The Spectral Radius of Maximum Weighted Unicyclic Graph

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Abstract

Let $\mathcal{U}_m^W$ be the set of unicyclic weighted graphs of size m with weight W. In this paper, we determine the weighted graph in $\mathcal{U}_m^W$ with maximum spectral radius.

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1 Introduction

Let $\mathcal{G}_{n,m}$ be the set of simple connected graphs with n vertices and m edges. For $G \in \mathcal{G}_{n,m}$, let $V(G) = \{v_1, v_2, \cdots, v_n\}$. Given weight multiset $W = \{w_1 \geq w_2 \geq \cdots \geq w_m > 0\}$ and let f be a bijection from $E(G)$ to W, which is denoted by $f(v_iv_j) = w_{ij}$ for every $v_iv_j \in E(G)$. A *weighted graph* of G(with respect to W and f), denoted by G_f^W, is the graph G along with weight $f(v_iv_j) = w_{ij}$ for every edge $v_iv_j \in E(G)$. G_f^W can also be represented by the *weighted adjacency matrix* $A_f^W = (a_{ij}^W)_{n\times n}$ of G_f, where

$$a_{ij}^W = \begin{cases} f(v_iv_j) = w_{ij} & if \quad v_iv_j \in \mathrm{E} \\ 0 & if \quad v_iv_j \notin \mathrm{E} \end{cases}$$

Clearly, A_f^W is non-negative and symmetric matrix. The spectrum of A_f^W is called the *spectrum of* G_f^W and the *spectral radius* of G_f^W is denoted by $\rho(G_f^W)$. It is well known that $\rho(G_f^W)$ is simple eigenvalue of A_f^W corresponding to positive eigenvector which is called *Perron-vector*. Given $G \in \mathcal{G}_{n,m}$ and weight multiset W, let $G^W = \{G_f^W \mid f : E(G) \longleftrightarrow W\}$ be all the

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weighted graph of G with respect to *weight function* f. We call a weighted graph $G^W_{f^*}$ the *maximal weighted graph* of G if $\rho(G^W_{f^*}) \geq \rho(G^W_f)$ for any $G^W_f \in G^W$. Denote by $\mathcal{G}^W_{n,m} = \cup_{G\in\mathcal{G}_{n,m}} G^W$ the class of weighted graphs of G in $\mathcal{G}_{n,m}$ with given weight multiset W. We call $G^W_{f^{**}}$ the *maximum weighted graph* if $\rho(G^W_{f^{**}}) \geq \rho(G^W_f)$ for any $G^W_f \in \mathcal{G}^W_{n,m}$.

A *unicyclic graph* is a connected graph with exactly one cycle. Let $\mathcal{U}_m$ be the class of all the unicyclic graphs each of them with m edges, and $\mathcal{U}^W_m$ the set of unicyclic graphs with given weight W.

Brualdi and Solheid [1] posed the following problem concerning the spectral radius of graphs: "given a set $\mathscr{S}$ of graphs, to find an upper bound for the spectral radius of graphs in $\mathscr{S}$ and characterize the graphs in which the maximal spectral radius is attained." The spectral radius of unicyclic graphs are investigated by some authors in [2, 3, 4, 5, 6]. However, the spectrum of weighted graph is less considered. There are few of results related on weighted unicyclic graphs. Yang et al [7] gave an upper bound of spectral radius of weighted trees. Yuan Jingsong et al [9] gave the spectral radius of the weighted double-star. In this paper, we will give the maximum weighted unicyclic graph in $\mathcal{U}^W_m$(see Theorem 2.2) whose spectral radius are determined by Theorem 2.9. The Corollary 2.10 shows that the spectral radius of maximum weighted unicyclic graph depends on the weight function, which is shown in the Table 1.

2 Main Results

As similar as the result for (unweighted) graph in [8], we have the following.

