Feedback Number of Generalized Kautz Digraphs GK(2,n)*

Lei Wang, Xirong Xu†, Yang Yuansheng,
Di Ming, Dong Xuezhi

Department of Computer Science
Dalian University of Technology
Dalian, 116024, P.R.China

Abstract

A subset of vertices of a graph G is called a feedback vertex set of G if its removal results in an acyclic subgraph. In this paper, we consider the feedback vertex set of generalized Kautz digraphs $GK(2,n)$. Let $f(2,n)$ denote the minimum cardinality over all feedback vertex sets of the Generalized Kautz digraph $GK(2,n)$, we obtain the upper bound of $f(2,n)$ as follows

$$f(2,n) \leq n - (\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{n-8}{9} \rfloor).$$

Keywords: Feedback vertex set, Feedback number, Generalized Kautz digraphs, Cycles, Acyclic subgraph.

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† Corresponding author: xirongxu@dlut.edu.cn

1 Introduction

Let $G = (V, E)$ be a graph or digraph without multiple edges, with vertex set $V(G)$ and edge set $E(G)$. It is well known that the cycle rank of a graph G is the minimum number of edges that must be removed in order to eliminate all of the cycles in the graph. That is, if G has υ vertices, ε edges, and ω connected components, then the minimum number of edges whose deletion from G leaves an acyclic graph equals the cycle rank (or Betti number) $\rho(G) = \varepsilon - \upsilon + \omega$.

A corresponding problem is the removal of vertices. A subset $F \subset V(G)$ is called a *feedback vertex set* if the subgraph $G - F$ is acyclic, that is, if $G - F$ is a forest. The minimum cardinality of a feedback vertex set is called *the feedback number* (or *decycling number* proposed first by Beineke and Vandell [5]) of G. A feedback vertex set of this cardinality is called a *minimum feedback vertex set.*

Determining the feedback number of a graph G is equivalent to finding the greatest order of an induced forest of G proposed first by Erdös, Saks and Sós [8], since the sum of the two numbers equals the order of G. A review of recent results and open problems on the decycling number is provided by Bau and Beineke [3].

Apart from its graph-theoretical interest, the minimum feedback vertex set problem has important application to several fields. For example, the problems are in operating systems to resource allocation mechanisms that prevent deadlocks [15], in artificial intelligence to the constraint satisfaction problem and Bayesian inference, in synchronous distributed systems to the study of monopolies and in optical networks to converters placement problem(see [7, 9]).

The minimum feedback set problem is known to be NP-hard for general graphs [11] and the best known approximation algorithm is one with an approximation ratio two [1]. Determining the feedback number is quite difficult even for some elementary graphs. However, the problem has been studied for some special graphs and digraphs, such as hypercubes, meshes, toroids, butterflies, cube-connected cycles and directed split-stars (see [1, 2, 4, 7, 9, 10, 13, 14, 16, 17, 18]).

In this paper, we consider the problem for a particular class of interconnection network, namely, generalized Kautz digraph $GK(d, n)$ $(d \geq 2, n \geq 1)$, which extends Kautz digraphs for general number of vertices. The vertex-set of $GK(d, n)$ is defined as the set $V(GK(d, n)) = \{0, 1, ..., n-1\}$ and the edge set

$$E = \{(i, j) | j \equiv -(id + r)(mod\ n), r = 1, 2, \ldots, d\}. \qquad (1.1)$$

Lemma 1.1. For $d \geq 2$ and $n \geq d$,

(1) $GK(d,n)$ is d-regular and $GK(d, d^n + d^{n-1}) = K(d,n)$;

(2) $GK(d,n)$ has no loops if and only if n is divisible by $(d+1)$;

(3) $GK(d,n)$ is strongly connected;

(4) $GK(d,n) \cong L(GK(d, \frac{n}{d}))$ if n is divisible by d.

Denote the minimum cardinality over all feedback vertex sets of $GK(2,n)$ by $f(2,n)$, and call it the feedback number of $GK(2,n)$. We obtain the upper bound of $f(2,n)$ as follows

$$f(2,n) \leq n - (\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{n-8}{9} \rfloor).$$

2 Some Notations and Lemma

Throughout this paper, we follow Xu [19] for graph-theoretical terminology and notation not defined here.

Let $G = (V,E)$ be a graph and $S \subset V(G)$. The symbol $N_G(S)$ denotes the set of neighbors of S, namely, $N_G(S) = \{x \in V(G-S) | xy \in E(G), y \in S\}$. The out-neighborhood of a vertex $v \in V(G)$ is denoted by $N^+(v) = \{u \in V(G) | vu \in E(G)\}$. For a set $S \subseteq V(G)$, $N^+(S) = \cup_{v \in S} N^+(v)$. The in-neighborhood of a vertex $v \in V(G)$ is denoted by $N^-(v) = \{u \in V(G) | uv \in E(G)\}$. For a set $S \subseteq V(G)$, $N^-(S) = \cup_{v \in S} N^-(v)$.

