Color degree and heterochromatic paths in edge-colored graphs

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Abstract

Let G be an edge-colored graphs. A heterochromatic path of G is such a path in which no two edges have the same color. Let $g^c(G)$ and $d^c(v)$ denote the heterochromatic girth and the color degree of a vertex v of G, respectively. In this paper, some color degree and heterochromatic girth conditions for the existence of heterochromatic paths are obtained.

Key words: heterochromatic path; heterochromatic girth; color degree.

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1 Terminology and Introduction

We only consider finite undirected simple graphs. Any undefined notations follow that of Bondy and Murty [1]. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph G, respectively. An edge coloring of G means a function $C : E(G) \to N$, the set of natural numbers. If G is assigned such a coloring, then we say that G is an edge colored graph. Denote the edge colored graph by (G, C), and call $C(e)$ the color of the edge $e \in E$. A heterochromatic(rainbow, or multicolored) path of G is such a

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path in which no two edges have the same color. A cycle in an edge colored graph is called heterochromatic if any two edges have distinct colors. The heterochromatic girth of G is the length of a shortest heterochromatic cycle in G, and we denote the heterochromatic girth of G by $g^c(G)$. For a vertex v of G, we say that color i is represented at vertex v if some edge incident with v has color i. The color degree $d^c(v)$ is the number of different colors that are represented at v, and the color neighborhood $CN(v)$ is the set of different colors that are represented at v. A maximum color neighborhood $N^c(v)$ is a color neighborhood of v with maximum size. Let P be a path, if u and v are two vertices on a path P, uPv denotes the segment of P from u to v, whereas $vP^{-1}u$ denotes the same segment but from v to u. A z-path is a path with z as one of its end-vertices. For a vertex $v \in V$ and a subgraph H of G, the neighbor set of v in H, $N_H(v)$, is defined to be the set of vertices of H adjacent to v, and the degree of v in H, $d_H(v)$, is defined as the size of $N_H(v)$.

In the middle of last century, many important theorems on long path and long cycles were given. Note that G contains either a Hamilton cycle or a cycle of length at least c, which implies that G contains either a Hamilton path or a path of length at least $c - 1$.

In 1952, Dirac showed the degree condition for a given graph G containing a path.

Theorem 1.1 [7] For a given graph G and a given integer d, if $d(v) \geq d$ for every vertex v of G. Then G contains a path of length at least d.

In 1963, pósa studied the parameter for every pair of nonadjacent vertices in G to obtain sufficient conditions for various kinds of cycles. The following conclusion is obtained.

Theorem 1.2 [9] For a 2-connected graph G and a given integer c, if $d(u) + d(v) \geq c$ for every pair of nonadjacent vertices u and v in G, then G contains either a Hamilton cycle or a cycle of length at least c.

In recent years, the research on long heterochromatic paths and long heterochromatic cycles has become an important domain. There are many existing literatures dealing with the existence of paths and cycles with special properties in edge colored graphs. In [4], Chen and Li showed that for an edge colored graph G with a $k-$*good coloring*, i.e., $d^c(v) \geq k$ for any $v \in V(G)$, if $3 \leq k \leq 7$, then G has a heterochromatic path of length at least $k-1$, whereas if $k \geq 8$, then G has a heterochromatic path of length at least $\lceil \frac{2k}{3} \rceil + 1$. In [5], Chen and Li studied the long heterochromatic paths in heterochromatic triangle free graphs. In [6], Chou, Manoussakis, Megalaki, et al. Showed that for a 2-edge colored graph and three specified vertices x, y and z, to decide whether there exists a color-alternating path from x to y passing through z is NP-complete. In [8], Li and Wang investigate the long heterochromatic cycles in edge-colored graphs. Thomason and Wagner [10] showed that for an integer $t \leq 5$, very few edge colorings of the complete graph k_n using at least t colors contain no rainbow path P_{t+1}.

In 2005, Broersma, Li, Woeginger, et al. Studied long heterochromatic paths in edge colored graphs, and obtained the following results.

Theorem 1.3 [2] For an edge colored graph G and an integer k, if $d^c(v) \geq k$ for every vertex v of G. Then for every vertex z of G there exists a heterochromatic z-path of length at least $\lceil \frac{k+1}{2} \rceil$.

Theorem 1.4 [2] For an edge colored graph G and an integer s, if $|CN(u) \cup CN(v)| \geq s \geq 1$ for every pair of vertices u and v of G, then G contains a heterochromatic path of length at least $\lceil \frac{s}{3} \rceil + 1$.

Very recently, Chen and Li [3] proposed the following Conjecture.

