, i_3) \mid$ *at least two of* i_1, i_2, i_3 *are* $1\}$, *if* $t > 1$.

(iii) $\{(1,1,1), (1,2,2), (2,1,2), (2,2,1)\}$, *if* $t > 1$.

Proof. Let M be a maximal clique. If $t = 1$, then obviously we have the case (i). Therefore, let $t > 1$. First suppose that M has two vertices which have the same entries in two coordinates. With no loss of generality, we let $(1,1,1), (1,1,2) \in M$. Then the remaining vertices in M are of the form (1) $(1, j, k)$ or (2) $(i, 1, k)$. If all vertices are of type (1) or all are of type (2), then we conclude that M is of the form (i). Otherwise, M is of the form (ii). Now assume that no two vertices in M coincide in two coordinates. With no loss of generality, let $(1,1,1), (1,2,2) \in M$. Then the remaining vertices in M are of the form (1) $(1, j, k)$, (2) $(i, 1, 2)$ or (3) $(i, 2, 1)$. If M has vertices of type (2) or (3), then clearly M is of the from (iii). Otherwise, we find that M is not maximal, a contradiction. □

The theorem above along with the preceding paragraph describe all possible maximal graphs G up to isomorphism. Note that in the case (i),

G has $r+s+t+rs$, $r+s+t+rt$, or $r+s+t+st$ vertices, in the case (ii), $2(r+s+t-1)$ vertices and in the case (iii), $r+s+t+4$ vertices.

Some small cases in Table 1 can be dealt with using computer search. For this purpose, Zoran Stanić and Nedeljko Stefanović have developed a software called *SCL (Star Complement Library)* which is available at [17]. The interested reader can download the software and examine the maximal graphs for small cases of Table 1. Here, for example, we present the results obtained from this program for the case $K_{1,2}+tK_1$, $t<5$. Let G be a maximal graph. As it seen from Table 1, there is no extension when $t=0$. For $t=1$, we have $G=P_5$, the path on 5 vertices. If $t=2$, then we obtain that $G=P_5+K_2$ or G is an irregular graph on 9 vertices with the spectrum $-2^3, -0.732051, 1^4, 2.73205$. For $t=3$, G is an irregular graph on 10 vertices with the spectrum $-2^2, -1^2, -0.414214, 1^4, 2.41421$ or G is an irregular graph on 14 vertices with the spectrum $-3^3, -2, -1.37228, 1^8, 4.37228$. Finally, if $t=4$, then G is an irregular graph on 14 vertices with the spectrum $-3, -2.70156, -2, -1^3, 1^7, 3.70156$ or G is an irregular graph on 23 vertices with the spectrum $-5^2, -4.66366, -4, -3, -2.79008, 1^{16}, 8.45374$.

4 Regular graphs

In this section we identify regular graphs which have H as a star complement for the eigenvalue 1. Suppose that G is a k-regular extension of H with star set X. Let $u \in X$. Then it is well known that $<\mathbf{b}_u, \mathbf{j}> = -1$. Therefore, if $H+u=H(a,b,c)$, then by (2), we have

$$a(s+1)+b(r+1)+(c+1)(1-rs)=0.$$

Using Theorem 2, we find the solutions of this equation. The results are given in the following Theorem.

Theorem 4 *Let $1 \le r \le s$, $(r,s) \neq (1,1)$ and let $H=K_{r,s}+tK_1$ be a star complement for the eigenvalue 1 in a regular graph G. Suppose that $H+u=H(a,b,c)$ is an extension by a vertex of H in G. Then the graph $H(a,b,c)$ has one of the forms presented in Table 2.*

Table 2 All possible extensions by a vertex of $H = K_{r,s} + tK_1$ for the eigenvalue 1 in a regular graph.

#	H	(a, b, c)
1	$K_{1,2} + tK_1$	$(0,2,3),(1,1,4),(0,1,1),(1,0,2)$
2	$K_{1,5} + tK_1$	$(0,2,0),(1,1,1),(0,4,1),(1,3,2)$
3	$K_{2,5} + tK_1$	$(1,1,0),(0,3,0),(2,2,1),(1,4,1)$
4	$K_{3,3} + tK_1$	$(1,1,0),(2,2,1)$
5	$K_{1,s} + tK_1$ (none of the above)	$(0,s-1,1),(1,s-2,2)$
6	$K_{r,s} + tK_1$ (none of the above)	$(r-1,s-1,1)$

A *strongly regular graph* with parameters (n, k, λ, μ) (a $\mathrm{srg}(n, k, \lambda, \mu)$, for short) is a k-regular graph on n vertices such that any two adjacent vertices have exactly λ common neighbors and any two nonadjacent vertices have exactly μ common neighbors. We refer the reader to [6, 7] which provide comprehensive surveys on strongly regular graphs. Given two positive integers n and k, *Kneser graph* $K(n, k)$ is the graph whose vertices represent the k-subsets of an n-set, and where two vertices are connected if and only if they correspond to disjoint subsets. The eigenvalues of Kneser graphs are known (see [14, Page 199]).