Lemma 2.1 ([9]). *Let u and v be two vertices of a connected weighted graph G^W_f with positive weights W. $v_1, v_2, ..., v_s$ $(1 \leq s \leq d(v))$ are some vertices of $N_G(v)\backslash N_G(u)$ and $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ is the Perron vector of G^W_f, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let $H^W_{f'}$ be the graph obtained from G^W_f by rotating the edges v_iv to the new edges v_iu together with the weights $f(v_iv)$ for $i = 1, 2, ..., s$. If $x_u \geq x_v$, then $\rho(G^W_f) < \rho(H^W_{f'})$.*

The weighted graph $H^W_{f'}$ stated in Lemma 2.1 is called the *transfiguration* from G^W_f by rotating v_iv to v_iu.

Theorem 2.2. *Let $G^W_{f^{**}}$ be a maximum weighted graph in $\mathcal{U}^W_m$. Then G must be the graph consisting of a 3-cycle $C_3 = v_1v_2v_3v_1$ such that v_1 joins $m - 3$ pendant vertices.*

Proof. Let $C = v_1v_2\cdots v_kv_1$ be the k-cycle of G, and $\mathbf{x}$ be the Perron vector of the maximum weighted graph $G^W_{f^{**}}$ whose entries is denoted by x_v at vertex v. Suppose that $k \geq 4$ and, without loss of generality, assume that $x_{v_1} \geq x_{v_4}$, we will get a transfiguration $H^W_{f'}$ from $G^W_{f^{**}}$ by rotating v_3v_4 to v_3v_1, then $\rho(G^W_{f^{**}}) < \rho(H^W_{f'})$ by Lemma 2.1. Clearly, $H^W_{f'} \in \mathcal{U}^W_m$ which contradicts the assumption of $G^W_{f^{**}}$. Thus $C = C_3$ is a 3-cycle.

If $d(v_1) > 2$ and $d(v_2) > 2$, without loss of generality assume that $x_{v_1} \geq x_{v_2}$, and let $N(v_2)\backslash V(C) = \{u_1, u_2, ..., u_r\}$. Let $H^W_{f'}$ be the transfiguration getting from $G^W_{f^{**}}$ by rotating u_iv_2 to u_iv_1 for $i = 1, 2, ..., r$. Then $\rho(G^W_{f^{**}}) < \rho(H^W_{f'})$ by Lemma 2.1, a contradiction. Thus there exists only one vertex in C_3 such that its degree greater than 2, say $d(v_1) > 3$ and $d(v_2) = d(v_3) = 2$.

At last we will show that the vertices in $U = V(G)\backslash V(C_3)$ must be adjacent to v_1. Otherwise, there exists a vertex $u \in U$ such that $uv_1 \notin E(G)$. Let P be a path connecting v_1 and u, and let u_1 be the vertex on P adjacent to u. If $x_{v_1} \geq x_{u_1}$ then rotating uu_1 to uv_1 we get $H^W_{f'} \in \mathcal{U}^W_m$ with large spectral radius by Lemma 2.1; if $x_{v_1} < x_{u_1}$ then rotating v_2v_1 and v_3v_1 to v_2u_1 and v_3u_1, respectively, we get $H^W_{f'} \in \mathcal{U}^W_m$ with large spectral radius by Lemma 2.1. It is a contradiction and the proof is completed. □

In what follows we denote by Δ the unicyclic graph of size m having a 3-cycle $v_1v_2v_3v_1$ with $m-3$ pendant vertices $v_4, ..., v_n$ joining to v_1, and $\Delta^W = \{\Delta^W_f \mid f : E(\Delta) \longleftrightarrow W\}$ where Δ^W_f is shown in Fig.1(a). Theorem 2.2 shows that the maximum graph $G^W_{f^{**}}$ of $\mathcal{U}^W_m$ belongs to Δ^W.

Lemma 2.3. *Let A^W_f be the weighted adjacency matrix of $\Delta^W_f \in \Delta^W$ where $f(v_iv_j) = w_{ij}$ for $v_iv_j \in E(\Delta)$. Then all the nonzero eigenvalues of A^W_f are the roots of the equation:*

$$\lambda^4 - (\sum_{v_i \sim v_j} w_{ij}^2)\lambda^2 - 2w_{12}w_{13}w_{23}\lambda + w_{23}^2(\sum_{4\leq i\leq n} w_{1i}^2) = 0 \qquad (1)$$