The subgraph induced by S is denoted by $G[S]$. The set S is cycle-free if $G[S]$ has no cycles. A cycle-free set S is maximal if $G[F]$ contains cycles for any $F \subseteq V(G)$ and $S \subset F$. It is clear that S is a minimal feedback vertex set if and only if $V(G) - S$ is a maximal cycle-free set.

By definition of $GK(d,n)$, the edge set of $GK(d,n)$ can also be written as the following definition:

$$E = \{(i,j) | j \equiv (d(n-1-i) + \beta)(mod\ n), \beta = 0, 1, \ldots, d-1\}. \quad (2.1)$$

Indeed, these two definitions are the same. To see this, let $\beta = d - r$. Then

$$r = 1, 2, \ldots, d \Leftrightarrow \beta = 0, 1, \ldots, d-1.$$

Substituting $\beta = d - r$ into (2.1) yields

$$j \equiv (d(n-1-i)+\beta)(mod\ n) = -(d+di-\beta)(mod\ n)$$
$$= -(d+di-d+r)(mod\ n) = -(id+r)(mod\ n).$$

This is (1.1). Similarly, substituting $r = d-\beta$ into (1.1) yields (2.1). Thus, for $i \in V(GK(d,n))$, $N(i) = N^+(i) \cup N^-(i)$, where

$$N^+(i) = \{(d(n-1-i)+\beta)(mod\ n) \mid \beta \in \{0,1,2,\dots,d-1\}\}$$
$$= \{(k+d)n-d+\beta-di \mid \beta \in \{0,1,2,\dots,d-1\}, k \in Z\},$$
$$N^-(i) = \{\tfrac{(d+k)n-i+\beta-d}{d} \mid \beta \in \{0,1,2,\dots,d-1\}, k \in Z\}.$$

Let $[a,b] = \{v_a, v_{a+1}, \dots, v_b\}$ for any two vertices $v_a, v_b \in V(GK(d,n))$ $(a \leq b)$. Let V_d be the subset of $V(GK(d,n))$ which contains the vertices deleted. The remaining vertex set of $V(GK(d,n))$ is denoted by V_r=$V(GK(d,n)) - V_d$. Let $G[V_r]$ be the subgraph of $GK(d,n)$ induced by the vertices set V_r.

Let V_c be the subset of V_r which contains the vertices lying on a cycle of induced subgraph $G[V_r]$. Let $V_{nc} = V(GK(d,n)) - V_c$. It implies that V_{nc} contains all the vertices without lying on any cycle of $G[V_r]$. If we prove that V_c is empty set, i.e., $V_c = \emptyset$, then it is equal to prove that $V_{nc} = V(GK(d,n))$, it means $G[V_r]$ is an acyclic subgraph. Inversely, if $G[V_r]$ is acyclic subgraph, then $V_c = \emptyset$. So, to prove $G[V_r]$ is acyclic subgraph, it suffices to prove that $V_c = \emptyset$ or $V_{nc} = V(GK(d,n))$.

In the following, we give a fundamental lemma.

Lemma 2.1. For arbitrary subset $S \subseteq V_r$, if $N^+(S) \subseteq V_{nc}$, then $S \subseteq V_{nc}$.

Proof. Suppose to the contrary that there exists a vertex $x \in S$ such that $x \notin V_{nc}$. Then there exists a cycle C in $G[V_r]$ containing x. It follows that C also containing a vertex $y \in N^+(x) \subseteq N^+(S)$, a contradiction to $N^+(S) \subseteq V_{nc}$. The lemma holds. □

3 Feedback Vertex Set of $GK(2, n)$

Now we consider the feedback vertex set of $GK(2,n)$. For any $i \in V(GK(2,n))$, the neighbour vertex set of i is $N(i) = N^+(i) \cup N^-(i)$, where

$$\begin{aligned} N^+(i) &= \{(k+2)n - 2 + \beta - 2i \mid \beta \in \{0,1\}, k \in Z\} \\ &= \{(k+2)n - 2 - 2i,\ (k+2)n - 1 - 2i \mid k \in Z\}, \\ N^-(i) &= \{\tfrac{(2+k)n-i+\beta-2}{2} \mid \beta \in \{0,1\}, k \in Z\} \\ &= \{\tfrac{(2+k)n-i-2}{2}, \tfrac{(2+k)n-i-1}{2} \mid k \in Z\}. \end{aligned}$$