Lemma 7 *The unique regular extension G of the case #6 in Table 2 is the complement of the line graph of $K_{r+1,s+1}$ if $t = 1$ and there is no regular extension if $t \neq 1$.*

Proof. We have $(a, b, c) = (r-1, s-1, 1)$. Suppose that X has p vertices. Then by the regularity of G, we have $r(k-s) = p(r-1)$, $s(k-r) = p(s-1)$ and $kt = p$. From the first two equations, we have $k(r-s) = p(r-s)$. This implies $k = p$, since if $r = s$, then from the second and third equations, we have $sk + kt = kst + rs$ which gives $t = 1$ and so $k = p$. From $k = p$, we have $t = 1$ and $p = rs$. We index the vertices of G as follows. The vertices of X are indexed by (i, j), where $1 \leq i \leq r$ and $1 \leq j \leq s$. The vertices of U and V are indexed by $(i, 0)$ $(1 \leq i \leq r)$ and $(0, j)$ $(1 \leq j \leq s)$, respectively. The vertex of W is indexed by $(0, 0)$. Then by the results of the previous section it is seen that (i, j) is joined to (i', j') in G if and only if $i \neq i'$ and $j \neq j'$. Therefore, G is the complement of the line graph of $K_{r+1,s+1}$. □

Lemma 8 *The regular extensions G of the case #5 in Table 2 are as below.*

(i) *If* $t = 1$, *then* G *is the the complement of line graph of of* $K_{2,s+3}$.

(ii) *If* $t = 2$, *then* G *is the complement of the line graph of* K_{s+3}.

(v) *If* $t = 0$ *or* $t > 2$, *then no regular extension exists.*

Proof. We have $(a, b, c) = (0, s-1, 1), (1, s-2, 2)$, where $s \neq 1, 2, 5$. Suppose that X has p vertices of type $(0, s-1, 1)$ and q vertices of type $(1, s-2, 2)$. By the regularity of G, we have $k - s = q$, $s(k-1) = p(s-1) + q(s-2)$ and $kt = p + 2q$. These equations give $(s^2 - s)(t-1) + tq(s-1) = 2qs$ which in turn yields $t \leq 3$. If $t = 3$, then $s \leq 2$, a contradiction. If $t = 1$, then $q = 0$ which has been dealt with in the preceding paragraph and hence G is the complement of the line graph of $K_{2,s+1}$. Now suppose that $t = 2$. Then $q = \binom{s}{2}$ and $p = 2s$. Therefore, G is of order $\binom{s+3}{2}$. Now it follows that G is the complement of the line graph of K_{s+3} (Kneser graph K$(s+3, 2)$) since it has a star complement $K_{1,s} + 2K_1$ for the eigenvalue 1 (see also [10]). □

Lemma 9 *The regular extensions* G *of the case* #1 *in Table* 2 *are as below.*

(i) *If* $t = 1$, *then* G *is the cycle* C_6.

(ii) *If* $t = 2$, *then* G *is the Petersen graph.*

(iii) *If* $t = 3$, *then* G *is the complement of Clebsch graph.*

(iv) *If* $t = 4$, *then* G *is a regular induced subgraph of the complement of the Schläfli graph.*

(v) *If* $t = 0$ *or* $t > 4$, *then no regular extension exists.*

Proof. Suppose that X has p_1, p_2, p_3, p_4 vertices of type $(0, 2, 3)$, $(1, 1, 4)$, $(0, 1, 1)$, $(1, 0, 2)$, respectively. By the regularity of G, we have

$$\begin{cases} k - 2 = p_2 + p_4, \\ 2(k-1) = 2p_1 + p_2 + p_3, \\ tk = 3p_1 + 4p_2 + p_3 + 2p_4. \end{cases}$$