Proof. Let $\lambda \neq 0$ be an eigenvalue of A^W_f and $\mathbf{x}$ the eigenvector of λ whose entries is denoted by x_{v_i} at vertex v_i. From $A^W_f\mathbf{x} = \lambda\mathbf{x}$, we have

$$\lambda x_{v_1} = w_{12}x_{v_2} + w_{13}x_{v_3} + \sum_{4\leq i\leq n} w_{1i}x_{v_i} \qquad (2)$$

$$\lambda x_{v_2} = w_{21}x_{v_1} + w_{23}x_{v_3} \qquad (3)$$

$$\lambda x_{v_3} = w_{31}x_{v_1} + w_{32}x_{v_2} \qquad (4)$$

$$\lambda x_{v_i} = w_{i1}x_{v_1} \quad (i = 4, ..., n) \qquad (5)$$

By multiplying λ to (2) and (3), we obtain the following (6) and (7)

$$\lambda^2 x_{v_1} = \lambda w_{12} x_{v_2} + \lambda w_{13} x_{v_3} + \lambda \sum_{4 \le i \le n} w_{1i} x_{v_i} \quad (6)$$

$$\lambda^2 x_{v_2} = \lambda w_{12} x_{v_1} + \lambda w_{23} x_{v_3} \quad (7)$$

Put (4) and (5) to (6), we obtain (8)

$$(\lambda^2 - \sum_{3 \le i \le n} w_{1i}^2) x_{v_1} = (\lambda w_{12} + w_{13} w_{23}) x_{v_2} \quad (8)$$

Put (4) to (7), we obtain (9)

$$(\lambda^2 - w_{23}^2) x_{v_2} = (\lambda w_{12} + w_{13} w_{23}) x_{v_1} \quad (9)$$

First we claim that x_{v_2} and x_{v_3} can not be zero simultaneously. In fact, if $x_{v_2} = x_{v_3} = 0$, we get $x_{v_1} = 0$ from (3), and so $x_{v_j} = 0$ by (5) for $j = 4, 5, ..., n$, thus $\mathbf{x} = \mathbf{0}$, a contradiction.

Next we claim that x_{v_1} and x_{v_2} can not be zero simultaneously. In fact, if $x_{v_1} = 0$ then $x_{v_i} = 0$ by (5) for $i = 4, 5, ..., n$, and return to (2), (3) and (4) we have

$$\begin{cases} w_{12} x_{v_2} + w_{13} x_{v_3} = 0 \\ \lambda x_{v_2} = w_{23} x_{v_3} \\ \lambda x_{v_3} = w_{32} x_{v_2} \end{cases} \quad (10)$$

If $x_{v_2} = 0$, then $x_{v_3} = 0$, and thus $\mathbf{x} = \mathbf{0}$, a contradiction. Similarly, x_{v_1} and x_{v_3} can not be zero simultaneously.

In addition, from (10) we know that if $x_{v_1} = 0$ and $x_{v_2} \ne 0$ then we have $w_{13} = w_{12}$ and $\lambda = -w_{23}$ that corresponds the eigenvector: $x_{v_1} = x_{v_j} = 0$ for $j = 4, 5, ..., m$ and $x_{v_2} = -x_{v_3} \ne 0$; if $x_{v_1} = 0$ and $x_{v_3} \ne 0$ then we also have $w_{13} = w_{12}$ and $\lambda = -w_{23}$ as the same as above.

According to above arguments, we need to consider two cases bellow.

Case 1. $x_{v_1} \ne 0$ and x_{v_2}, x_{v_3} can not equal zero simultaneously.

Without loss of generality, we assume $x_{v_2} \ne 0$. Eliminating x_{v_1} and x_{v_2} from (8) and (9), we obtain (1).

Case 2. $x_{v_1} = 0$, $x_{v_2} \ne 0$ and $x_{v_3} \ne 0$.

In this case, we know that $\lambda = -w_{23}$ and $w_{12} = w_{13}$. Thus one can verify that $\lambda = -w_{23}$ is also a root of (1).