Define six subsets of $V(GK(2,n))$ as follows:

$$\begin{aligned} F_1 &= [0, \lfloor \tfrac{n}{3} \rfloor - 1], \\ F_2 &= [\lfloor \tfrac{n}{3} \rfloor, \lfloor \tfrac{n}{3} \rfloor], \\ F_3 &= [\lfloor \tfrac{n}{3} \rfloor + 1, \lfloor \tfrac{n}{3} \rfloor + \lfloor \tfrac{n-2}{3} \rfloor], \\ F_4 &= [\lfloor \tfrac{n}{3} \rfloor + \lfloor \tfrac{n-2}{3} \rfloor + 1, \lfloor \tfrac{n}{3} \rfloor + \lfloor \tfrac{n-2}{3} \rfloor + 1], \\ F_5 &= [\lfloor \tfrac{n}{3} \rfloor + \lfloor \tfrac{n-2}{3} \rfloor + 2, \lfloor \tfrac{n}{3} \rfloor + \lfloor \tfrac{n-2}{3} \rfloor + \lfloor \tfrac{n-8}{9} \rfloor + 1], \\ F_6 &= [\lfloor \tfrac{n}{3} \rfloor + \lfloor \tfrac{n-2}{3} \rfloor + \lfloor \tfrac{n-8}{9} \rfloor + 2, n-1]. \end{aligned}$$

Apparently, $F_1, F_2, \ldots, F_6$ is a partition of $V(GK(2,n))$, so we have $F_i \cap F_j = \emptyset$ for $1 \leq i \neq j \leq 6$, and $V(GK(2,n)) = F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 \cup F_6$.

Suppose that $V_d = F_2 \cup F_4 \cup F_6$, then $V_r = F_1 \cup F_3 \cup F_5$ since $F_1, \ldots, F_6$ is a partition of $V(GK(2,n))$. In the following we will prove that $G[F_1 \cup F_3 \cup F_5] = G[V_r]$ is acyclic, which implies that $F_2 \cup F_4 \cup F_6$ is a feedback vertex set of $GK(2,n)$.

Lemma 3.1. $G[V_r]$ is acyclic.

Proof. Recall that $G[V_r]$ is acyclic if and only if $V_c = \emptyset$ or $V_{nc} = V(GB(2,n))$. To prove $G[V_r]$ is acyclic, it suffices to show $V_{nc} = V(GK(2,n))$.

Let $n \equiv t (mod\ 9)$, $0 \leq t \leq 8$. According to the value of t, we will consider nine cases. We just prove two cases for $t = 0, 2$, other cases omit.

Case1. $t = 0$.

Since $n \equiv 0 (mod\ 9)$, that is, $n = 9m, m \in Z_+$. Then we have

$$\begin{aligned}
F_1 &= [0, \lfloor\tfrac{n}{3}\rfloor - 1] = [0, 3m-1],\\
F_2 &= [\lfloor\tfrac{n}{3}\rfloor, \lfloor\tfrac{n}{3}\rfloor] = [3m, 3m],\\
F_3 &= [\lfloor\tfrac{n}{3}\rfloor + 1, \lfloor\tfrac{n}{3}\rfloor + \lfloor\tfrac{n-2}{3}\rfloor] = [3m+1, 6m-1],\\
F_4 &= [\lfloor\tfrac{n}{3}\rfloor + \lfloor\tfrac{n-2}{3}\rfloor + 1, \lfloor\tfrac{n}{3}\rfloor + \lfloor\tfrac{n-2}{3}\rfloor + 1] = [6m, 6m],\\
F_5 &= [\lfloor\tfrac{n}{3}\rfloor + \lfloor\tfrac{n-2}{3}\rfloor + 2, \lfloor\tfrac{n}{3}\rfloor + \lfloor\tfrac{n-2}{3}\rfloor + \lfloor\tfrac{n-8}{9}\rfloor + 1] = [6m+1, 7m-1],\\
F_6 &= [\lfloor\tfrac{n}{3}\rfloor + \lfloor\tfrac{n-2}{3}\rfloor + \lfloor\tfrac{n-8}{9}\rfloor + 2, n-1] = [7m, 9m-1].
\end{aligned}$$

$\forall x \in V_d = F_2 \cup F_4 \cup F_6 = [3m, 3m] \cup [6m, 6m] \cup [7m, 9m-1]$, since V_d contains the vertices deleted from $V(GK(2,n))$ then $x \notin V_r$, i.e., $x \notin V_c$, that is $x \in V_{nc}$. Thus $V_d \subseteq V_{nc}$, i.e.,

(1) $[3m, 3m] \cup [6m, 6m] \cup [7m, 9m-1] \subseteq V_{nc}$.