These equations yield $t \leq 5$. Let $t = 5$. Then $p_i \leq 10$ for $1 \leq i \leq 4$. We have $p_2 = k - p_4 - 2$ and $p_3 = k - p_4 - 16$. Therefore, $k - p_4 \geq 16$ and so $p_2 \geq 14$, a contradiction. Hence, $1 \leq t \leq 4$. If $t = 1$, then $p_i = 0$ for $i = 1, 2, 4$ and $k = p_3 = 2$ and we have $G = C_6$. If $t = 2$, then the case coincides with the case #5 and hence G is the Petersen graph (K(5, 2)). Let $t = 3$. Then $p_2 = 0$, $p_1 \leq 1$, $p_3 \leq 6$ and $p_4 \leq 3$. Also we have $p_1 = 6 - k$ which means $k = 5, 6$. Now from $p_4 = k - 2 \leq 3$, it is obtained that $k = 5$ and hence $p_1 = 1, p_3 = 6$ and $p_4 = 3$. The unique graph we obtain is a srg(16, 5, 0, 2). But there is a unique strongly regular graph with these parameter which is the complement of the Clebsch graph [6]. Its eigenvalues are $5^1, 1^{10}, -3^5$. Finally, let $t = 4$. By taking all possibilities for vertices of any type, we obtain a srg(27, 10, 1, 5). There is a unique strongly regular graph with these parameter which is the complement of the Schläfli graph [6]. Its eigenvalues are $10^1, 1^{20}, -5^6$. □

Lemma 10 *The regular extensions G of the case #2 in Table 2 are as below.*

(*i*) *If $t = 0$, then G is the complement of the Clebsch graph.*

(*ii*) *If $t = 1$, then G is a regular induced subgraph of the complement of the Schläfli graph.*

(*iii*) *If $t = 2$, then G is the complement of the line graph of K_8*

(*iv*) *If $t > 2$, then no regular extension exists.*

Proof. Suppose that X has p_1, p_2, p_3, p_4 vertices of type (0, 2, 0), (1, 1, 1), (0, 4, 1), (1, 3, 2), respectively. By the regularity of G, we have

$$\begin{cases} k - 5 = p_2 + p_4, \\ 5(k-1) = 2p_1 + p_2 + 4p_3 + 3p_4, \\ tk = p_2 + p_3 + 2p_4. \end{cases}$$

These equations yield $t \leq 2$. Let $t = 0$. Then $p_i = 0$ for $2 \leq i \leq 4$, $k = 5$ and $p_1 = 10$. The unique graph we obtain is a srg(16, 5, 0, 2), i.e. the complement of the Clebsch graph. If $t = 1$, then $p_4 = 0$ and if we take all possibilities for vertices of other types, then we find a srg(27, 10, 1, 5), i.e. the complement of the Schläfli graph. Finally, let $t = 2$. Then $p_i \leq 10$ for $1 \leq i \leq 4$. Since $p_3 = p_2 + 10$, we have $p_3 = 10$, $p_2 = 0$, $p_1 = k - 15$ and $p_4 = k - 5$. This implies $p_4 = 10$ and $p_1 = 0$. Therefore, in this situation the case coincides with the case #5 and G is K(8, 2). □

Lemma 11 *The regular extensions G of the case #4 in Table 2 are as follows.*

(*i*) *If* $t = 0$, *then* G *is a regular induced subgraph of the line graph of* K_6.

(*ii*) *If* $t = 1$, *then* G *is the complement of the line graph of* $K_{4,4}$.

(*iii*) *If* $t > 1$, *then no regular extension exists.*

Proof. Suppose that X has p_1 and p_2 vertices of type $(1, 1, 0)$ and $(2, 2, 1)$, respectively. By the regularity of G, we have

$$\begin{cases} 3(k-3) = p_1 + 2p_2, \\ tk = p_2. \end{cases}$$

These equations yield $t \leq 1$. If $t = 0$, then $p_2 = 0$ and if we take all possibilities for vertices of type 1, then we find a srg$(15, 6, 1, 3)$. There is a unique strongly regular graph with these parameters which is K$(6, 2)$ [6]. Its eigenvalues are $6^1, 1^9, -3^5$. Now let $t = 1$. Then $p_1, p_2 \leq 9$. We have $p_1 = k - 9$ and $p_2 = k$ which yield $p_1 = 0$ and $p_2 = 9$. Thus we have the case #6 and G is the complement of the line graph of $K_{4,4}$. □