Conversely, let λ be a root of (1), we will show that there exists $\mathbf{x} = (x_{v_1}, x_{v_2}, ..., x_{v_n}) \ne 0$ such that $A_f^W \mathbf{x} = \lambda \mathbf{x}$. Now we need to consider two cases. If $\lambda \ne -w_{23}$, then we take $x_{v_1} \ne 0$ and get $x_{v_i} = \frac{w_{i1}}{\lambda} x_{v_1}$ for $i = 4, 5, ..., m$, $x_{v_2} = \frac{w_{13} w_{23} + \lambda w_{12}}{\lambda^2 - w_{23}^2} x_{v_1}$, $x_{v_3} = \frac{w_{12} w_{23} + \lambda w_{13}}{\lambda^2 - w_{23}^2} x_{v_1}$ from (5), (3)

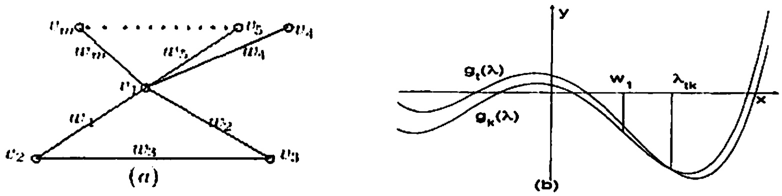

Figure 1: Δ_f^W is shown in (a), and $g_t(\lambda_{tk}) = g_k(\lambda_{tk}) < 0$ are shown in (b)

and (4) respectively. From the fact that λ is a root of (1), we can verify $\mathbf{x}$ satisfy (2). Hence $A_f^W \mathbf{x} = \lambda \mathbf{x}$. If $\lambda = -w_{23}$ then $w_{12} = w_{13}$ from (1). Set $x_{v_1} = x_{v_4} = \cdots = x_{v_n} = 0$ and $x_{v_2} = -x_{v_3} \neq 0$. We can verify the equations of (2), (3), (4) and (5). Hence $A_f^W \mathbf{x} = \lambda \mathbf{x}$.

We end the proof. □

Lemma 2.4. *If $\Delta_{f^*}^W$ is maximal in Δ^W, then $\{f^*(v_1v_2), f^*(v_1v_3)\} = \{w_1, w_2\}$ where W is ordered as $w_1 \geq w_2 \geq \cdots \geq w_m$.*

Proof. Let $\mathbf{x} = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T$ be the Perron vector of $\Delta_{f^*}^W$, and $A_{f^*}^W$ the weighted adjacent matrix. We first prove $x_{v_1} > x_{v_j}$ $(j = 2, 3, \ldots, n)$. Otherwise, without loss of generality, assume that $x_{v_2} \geq x_{v_1}$. By rotating $v_1v_4, \ldots, v_1v_n$ to $v_2v_4, \ldots, v_2v_n$, respectively, we get a weighted graph Δ_f^W from $\Delta_{f^*}^W$ such that $\rho(\Delta_f^W) > \rho(\Delta_{f^*}^W)$ by Lemma 2.1, a contradiction. By the same way, we can also prove that $x_{v_2}, x_{v_3} > x_{v_i}$ $(i = 4, 5, \ldots, n)$. Then $x_{v_1}x_{v_2}, x_{v_1}x_{v_3} > x_{v_1}x_{v_j}$ $(j = 4, 5, \ldots, n)$, $x_{v_1}x_{v_2}, x_{v_1}x_{v_3} > x_{v_2}x_{v_3}$.

Now suppose that $f^*(v_2v_3) > f^*(v_1v_2)$, by exchanging the weights on v_1v_2 and v_2v_3, we get $\Delta_{f'}^W$ from $\Delta_{f^*}^W$ with weighted adjacent matrix $A_{f'}^W$. Then

$$x^T A_{f'}^W x - x^T A_{f^*}^W x = 2(x_{v_1}x_{v_2} - x_{v_2}x_{v_3})(f^*(v_2v_3) - f^*(v_1v_2)) > 0$$

The Rayleigh principle implies $\rho(\Delta_{f'}^W) > \rho(\Delta_{f^*}^W)$, a contradiction. Similarly, $f^*(v_2v_3) < f^*(v_1v_3)$. Thus we obtain our result. □