(2) Since $[0, m-1] \subseteq V_r$ and

$$N^+([0, m-1]) = [7m, 9m-1] \subseteq V_{nc},$$

by Lemma 2.1, we have $[0, m-1] \subseteq V_{nc}$, which and (1) imply that $[0, m-1] \cup [3m, 3m] \cup [6m, 6m] \cup [7m, 9m-1] \subseteq V_{nc}$.

(3) Let $k = \lceil \frac{1}{2} \log \frac{3m-1}{8} \rceil$, then

$$\begin{array}{lcl}
k & \geq & \frac{1}{2}\log\frac{3m-1}{8},\\
2^{2k+3} & \geq & 3m-1,\\
\frac{2^{2k+3}+1}{3} & \geq & m,\\
6m + \frac{2^{2k+3}+1}{3} & \geq & 7m,\\
k & < & \frac{1}{2}\log\frac{3m-1}{8} + 1,\\
2^{2k+1} & < & 3m-1,\\
\frac{2^{2k+2}+2}{3} & < & 2m.
\end{array}$$

Since $[4m, 6m - \frac{2^{2k+2}+2}{3} - 1] \subseteq V_r$ and

$$\begin{aligned}
&N^+([4m, 6m - \tfrac{2^{2k+2}+2}{3} - 1])\\
&= [6m + \tfrac{2^{2k+3}+1}{3} + 1, 9m-1] \cup [0, m-1]\\
&\subseteq [7m, 9m-1] \cup [0, m-1] \subseteq V_{nc},
\end{aligned}$$

by Lemma 2.1, we have $[4m, 6m - \frac{2^{2k+2}+2}{3} - 1] \subseteq V_{nc}$.

Since $[6m + \frac{2^{2k+1}+1}{3}, 7m-1] \subseteq V_r$ and

$$N^+([6m + \tfrac{2^{2k+1}+1}{3}, 7m-1]) = [4m, 6m - \tfrac{2^{2k+2}+2}{3} - 1] \subseteq V_{nc},$$

by Lemma 2.1, we have $[6m + \frac{2^{2k+1}+1}{3}, 7m-1] \subseteq V_{nc}$.

Since $[6m - \frac{2^{2k+2}+2}{3}, 6m - \frac{2^{2k}+2}{3} - 1] \subseteq V_r$ and

$$\begin{aligned} & N^+([6m - \tfrac{2^{2k+2}+2}{3}, 6m - \tfrac{2^{2k}+2}{3} - 1]) \\ & = [6m + \tfrac{2^{2k+1}+1}{3} + 1, 6m + \tfrac{2^{2k+3}+1}{3}] \\ & \subseteq [6m + \tfrac{2^{2k+1}+1}{3}, 9m-1] \subseteq V_{nc}, \end{aligned}$$

by Lemma 2.1, we have $[6m - \frac{2^{2k+2}+2}{3}, 6m - \frac{2^{2k}+2}{3} - 1] \subseteq V_{nc}$.

Since $[6m + \frac{2^{2k-1}+1}{3}, 6m + \frac{2^{2k+1}+1}{3} - 1] \subseteq V_{nc}$ and

$$\begin{aligned} & N^+([6m + \tfrac{2^{2k-1}+1}{3}, 6m + \tfrac{2^{2k+1}+1}{3} - 1]) \\ & = [6m - \tfrac{2^{2k+2}+2}{3}, 6m - \tfrac{2^{2k}+2}{3} - 1] \\ & \subseteq [4m, 6m - \tfrac{2^{2k}+2}{3} - 1] \subseteq V_{nc}, \end{aligned}$$

by Lemma 2.1, we have $[6m + \frac{2^{2k-1}+1}{3}, 6m + \frac{2^{2k+1}+1}{3} - 1] \subseteq V_{nc}$.

Continuing in this way, we have $[6m - \frac{2^{2l+2}+2}{3}, 6m - \frac{2^{2l}+2}{3} - 1] \subseteq V_{nc}$ and $[6m + \frac{2^{2l-1}+1}{3}, 6m + \frac{2^{2l+1}+1}{3} - 1] \subseteq V_{nc}$ for $l = k, k-1, \ldots, 1$. It follows that $[4m, 6m-3] \subseteq V_{nc}$ and $[6m+1, 7m-1] \subseteq V_{nc}$, which and (2) imply that $[0, m-1] \cup [3m, 3m] \cup [4m, 6m-3] \cup [6m, 9m-1] \subseteq V_{nc}$.

(4) Since $[6m-2, 6m-1] \subseteq V_r$ and

$$N^+([6m-2, 6m-1]) = [6m, 6m+3] \subseteq V_{nc},$$

by Lemma 2.1, we have $[6m-2, 6m-1] \subseteq V_{nc}$, which and (3) imply that $[0, m-1] \cup [3m, 3m] \cup [4m, 9m-1] \subseteq V_{nc}$.