Conjecture 1.5 [3] Let G be an edge colored graph and $k \geq 3$ is an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Then G has a heterochromatic path of length at least $k-1$.

In this paper, based on conjecture 1.5, we conclude the following result.

Theorem A. Let G be an edge colored graph with $g^c(G) \geq k+1$, where $k \geq 3$ is an integer. If $d^c(v) \geq d$ for any vertex $v \in G$. Then G has a heterochromatic path of length at least $\lceil \frac{kd}{k+1} \rceil$.

In Section 2, we will prove some lemmas that will be used in this proof of our main result. In Section 3, we will prove Theorem A.

2 Lemmas

Lemma 2.1 Suppose G is an edge colored graph with $g^c(G) \geq k+1$, where $k \geq 3$ is an integer, and $P = u_0 u_1 \ldots u_l$ is a heterochromatic path of length l. If two edges $u_0 u_i$ and $u_0 u_j$ $(k \leq i < i+k-2 < j \leq l)$ exist and their colors are distinct and do not appear on the path P. Let $G' = G[\{u_0, u_i, u_{i+1}, \ldots, u_j\}]$, then $d^c_{G'}(u_0) \leq j - i - k + 3$.

Proof. First let $N^c_{G'}(u_0) = \{u_{y_1} = u_i, u_{y_2}, \ldots, u_{y_{t-1}}, u_{y_t} = u_j$, where $i = y_1 < y_2 < \ldots < y_t = j\}$ be a maximum color neighborhood of u_0 in (G', C). We need to show that there exists s such that $u_{y_{s+1}} - u_{y_s} \geq k-1$, where $1 \leq s \leq t-1$. Otherwise, we can assume $u_{y_{s+1}} - u_{y_s} \leq$

$k-2$ for all $1 \leq s \leq t-1$. Since $g^c(G) \geq k+1$, it is easy to see that $g^c(G') \geq k+1$. Since the cycle $u_0u_{y_{t-1}}Pu_{y_t}u_0$ is not heterochromatic, by $C(u_{y_t}u_0) \notin C(P)$, we can conclude that $C(u_0u_{y_{t-1}}) \in C(u_{y_{t-1}}Pu_{y_t})$. Similarly, the cycle $u_0u_{y_{t-2}}Pu_{y_{t-1}}u_0$ is not heterochromatic, we must have $C(u_0u_{y_{t-2}}) \in C(u_{y_{t-2}}Pu_{y_{t-1}})$.
In the same way, we can get, orderly, $C(u_0u_{y_s}) \in C(u_{y_s}Pu_{y_{s+1}})$, $1 \leq s \leq t-1$. Then the cycle $u_0u_{y_1}Pu_{y_2}u_0$ is heterochromatic and length at most k, a contradiction. This implies that there exists $s \in \{1, 2, \ldots, t-1\}$ such that $u_{y_{s+1}} - u_{y_s} \geq k-1$. Then we get that $d^c_{G'}(u_0) \leq j-i-k+3$. □

In a similar way, we can get the following property.

Lemma 2.2 Suppose G is an edge colored graph with $g^c(G) \geq k+1$, where $k \geq 3$ is an integer, and $P = u_0u_1 \ldots u_l$ is a heterochromatic path of length l. If the edge u_0u_i exists and its color does not appear on the path P. Let $G' = G[\{u_0, u_1, u_2, \ldots, u_i\}]$ and $k \leq i \leq l$, then $d^c_{G'}(u_0) \leq i-k+2$.

Now we can state our main theorem.

3 Proof of Theorem 1.3

Proof. First suppose $P = u_0u_1u_2 \ldots u_l$ is one of the longest heterochromatic paths in G, then $CN(u_0) \subseteq C(P) \cup \{C(u_0u_{x_i}), i = 1, 2, \ldots, s\}$ and $|\{C(u_0u_{x_i}), i = 1, 2, \ldots, s\} \backslash C(P)| = s$, $CN(u_l) \subseteq C(P) \cup \{C(u_{y_i}u_l), i = 1, 2, \ldots, t\}$ and $|\{C(u_{y_i}u_l), i = 1, 2, \ldots, t\} \backslash C(P)| = t$, where $k \leq x_1 < x_2 < \ldots < x_s \leq l$, $0 \leq y_1 < y_2 < \ldots < y_t \leq l-k$. Since $g^c(G) \geq k+1$, i.e., there exists no heterochromatic cycles of length k in G, we have $x_1 \geq k$, $x_{i+1} > x_i + k - 2$, for $i = 1, 2, \ldots, s-1$; $y_t \leq l-k$, $y_{j+1} > y_j + k - 2$ for $j = 1, 2, \ldots, t-1$.