In what follows W is ordered as non-increased sequence: $w_1 \geq w_2 \geq \cdots \geq w_m$. According to Lemma 2.3, we may assume that $f^*(v_1v_2) = w_1$, $f^*(v_1v_3) = w_2$. Therefore, to determine the maximal weighted graph $\Delta_{f^*}^W$, we only need to decide the value $f^*(v_2v_3) \in \{w_3, w_4, \ldots, w_n\}$. We now focus to consider weight graph $\Delta_{f_k}^W$ in Δ^W where f_k is defined by

$$f_k(e) = \begin{cases} w_1, & \text{if } e = v_1v_2 \\ w_2, & \text{if } e = v_1v_3 \\ w_k, & \text{if } e = v_2v_3 \end{cases} \text{ and } f_k : \{v_1v_l \mid 3 \leq l \leq m\} \longleftrightarrow W \backslash \{w_1, w_2, w_k\}.$$

In what follows we will determine $f^* = f_t$ for some $t \geq 3$. To this end, let us define

$$g_k(\lambda) = \lambda^4 - \sum_{1\leq i\leq m} w_i^2\lambda^2 - 2w_1w_2w_k\lambda + w_k^2 \sum_{3\leq i\neq k\leq m} w_i^2 \quad (11)$$

which is obtained from (1) by replacing w_{12}, w_{13}, w_{23} with w_1, w_2, w_k respectively. Thus, according to Lemma 2.3 and Lemma 2.4, we have the following.

Corollary 2.5. *Let $\Delta^W_{f^*}$ be maximal in Δ^W. Then $\rho(\Delta^W_{f^*}) = max\{\rho(\Delta^W_{f_k}) \mid 3 \leq k \leq m\} = \max\{\lambda \mid g_k(\lambda) = 0, 3 \leq k \leq m\}$.*

Taking any $w_t \neq w_k (3 \leq k \neq t \leq m)$, from (11) we have

$$g_t(\lambda) - g_k(\lambda) = -2w_1w_2(w_t - w_k)\lambda + (w_t^2 - w_k^2)\Big(\sum_{3\leq i\neq k,t\leq m} w_i^2\Big) \quad (12)$$

Let λ_{tk} be the unique root of equation (12) which is adhered in what follows. Then

$$\lambda_{tk} = \frac{(w_t + w_k)(\sum_{3\leq i\neq k,t\leq m} w_i^2)}{2w_1w_2} > 0 \quad (13)$$

Lemma 2.6. *$w_t > w_k$ if and only if $g_t(\lambda) - g_k(\lambda) < 0$ whenever $\lambda > \lambda_{tk}$; $w_t = w_k$ if and only if $g_t(\lambda) = g_k(\lambda)$.*

Proof. From (12), we know that $g_t(\lambda_{tk}) - g_k(\lambda_{tk}) = 0$ and the sign of $g_t(\lambda) - g_k(\lambda)$ for $\lambda > \lambda_{tk}$ is accordance with that of the coefficient of λ in (12). Thus if $w_t > w_k$ then $g_t(\lambda) - g_k(\lambda) < 0$ whenever $\lambda > \lambda_{tk}$. Conversely, if $g_t(\lambda) - g_k(\lambda) < 0$ whenever $\lambda > \lambda_{tk}$ then $g_t(\lambda) - g_k(\lambda) \longrightarrow -\infty$ while λ tends to $+\infty$, which implies that the coefficient of λ in (12) is negative and so $w_t > w_k$.

The next part of this lemma is obvious. □

Notice that $w_t = w_k$ if and only if $g_t(\lambda) = g_k(\lambda)$. To find the largest root of $g_t(\lambda)$ we may assume that the given weights are distinct, i.e., $w_1 > w_2 > \cdots > w_m$. Additionally, the largest root of $g_k(\lambda)$ is simple by Perron-Frobinius theorem. Further, we have the following.