(5) Let $k = \lfloor \frac{1}{2} \log 4(m-1) \rfloor$, then

$$\begin{array}{lcl} k & \leq & \frac{1}{2}\log 4(m-1), \\ 2^{2k} & \leq & 4(m-1), \\ 3m + 2^{2k} & \leq & 7m-4, \\ 2^{2k-2} & \leq & m-1, \\ 3m + 2^{2k-2} & \leq & 4m-1, \\ k & > & \frac{1}{2}\log 4(m-1) - 1, \\ 2^{2k} & > & m-1, \\ 3m + 2^{2k} & > & 4m-1, \\ 3m + 2^{2k} & \geq & 4m. \end{array}$$

(5.1) Since $[m, 3m - 2^{2k-1} - 1] \subseteq V_r$ and

$$N^+([m, 3m - 2^{2k-1} - 1]) = [3m + 2^{2k}, 7m - 1] \subseteq [4m, 7m - 1] \subseteq V_{nc},$$

by Lemma 2.1, we have $[m, 3m - 2^{2k-1} - 1] \subseteq V_{nc}$, which and (4) imply that $[0, 3m - 2^{2k-1} - 1] \cup [3m, 3m] \cup [4m, 9m - 1] \subseteq V_{nc}$.

(5.2) Since $[3m + 2^{2k-2}, 4m - 1] \subseteq V_r$ and

$$N^+([3m + 2^{2k-2}, 4m - 1]) = [m, 3m - 2^{2k-1} - 1] \subseteq V_{nc},$$

by Lemma 2.1, we have $[3m + 2^{2k-2}, 4m - 1] \subseteq V_{nc}$, which and (5.1) imply that $[0, 3m - 2^{2k-1} - 1] \cup [3m, 3m] \cup [3m + 2^{2k-2}, 9m - 1] \subseteq V_{nc}$.

(5.3) Since $[3m - 2^{2k-1}, 3m - 2^{2k-3} - 1] \subseteq V_r$ and

$$N^+([3m - 2^{2k-1}, 3m - 2^{2k-3} - 1]) = [3m + 2^{2k-2}, 3m + 2^{2k} - 1] \subseteq [3m + 2^{2k-2}, 9m - 1] \subseteq V_{nc},$$

by Lemma 2.1, we have $[3m - 2^{2k-1}, 3m - 2^{2k-3} - 1] \subseteq V_{nc}$.

Since $[3m + 2^{2k-4}, 3m + 2^{2k-2} - 1] \subseteq V_r$ and

$$N^+([3m + 2^{2k-4}, 3m + 2^{2k-2} - 1]) = [3m - 2^{2k-1}, 3m - 2^{2k-3} - 1] \subseteq [0, 3m - 2^{2k-3} - 1] \subseteq V_{nc},$$

by Lemma 2.1, we have $[3m + 2^{2k-4}, 3m + 2^{2k-2} - 1] \subseteq V_{nc}$.

Continuing in this way, we have $[3m - 2^{2l-1}, 3m - 2^{2l-3} - 1] \subseteq V_{nc}$ and $[3m + 2^{2l-4}, 3m + 2^{2l-2} - 1] \subseteq V_{nc}$ for $l = k, k-1, \ldots, 2$. It follows $[3m - 2^{2k-1}, 3m - 3] \subseteq V_{nc}$ and $[3m + 1, 3m + 2^{2k-2} - 1] \subseteq V_{nc}$, which and (5.2) imply that $[0, 3m - 3] \cup [3m, 9m - 1] \subseteq V_{nc}$.

(6) Since $[3m - 2, 3m - 1] \subseteq V_r$ and

$$N^+([3m - 2, 3m - 1]) = [3m, 3m + 3] \subseteq V_{nc},$$

by Lemma 2.1, we have $[3m - 2, 3m - 1] \subseteq V_{nc}$, which and (5.3) imply that $[0, 9m - 1] \subseteq V_{nc}$, i.e., $V_c = \emptyset$.

Case2. $t = 2$.

Since $n \equiv 2 (mod\ 9)$, that is, $n = 9m + 2, m \in Z_+$. Then we have

$F_1 = [0, \lfloor \frac{n}{3} \rfloor - 1] = [0, 3m - 1]$,
$F_2 = [\lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{3} \rfloor] = [3m, 3m]$,
$F_3 = [\lfloor \frac{n}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n-2}{3} \rfloor] = [3m + 1, 6m]$,
$F_4 = [\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n-2}{3} \rfloor + 1, \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n-2}{3} \rfloor + 1] = [6m + 1, 6m + 1]$,
$F_5 = [\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n-2}{3} \rfloor + 2, \lfloor \frac{n}{3} \rfloor + \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{n-8}{9} \rfloor + 1] = [6m + 2, 7m]$,
$F_6 = [\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n-2}{3} \rfloor + \lfloor \frac{n-8}{9} \rfloor + 2, n - 1] = [7m + 1, 9m + 1]$.