By Lemma 2.2, we have that

$$|\{C(u_0u_2), C(u_0u_3), \ldots, C(u_0u_{x_1})\}| \leq x_1 - k + 1.$$

We can also get from Lemma 2.1 that for any $1 \leq i \leq s-1$,

$$|\{C(u_0u_{x_i+1}), C(u_0u_{x_i+2}), \ldots, C(u_0u_{x_{i+1}-1}), C(u_0u_{x_{i+1}})\}| \leq x_{i+1} - x_i - k + 2.$$

So

$$
\begin{aligned}
&|\{C(u_0u_1), C(u_0u_2), \ldots, C(u_0u_{l-1}), C(u_0u_l)\}| \\
&\leq |\{C(u_0u_1)\}| + |\{C(u_0u_2), C(u_0u_3), \ldots, C(u_0u_{x_1-1}), C(u_0u_{x_1})\}| \\
&\quad + |\{C(u_0u_{x_1+1}), C(u_0u_{x_1+2}), \ldots, C(u_0u_{x_2-1}), C(u_0u_{x_2})\}| \\
&\quad + |\{C(u_0u_{x_2+1}), C(u_0u_{x_2+2}), \ldots, C(u_0u_{x_3-1}), C(u_0u_{x_3})\}| \\
&\quad + \ldots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1) \\
&\quad + |\{C(u_0u_{x_{s-1}+1}), C(u_0u_{x_{s-1}+2}), \ldots, C(u_0u_{x_s-1}), C(u_0u_{x_s})\}| \\
&\quad + |\{C(u_0u_{s+1}), C(u_0u_{s+2}), \ldots, C(u_0u_{l-1}), C(u_0u_l)\}| \\
&\leq 1 + (x_1 - k + 1) + (x_2 - x_1 - k + 2) + (x_3 - x_2 - k + 2) + \ldots + \\
&\quad + (x_s - x_{s-1} - k + 2) + (l - x_s) \\
&= l - s(k-2).
\end{aligned}
$$

On the other hand, for any vertex v which is adjacent to u_0 but does not belong to the path P, the color of the edge u_0v is not same as the color of the edge $u_{y_j}u_{y_j+1}$ for any $1 \leq j \leq t$, for otherwise, $vu_0Pu_{y_j}u_lP^{-1}u_{y_j+1}$ is a heterochromatic path of length $l+1$, a contradiction. So we have $CN(u_0)\backslash\{C(u_0u_i) : 1 \leq i \leq l\} \subseteq C(P)\backslash\{C(u_{y_j}u_{y_j+1}) : 1 \leq j \leq t\}$, and then

$$|CN(u_0)\backslash\{C(u_0u_i) : 1 \leq i \leq l\}| \leq l - t. \qquad (2)$$

From Inequalities (1) and (2), we have

$$
\begin{aligned}
d &\leq |CN(u_0)| \\
&\leq |CN(u_0)\backslash\{C(u_0u_i) : 1 \leq i \leq l\}| + |\{C(u_0u_i) : 1 \leq i \leq l\}| \qquad (3) \\
&\leq (l-t) + [l - s(k-2)] = 2l - s(k-2) - t.
\end{aligned}
$$

On the other hand, since the color degrees of the two vertices u_0 and u_l are both at least d, and because of the assumption that P is one of the longest heterochromatic paths, we have that $d^c(u_0) \geq d$, $d^c(u_l) \geq d$, $l + s \geq d$ and $l + t \geq d$, respectively. This implies that $s \geq d - l$ and $t \geq d - l$. Now we can get from Inequality (3) that

$$
\begin{aligned}
d &\leq 2l - s(k-2) - t \\
&\leq 2l - (d-l)(k-2) - (d-l) \\
&= 2l - (d-l)(k-1) \qquad\qquad (4) \\
&= 2l - d(k-1) + l(k-1) \\
&= l(k+1) - d(k-1).
\end{aligned}
$$

So we have $dk \leq l(k+1)$, and $l \geq \lceil \frac{dk}{k+1} \rceil$. This proves Theorem A. □

The following is an immediate result of the preceding theorem.

4 Some results based on the Theorem

Corollary 4.1. Let G be an edge colored graph, and $d^c(v) \geq d$ for any vertex $v \in V(G)$. If G contains no the heterochromatic cycles. Then G has a heterochromatic path of length at least d.

Corollary 4.2. Let G be an edge colored graph with $g^c(G) \geq k+1$, where $k \geq 3$ is an integer. If $d^c(v) \geq d$ and $d \leq k+1$ for any $v \in V(G)$. Then G has a heterochromatic path of length at least $d-1$.

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