Lemma 2.7. *Let $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ be the roots of $g_k(\lambda)(3 \leq k \leq m)$. Then $\lambda_1 > \lambda_2 > 0 > \lambda_3 \geq \lambda_4$ and $w_1 \in (\lambda_2, \lambda_1)$.*

Proof. First from (11) we know that $g_k(\lambda)$ has no zero root, and

$$\begin{aligned} g_k(w_1) &= w_1^4 - \Big(\sum_{1\leq i\leq m} w_i^2\Big)w_1^2 - 2w_1^2w_2w_k + w_k^2\Big(\sum_{3\leq j\neq k\leq m} w_j^2\Big) \\ &= -w_1^2\Big(\sum_{2\leq i\leq m} w_i^2\Big) - 2w_1^2w_2w_k + w_k^2\Big(\sum_{3\leq j\neq k\leq m} w_j^2\Big) < 0 \end{aligned}$$

Since $\lambda_1 \geq \max\{|\lambda_2|, |\lambda_3|, |\lambda_4|\}$ and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, we have $\lambda_3 < 0$. Additionally, the image of the function $g_k(\lambda)$ is of "W" type and $g_k(0) > 0$. We claim that $w_1 \in (\lambda_2, \lambda_1)$ and $\lambda_1 > \lambda_2 > 0 > \lambda_3 \geq \lambda_4$. □

Lemma 2.8. *Let $3 \leq k \neq t \leq m$, and $w_1 > w_2 > \cdots > w_m$. Then*

(1) *Suppose $g_t(\lambda_{tk}) < 0$. Then $w_k < w_t$ if and only if $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$.*

(2) *Suppose $g_t(\lambda_{tk}) > 0$. Then $(w_k - w_t)((w_k + w_t)\sum_{3\leq i\neq t,k\leq m} w_i^2 - 2w_1^2 w_2) > 0$ if and only if $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$.*

(3) *Suppose $g_t(\lambda_{tk}) = 0$. Then $(w_k + w_t)\sum_{3\leq i\neq t,k\leq m} w_i^2 - 2w_1^2 w_2 < 0$ and $w_k < w_t$ if and only if $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$, and $(w_k + w_t)\sum_{3\leq i\neq t,k\leq m} w_i^2 - 2w_1^2 w_2 > 0$ if and only if $\rho(\Delta^W_{f_k}) = \rho(\Delta^W_{f_t})$*

Proof. (1). If $w_t > w_k$ then $g_t(\lambda) - g_k(\lambda) < 0$ whenever $\lambda > \lambda_{tk}$ by Lemma 2.6, and hence the image of $g_t(\lambda)$ is below that of $g_k(\lambda)$ whenever $\lambda > \lambda_{tk}$(see Fig.1(b)). Since $g_t(\lambda)$($g_k(\lambda)$) has only two positive eigenvalues by Lemma 2.7, and $g_t(0) > 0$($g_k(0) > 0$), $g_t(\lambda_{tk}) < 0(g_k(\lambda_{tk}) < 0)$. Thus we claim that $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$.

Conversely, first suppose that $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$. By the way of contradiction, let $w_t < w_k$. Then $g_t(\lambda) - g_k(\lambda) > 0$ whenever $\lambda > \lambda_{tk}$ by Lemma 2.6, and hence the image of $g_t(\lambda)$ is up that of $g_k(\lambda)$ whenever $\lambda > \lambda_{tk}$. Since $g_t(\lambda_{tk}) < 0$, we have $\rho(\Delta^W_{f_k}) > \rho(\Delta^W_{f_t})$ as above arguments, a contradiction.

(2). Taking $\lambda = w_1$ in (12), we have

$$g_k(w_1) - g_t(w_1) = (w_k - w_t)((w_k + w_t)\sum_{3\leq i\neq t,k\leq m} w_i^2 - 2w_1^2 w_2).$$

By assumption, $g_k(w_1) - g_t(w_1) > 0$. Then the image of $g_t(\lambda)$ is below that of $g_k(\lambda)$ between their second and first large eigenvalues by Lemma 2.7(see Fig.2). Thus $\rho(\Delta^W_{f_k}) \leq \rho(\Delta^W_{f_t})$, and further $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$ since $g_t(\lambda_{tk}) > 0$. Conversely, suppose that $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$. If $g_k(w_1) - g_t(w_1) < 0$ then $g_t(w_1) - g_k(w_1) > 0$. Since $g_k(\lambda_{tk}) = g_t(\lambda_{tk}) > 0$(note that $\lambda_{tk} = \lambda_{kt}$ by definition), we claim that $\rho(\Delta^W_{f_t}) < \rho(\Delta^W_{f_k})$ as the arguments above, a contradiction; if $g_k(w_1) - g_t(w_1) = 0$ then $\lambda_{tk} = w_1$ by (12). However, $g_k(\lambda_{tk}) = g_t(\lambda_{tk}) > 0$ and $g_k(w_1) = g_t(w_1) < 0$ by Lemma 2.7, a contradiction. Hence $g_k(w_1) - g_t(w_1) > 0$.