Lemma 12 *The regular extensions* G *of the case #3 in Table 2 are as follows.*

(*i*) *If* $t = 0$, *then* G *is a regular induced subgraph of the complement of the Schläfli graph.*

(*ii*) *If* $t = 1$, *then* G *is a regular induced subgraph of the complement of a Chang graph.*

(*iii*) *If* $t > 1$, *then no regular extension exists.*

Proof. This case is somewhat different from the other cases. Suppose that X has p_1, p_2, p_3, p_4 vertices of type $(1, 1, 0)$, $(0, 3, 0)$, $(2, 2, 1)$, $(1, 4, 1)$, respectively. By the regularity of G, we have

$$\begin{cases} 2(k-5) = p_1 + 2p_3 + p_4, \\ 5(k-2) = p_1 + 3p_2 + 2p_3 + 4p_4, \\ tk = p_3 + p_4. \end{cases}$$

These equations yield $t \leq 1$. Let $t = 0$. Then $p_3 = p_4 = 0$. If we take all possibilities for vertices of types 1 and 2, then we find a srg$(27, 10, 1, 5)$, i.e. the complement of the Schläfli graph. Now assume that $t = 1$. Note that $p_i \leq 10$ for $1 \leq i \leq 4$. We have $p_4 = p_1 + 10$ which yields $p_4 = 10$,

$p_1 = 0$ and $p_2 = p_3 = k - 10$. It implies any regular graph containing $K_{2,5} + K_1$ as star complement for 1, has a 10-regular induced subgraph F, that is, the subgraph induced on 10 vertices of type 4. We index a vertex of type 2 by the triple $\{i, j, k\}$ if it is adjacent to v_i, v_j, v_k in V. Similarly, we index a vertex of type 3 by the pair $\{i, j\}$ if it is adjacent to v_i, v_j in V. From (2), it is seen that any two vertices of X, one of type 2 and the other of type 3 must have intersecting neighborhoods in V. This implies that $p_2, p_3 \leq 5$ and so $k \leq 15$. Suppose that $k = 15$. Then $p_2 = p_3 = 5$. Our goal is to find a 15-regular graph, say G, on 28 vertices containing F as an induced subgraph and having 5 vertices for each of types 2 and 3. By (2), we see that a vertex of type 2 (3) is adjacent to a vertex of type 4 in the compatibility graph if and only if they have 2 (1) or 3 (2) common neighbors in V and moreover a vertex of type 2 (3) is adjacent to a vertex of type 4 in G if and only if they have 2 (1) common neighbors in V. Now an easy analysis shows that up to isomorphism the following cases may occur for the vertices of type 2 and 3 in G:

1. type 2: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$,
   type 3: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$;

2. type 2: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$ $\{2,3,4\}$,
   type 3: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$ $\{2,4\}$;

3. type 2: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,5\}$ $\{2,3,4\}$,
   type 3: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$ $\{2,5\}$;

4. type 2: $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$ $\{2,3,4\}$,
   type 3: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$ $\{4,5\}$;

5. type 2: $\{1,2,4\}$, $\{1,2,5\}$, $\{1,4,5\}$, $\{1,3,5\}$ $\{2,3,4\}$,
   type 3: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,5\}$ $\{4,5\}$;

6. type 2: $\{1,2,4\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{2,3,5\}$ $\{2,4,5\}$,
   type 3: $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{4,5\}$ $\{1,5\}$.

Since the vertices of types 2 and 3 in G induces a 6-regular graph, only the case 6 can hold. The graph we obtain is a srg$(28, 15, 6, 10)$. There are four strongly regular graphs with these parameters, one is Kneser graph K$(8, 2)$ and the other three are the complements of the Chang graphs (see [7, page 258] and [6]). Since K$(8, 2)$ has no induced subgraph $K_{2,5} + K_1$, G must be the complement of a Chang graph. Similarly, we deal with the other values of k and we find that there are solutions for $k = 10, 12, 13$ and they are induced subgraphs of the one with $k = 15$. In the following we demonstrate the choices for vertices of type 2 and 3 in each case.

- $k = 10$, $p_2 = p_3 = 0$.
- $k = 12$, type 2: $\{1,3,5\}, \{2,4,5\}$, type 3: $\{1,2\}, \{3,4\}$.
- $k = 13$, type 2: $\{1,2,4\}, \{1,3,5\}, \{2,3,5\}$, type 3: $\{1,2\}, \{3,4\}, \{4,5\}$.

□

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