$\forall x \in V_d = F_2 \cup F_4 \cup F_6$, since V_d contains the vertices deleted from $V(GK(2,n))$ then $x \notin V_r$, i.e., $x \notin V_c$, that is $x \in V_{nc}$. Thus $V_d \subseteq V_{nc}$, i.e.,

(1) $[3m,3m] \cup [6m+1,6m+1] \cup [7m+1,9m+1] \subseteq V_{nc}$.

(2) Since $[0,m-1] \subseteq V_r$ and

$$N^+([0,m-1]) = [7m+2, 9m+1] \subseteq V_{nc},$$

by Lemma 2.1 we have $[0,m-1] \subseteq V_{nc}$, which and (1) imply that $[0,m-1] \cup [3m,3m] \cup [6m+1,6m+1] \cup [7m+1,9m+1] \subseteq V_{nc}$.

(3) Let $k = \lceil \frac{1}{2} \log \frac{3m-1}{8} \rceil$, then

$$\begin{array}{lcl} k & \geq & \frac{1}{2}\log\frac{3m-1}{8}, \\ 2^{2k+3} & \geq & 3m-1, \\ \frac{2^{2k+3}+1}{3} & \geq & m, \\ 6m + \frac{2^{2k+3}+1}{3} & \geq & 7m, \\ k & < & \frac{1}{2}\log\frac{3m-1}{8} + 1, \\ 2^{2k+1} & < & 3m-1, \\ \frac{2^{2k+2}+2}{3} & < & 2m. \end{array}$$

(3.1) Since $[4m+1, 6m - \frac{2^{2k+2}+2}{3} + 1] \subseteq V_r$ and

$$\begin{aligned} & N^+([4m+1, 6m - \tfrac{2^{2k+2}+2}{3} + 1]) \\ & = [6m + \tfrac{2^{2k+3}+1}{3} + 1, 9m+1] \cup [0, m-1] \\ & \subseteq [7m+1, 9m+1] \cup [0,m-1] \subseteq V_{nc}, \end{aligned}$$

by Lemma 2.1, we have $[4m+1, 6m - \frac{2^{2k+2}+2}{3} + 1] \subseteq V_{nc}$.

Since $[6m + \frac{2^{2k+1}+1}{3} + 1, 7m] \subseteq V_r$ and

$$N^+([6m + \tfrac{2^{2k+1}+1}{3} + 1, 7m]) = [4m+2, 6m - \tfrac{2^{2k+2}+2}{3} + 1] \subseteq V_{nc},$$

by Lemma 2.1, we have $[6m + \frac{2^{2k+1}+1}{3} + 1, 7m] \subseteq V_{nc}$, which and (2) imply that $[0,m-1] \cup [3m,3m] \cup [4m+1, 6m - \frac{2^{2k+2}+2}{3} + 1] \cup [6m+1, 6m+1] \cup [6m + \frac{2^{2k+1}+1}{3} + 1, 9m+1]$.

(3.2) Since $[6m - \frac{2^{2k+2}+2}{3} + 2, 6m - \frac{2^{2k}+2}{3} + 1] \subseteq V_r$ and

$$\begin{aligned} & N^+([6m - \tfrac{2^{2k+2}+2}{3} + 2, 6m - \tfrac{2^{2k}+2}{3} + 1]) \\ & = [6m + \tfrac{2^{2k+1}+1}{3} + 1, 6m + \tfrac{2^{2k+3}+1}{3}] \\ & \subseteq [6m + \tfrac{2^{2k+1}+1}{3} + 1, 9m+1] \subseteq V_{nc}, \end{aligned}$$

by Lemma 2.1, we have $[6m - \frac{2^{2k+2}+2}{3} + 2, 6m - \frac{2^{2k}+2}{3} + 1] \subseteq V_{nc}$.

Since $[6m + \frac{2^{2k-1}+1}{3} + 1, 6m + \frac{2^{2k+1}+1}{3}] \subseteq V_r$ and

$$\begin{aligned} &N^+([6m + \tfrac{2^{2k-1}+1}{3} + 1, 6m + \tfrac{2^{2k+1}+1}{3}]) \\ &= [6m - \tfrac{2^{2k+2}+2}{3} + 2, 6m - \tfrac{2^{2k}+2}{3} + 1] \subseteq V_{nc}, \end{aligned}$$

by Lemma 2.1, we have $[6m + \frac{2^{2k-1}+1}{3} + 1, 6m + \frac{2^{2k+1}+1}{3}] \subseteq V_{nc}$.