(3). Since $g_t(\lambda_{tk}) = 0$, we have λ_{tk} is the first large of second large root of both $g_t(\lambda) = 0$ and $g_k(\lambda) = 0$.

If $(w_k + w_t)\sum_{3\leq i\neq t,k\leq m} w_i^2 - 2w_1^2 w_2 < 0$, that is, $\lambda_{tk} < w_1$, then λ_{tk} is the second large root by Lemma 2.7. If $w_k < w_t$ then $g_t(\lambda) - g_k(\lambda) < 0$ whenever $\lambda > \lambda_{tk}$ by Lemma 2.6, and hence the image of $g_t(\lambda)$ is below that of $g_k(\lambda)$ whenever $\lambda > \lambda_{tk}$(see Fig.1(b)). Thus we claim $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$. Conversely, if $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$, then λ_{tk} must be the second large root since $0 = g_t(\lambda_{tk}) = g_k(\lambda_{tk})$. Hence $g_t(\lambda) - g_k(\lambda) < 0$ for $\lambda > \lambda_{tk}$, and $\lambda_{tk} < w_1$ by Lemma 2.7. The former gives $w_t > w_k$ from Lemma 2.6, and the latter gives $\lambda_{tk} < w_1$, that is, $(w_k + w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 < 0$.

If $(w_k + w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 > 0$, that is, $\lambda_{tk} > w_1$, then λ_{tk} is the first large root by Lemma 2.7. It follows that $\rho(\Delta^W_{f_k}) = \rho(\Delta^W_{f_t}) = \lambda_{tk}$, and the vice verse.

We complete the proof. □

From Lemma 2.8, we obtain the necessary and sufficiency condition for $\Delta^W_{f_t}$ to be the maximal graph in Δ^W.

Theorem 2.9. *Let $\Delta^W_{f_t} \in \Delta^W (3 \le t \le m)$, where $m \ge 4$ and $W = \{w_1 > w_2 > \cdots > w_m\}$. Then $\Delta^W_{f_t}$ is maximum graph in $\mathcal{U}^W_m$ if and only if (14) holds.*

$$\begin{cases} (w_k+w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 > 0 \text{ and } g_t(\lambda_{tk}) \ge 0, \text{ for } 3\le k\le t-1 \text{ (if any)} \\ (w_k+w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 < 0 \text{ if } g_t(\lambda_{tk}) > 0, \text{ for some } t+1\le k\le m \end{cases} \quad (14)$$

Proof. Suppose that $\Delta^W_{f_t}$ is the maximal graph in Δ^W for some $3 \le t \le m$. For $3 \le k \le t-1$, $w_k > w_t$. Then $g_t(\lambda_{tk}) \ge 0$ by Lemma 2.8(1). If $g_t(\lambda_{tk}) > 0$ then $(w_k + w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 > 0$ Lemma 2.8(2); if $g_t(\lambda_{tk}) = 0$ then $(w_k+w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 > 0$ by the latter condition of Lemma 2.8(3). Thus the first term of (14) holds. For $t + 1 \le k \le m$, $w_k < w_t$. If $g_t(\lambda_{tk}) > 0$, then $(w_k - w_t)((w_k + w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2) > 0$ by Lemma 2.8(2), which gives the second term of (14).

Conversely, we consider two cases. First we suppose that $3 \le k \le t - 1$. Then $(w_k + w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 > 0$ and $g_t(\lambda_{tk}) \ge 0$ by assumption. If $g_t(\lambda_{tk}) > 0$ then $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$ by Lemma 2.8(2) since $w_k > w_t$; if $g_t(\lambda_{tk}) = 0$ then $\rho(\Delta^W_{f_k}) = \rho(\Delta^W_{f_t})$ by the latter condition of Lemma 2.8(3). Next we suppose that $t + 1 \le k \le m$. If $g_t(\lambda_{tk}) > 0$ then $(w_k+w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 < 0$ by assumption. In this situation we have $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$ by Lemma 2.8(2) since $w_t > w_k$. If $g_t(\lambda_{tk}) < 0$ then $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$ by Lemma 2.8(1) since $w_t > w_k$. For $g_t(\lambda_{tk}) = 0$, we again divide three subcases. If $(w_k + w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 < 0$, then $\rho(\Delta^W_{f_k}) < \rho(\Delta^W_{f_t})$ by the first condition of Lemma 2.8(3) since $w_t > w_k$; if $(w_k+w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 > 0$, then $\rho(\Delta^W_{f_k}) = \rho(\Delta^W_{f_t})$ by the second condition of Lemma 2.8(3); if $(w_k + w_t)\sum_{3\le i\ne t,k\le m} w_i^2 - 2w_1^2 w_2 = 0$ then $\lambda_{tk} = w_1$, and so $0 = g_t(\lambda_{tk}) = g_t(w_1) < 0$ by Lemma 2.7, a contradiction.

We complete the proof. □

Clearly, the extremal situations of Theorem 2.9 can be simplified as bellow.

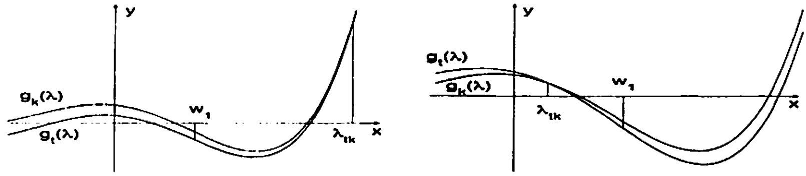

Figure 2: $g_t(\lambda_{tk})>0$ and $g_t(w_1) - g_k(w_1) < 0$ are shown in Fig.2

Corollary 2.10. *$\Delta^W_{f_3}$is a maximal in Δ^Wif and only if $(w_k+w_3)\sum_{4\leq i\neq k\leq m} w_i^2 < 2w_1^2w_2$ if $g_3(\lambda_{3k}) > 0$ for some $4 \leq k \leq m$. $\Delta^W_{f_m}$ is a maximal in Δ^W if and only if $(w_k + w_m)\sum_{4\leq i\neq k\leq m} w_i^2 > 2w_1^2w_2$ and $g_t(\lambda_{tk}) \geq 0$ for any $3 \leq k \leq m-1$.*

The condition "$g_t(\lambda_{tk}) > 0$" in (14) is simple since both $g_t(\lambda)$ and λ_{tk} are clearly expressed in (11) and (13) respectively. At last, by putting weighted adjacency matrix of Δ^W_f to MATLAB program, we give the Table 1 to illustrate that the maximum graph $\Delta^W_{f_t} \in \Delta^W$ can achieve at any possible value $3 \leq t \leq m$, which can also be verified by our Theorem 2.9 and Corollary 2.10. From Table 1 we know that $\Delta^W_{f_8}$, $\Delta^W_{f_3}$ and $\Delta^W_{f_6}$ are three maximum weighted unicyclic graphs corresponding to different weights W.

Table 1:

k	1	2	3	4	5	6	7	8	
$W = \{w_k \mid k\}$	2	2	1.9	1.8	1.7	1.7	1.7	1.6	max
$\rho(\Delta^W_{f_k})$			5.20306	5.20303	5.20310	5.20310	5.20310	5.20312	$\rho(\Delta^W_{f_8})$
$W = \{w_k \mid k\}$	3	2	1.9	1.8	1.7	1.7	1.7	1.6	max
$\rho(\Delta^W_{f_k})$			5.78450	5.77739	5.77035	5.77035	5.77035	5.76327	$\rho(\Delta^W_{f_3})$
$W = \{w_k \mid k\}$	2	2	1.9	1.8	1.8	1.7	1.7	1.5	max
$\rho(\Delta^W_{f_k})$			5.20645	5.20646	5.20646	5.20658	5.20658	5.20649	$\rho(\Delta^W_{f_6})$

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