Continuing in this way, we have $[6m - \frac{2^{2l+2}+2}{3} + 2, 6m - \frac{2^{2l}+2}{3} + 1] \subseteq V_{nc}$ and $[6m + \frac{2^{2l-1}+1}{3} + 1, 6m + \frac{2^{2l+1}+1}{3}] \subseteq V_{nc}$ for $l = k, k-1, \ldots, 1$.

It follows that $[6m - \frac{2^{2k+2}+2}{3} + 2, 6m - 1] \subseteq V_{nc}$ and $[6m + 2, 6m + \frac{2^{2k+1}+1}{3}] \subseteq V_{nc}$, which and (3.1) imply that $[0, m-1] \cup [3m, 3m] \cup [4m + 1, 6m - 1] \cup [6m + 1, 9m + 1] \subseteq V_{nc}$.

(4) Since $[6m, 6m] \subseteq V_r$ and

$$N^+([6m, 6m]) = [6m + 2, 6m + 3] \subseteq V_{nc},$$

by Lemma 2.1, we have $[6m, 6m] \subseteq V_{nc}$, which and (3.2) imply that $[0, m-1] \cup [3m, 3m] \cup [4m + 1, 9m + 1] \subseteq V_{nc}$.

(5) Let $k = \lceil \frac{1}{2} \log \frac{m}{2} \rceil$, then

$$\begin{array}{lcl} k & \geq & \frac{1}{2} \log \frac{m}{2}, \\ 2^{2k} & \geq & \frac{m}{2}, \\ 3m + 2^{2k+1} & \geq & 4m, \\ 3m + 2^{2k+1} + 2 & \geq & 4m + 2, \\ k & < & \frac{1}{2} \log(2m - 1), \\ 2^{2k} & < & 2m - 1, \\ 3m - 2^{2k} - 1 & > & m. \end{array}$$

(5.1) Since $[m, 3m - 2^{2k} - 1] \subseteq V_r$ and

$$N^+([m, 3m - 2^{2k} - 1]) = [3m + 2^{2k+1} + 2, 7m + 1] \subseteq [4m + 1, 7m + 1] \subseteq V$$

by Lemma 2.1, we have $[m, 3m - 2^{2k} - 1] \subseteq V_{nc}$, which and (4) imply that $[0, 3m - 2^{2k} - 1] \cup [3m, 3m] \cup [4m + 1, 9m + 1] \subseteq V_{nc}$.

(5.2) Since $[3m + 2^{2k-1} + 1, 4m] \subseteq V_r$ and

$$N^+([3m + 2^{2k-1} + 1, 4m]) = [m, 3m - 2^{2k} - 1] \subseteq [0, 3m - 2^{2k} - 1] \subseteq V_{nc},$$

by Lemma 2.1, we have $[3m+2^{2k-1}+1, 4m] \subseteq V_{nc}$, which and (5.1) imply that $[0, 3m-2^{2k}-1] \cup [3m, 3m] \cup [3m+2^{2k-1}+1, 9m+1] \subseteq V_{nc}$.

(5.3) Since $[3m-2^{2k}, 3m-2^{2k-2}-1] \subseteq V_r$ and

$$\begin{aligned} & N^+([3m-2^{2k}, 3m-2^{2k-2}-1]) \\ = \ & [3m+2^{2k-1}+2, 3m+2^{2k+1}+1] \subseteq [3m+2^{2k-1}+1, 9m+1] \subseteq V_{nc}, \end{aligned}$$

by Lemma 2.1, we have $[3m-2^{2k}, 3m-2^{2k-2}-1] \subseteq V_{nc}$, which and (5.2) imply that $[0, 3m-2^{2k-2}-1] \cup [3m, 3m] \cup [3m+2^{2k-1}+1, 9m+1] \subseteq V_{nc}$.

(5.4) Since $[3m+2^{2k-3}+1, 3m+2^{2k-1}] \subseteq V_r$ and

$$\begin{aligned} & N^+([3m+2^{2k-3}+1, 3m+2^{2k-1}]) \\ = \ & [3m-2^{2k}, 3m-2^{2k-2}-1] \subseteq [0, 3m-2^{2k-2}-1] \subseteq V_{nc}, \end{aligned}$$

by Lemma 2.1, we have $[3m+2^{2k-3}+1, 3m+2^{2k-1}] \subseteq V_{nc}$, which and (5.3) imply that $[0, 3m-2^{2k-2}-1] \cup [3m, 3m] \cup [3m+2^{2k-3}+1, 9m+1] \subseteq V_{nc}$.

Continuing in this way, we have $[3m-2^{2l}, 3m-2^{2l-2}-1] \subseteq V_{nc}$ and $[3m+2^{2l-3}+1, 3m+2^{2l-1}] \subseteq V_{nc}$ for $l = k, k-1, \ldots, 2$.

It follows that $[3m-2^{2k}, 3m-5] \subseteq V_{nc}$ and $[3m+3, 3m+2^{2k-1}] \subseteq V_{nc}$ which and (5.4) imply that $[0, 3m-5] \cup [3m, 3m] \cup [3m+3, 9m+1] \subseteq V_{nc}$.

(6) Since $[3m-4, 3m-2] \subseteq V_r$ and

$$N^+([3m-4, 3m-2]) = [3m+4, 3m+9] \subseteq V_{nc},$$

by Lemma 2.1, we have $[3m-4, 3m-3] \subseteq V_{nc}$, which and (5.4) imply that $[0, 3m-2] \cup [3m, 3m] \cup [3m+3, 9m+1] \subseteq V_{nc}$.

(7) Since $[3m+2, 3m+2] \subseteq V_r$ and

$$N^+([3m+2, 3m+2]) = [3m-4, 3m-3] \subseteq V_{nc},$$

by Lemma 2.1, we have $[3m+2, 3m+2] \subseteq V_{nc}$, which and (6) imply that $[0, 3m-2] \cup [3m, 3m] \cup [3m+2, 9m+1] \subseteq V_{nc}$.

(8) Since $[3m-1, 3m-1] \subseteq V_r$ and

$$N^+([3m-1, 3m-1]) = [3m+2, 3m+3] \subseteq V_{nc},$$

by Lemma 2.1, we have $[3m-1, 3m-1] \subseteq V_{nc}$, which and (7) imply that $[0, 3m] \cup [3m+2, 9m+1] \subseteq V_{nc}$.

(9) Since $[3m+1, 3m+1] \subseteq V_r$ and

$$N^+([3m+1, 3m+1]) = [3m-2, 3m-1] \subseteq V_{nc},$$

by Lemma 2.1, we have $[3m+1, 3m+1] \subseteq V_{nc}$, which and (8) imply that $[0, 9m+1] \subseteq V_{nc}$, i.e., $V_c = \emptyset$. Thus, $G[V_r]$ is acyclic.

□

4 Upper Bound of the Minimum Feedback Vertex Number of $GK(2, n)$

Let $f(2, n)$ denote the minimum cardinality over all feedback vertex sets of $GK(2, n)$. By Lemma 3.1, we obtain the upper bound of the minimum feedback vertex number of $GK(2, n)$ as follows

Theorem 4.1. *The minimum feedback vertex set of $GK(2, n)$ is of size*

$$f(2, n) \leq n - (\lfloor\frac{n}{3}\rfloor + \lfloor\frac{n-2}{3}\rfloor + \lfloor\frac{n-8}{9}\rfloor).$$

Proof. By Lemma 3.1, we have $V_d = F_2 \cup F_4 \cup F_6$ is a feedback vertex set of $GK(2, n)$. Since

$$|F_2| = |[\lfloor\tfrac{n}{3}\rfloor, \lfloor\tfrac{n}{3}\rfloor]| = 1,$$
$$|F_4| = |[\lfloor\tfrac{n}{3}\rfloor + \lfloor\tfrac{n-2}{3}\rfloor + 1, \lfloor\tfrac{n}{3}\rfloor + \lfloor\tfrac{n-2}{3}\rfloor + 1]| = 1,$$
$$|F_6| = |[\lfloor\tfrac{n}{3}\rfloor + \lfloor\tfrac{n-2}{3}\rfloor + \lfloor\tfrac{n-8}{9}\rfloor + 2, n-1]| = n - (\lfloor\tfrac{n}{3}\rfloor + \lfloor\tfrac{n-2}{3}\rfloor + \lfloor\tfrac{n-8}{9}\rfloor + 2$$

Thus,

$$\begin{aligned} f(2,n) \leq |V_d| &= |F_2 \cup F_4 \cup F_6| = |F_2| + |F_4| + |F_6| \\ &= 1 + 1 + n - (\lfloor\frac{n}{3}\rfloor + \lfloor\frac{n-2}{3}\rfloor + \lfloor\frac{n-8}{9}\rfloor + 2) \\ &= n - (\lfloor\frac{n}{3}\rfloor + \lfloor\frac{n-2}{3}\rfloor + \lfloor\frac{n-8}{9}\rfloor). \end{aligned}$$

□

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