Super edge-magic total labeling of subdivision of stars

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Abstract. An edge-magic total (EMT) labeling on a graph G is a one-to-one mapping $\lambda : V(G) \cup E(G) \longrightarrow \{1, 2, \cdots, |V(G)| + |E(G)|\}$ such that the set of edge weights is one point set, i.e. for any edge $xy \in G$, $w(xy) = \{a\}$ where $a = \lambda(x) + \lambda(y) + \lambda(xy)$ is called a magic constant. If $\lambda(V(G)) = \{1, 2, \cdots, |V(G|\}$ then an edge-magic total labeling is called a *super edge-magic total labeling*. In this paper, we formulate a super edge-magic total labeling for a particular tree family called subdivided star $T(l_1, l_2, \cdots, l_p)$ for $p > 3$.

Keywords : Super edge-magic total labeling, subdivided star.

1 Introduction

All graphs in this paper are finite, simple, planar and undirected. A graph G is a collection of vertices and edges and one can write $G = (V(G), E(G))$ where $V(G)$ is the vertex set and $E(G)$ is the edge set. A general reference for graph-theoretic ideas can be seen in [17]. An association of graph elements with a number set under some certain condition is called a *labeling* (or *valuation*) on a graph. For a brief study of labeling one can see [7]. In this paper, we focus on one type of labeling called the *super edge-magic total labeling (SEMTL)*. A super edge-magic total labeling on a graph G is a one-to-one mapping $\lambda : V(G) \cup E(G) \longrightarrow \{1, 2, \cdots, |V(G)| + |E(G)|\}$ such that $\lambda(V(G)) = \{1, 2, \cdots, |V(G|\}$ and the set of edge weights is a one point

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set, i.e. for any edge $xy \in G$, $w(xy) = \{a\}$ where $a = \lambda(x) + \lambda(y) + \lambda(xy)$ is called magic constant.

The subject of edge-magic total labeling of graphs has its origin in the works of Kotzig and Rosa [13, 14], on what they called magic valuations of graphs. The notion of super edge-magic total labeling was introduced by Enomoto et al. in [4]. The following conjecture and lemma give motivation to the people to find the labeling of trees.

Conjecture 1 *[4] Every tree admits a super edge-magic total labeling.*

Lemma 1 *[5]* A graph G with v vertices and e edges is super edge-magic total if and only if there exists a bijective function $\lambda : V(G) \to \{1, 2, \cdots, v\}$ such that the set $S = \{\lambda(x) + \lambda(y) | xy \in E(G)\}$ consists of e consecutive integers. In such a case, λ extends to a super edge-magic total labeling of G with magic constant $a = v + e + s$, where $s = min(S)$ and

$$S = \{\lambda(x) + \lambda(y) | xy \in E(G)\} = \{a - (v+1), a - (v+2), \cdots, a - (v+e)\}.$$

In the effort of attacking this conjecture, many authors have considered super edge-magic total labeling for some particular classes of trees for example [1, 6, 8, 11, 16]. Lee and Shah [15] have verified this conjecture by a computer search for trees on at most 17 vertices. Earlier, in [13] Kotzig and Rosa proved that every caterpillar is super edge-magic total. However, conjecture 1 still remains open.

Definition 1 *Consider the star graph with p branches, $K_{1,p}$. A subdivided star $G = T(l_1, l_2, \cdots, l_p)$ be a graph obtained by inserting $l_i - 1$ vertices to the i_{th} branch of the star $K_{1,p}$. The graph $T(1, 1, \cdots, 1)$ is simply a star.*

Super edge-magic total labeling of subdivided star $T(l_1, l_2, l_3)$ was studied by Baskoro et al. [2] and they discuss different combinations of (l_1, l_2, l_3). In [12, 9] M. Hussain et al. established a super edge-magic total labeling of $T(l_1, l_2, \cdots, l_p)$ for $p \geq 4$. In [10] M. Javaid et al. discussed super (a, d) edge-antimagic total labeling of $T(l_1, l_2, \cdots, l_p)$ for $p \geq 4$ and $d = 0, 1, 2$. In all these said references the authors discussed some particular cases. However, super edge-magic total labeling of $T(l_1, l_2, \cdots, l_p)$ for different combination of $p - tuple$ is still open. In this paper we will present a super edge-magic total labelings on some more subdivisions of star.

2 Main Results

Through out this section, we will use the following representation for the elements of $T(l_1, l_2, \cdots, l_p)$,

$$\begin{aligned} V(G) &= \{c\} \cup \{x_{ij} | 1 \leq i \leq p\,;\ 1 \leq j \leq l_i\} \\ E(G) &= \{cx_{i1} | 1 \leq i \leq p\} \cup \{x_{ij}x_{i(j+1)} | 1 \leq i \leq p\,; 1 \leq j \leq l_i - 1\} \end{aligned}$$

where c is the center of $T(l_1, l_2, \cdots, l_p)$.

Theorem 1 *The graph $G \cong T(n, n-1, l, l+2)$ admits a super edge-magic total labeling with magic constant $a = 5(n+l)+6$, where n, l both are even and $l > n$.*

Proof. Let us denote $v = |V(G)|$ and $e = |E(G)|$, so we have

$$v = 2(n+l+1) \quad , \quad e = 2(n+l)+1.$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, ..., v+e\}$ as follows:

$$\lambda(c) = 2n+l+1$$

$$\text{For odd } j, \quad \lambda(x_{ij}) = \begin{cases} \frac{j+1}{2} & \text{for } i=1 \\ n+\frac{1-j}{2} & \text{for } i=2 \\ n+\frac{j+1}{2} & \text{for } i=3 \\ n+l+1+\frac{1-j}{2} & \text{for } i=4 \end{cases}$$

$$\text{For even } j, \quad \lambda(x_{ij}) = \begin{cases} n+l+\frac{j+2}{2} & \text{for } i=1 \\ 2n+l+\frac{2-j}{2} & \text{for } i=2 \\ 2n+l+1+\frac{j}{2} & \text{for } i=3 \\ 2n+2l+3-\frac{j}{2} & \text{for } i=4 \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s = n+l+3, n+l+4, \cdots, 3(n+l)+2, 3(n+l)+3$. Therefore, by Lemma [1] λ can be extended to a super edge-magic total labeling and we obtain the magic constant $a = v+e+s = 5(n+l)+6$.

Theorem 2 *The graph $G \cong T(n, n-1, l, l+2, 2l+4)$ admits a super edge-magic total labeling with magic constant $a = 5(n+2l)+16$, where n, l both are even and $l > n$.*

Proof. Let us denote $v = |V(G)|$ and $e = |E(G)|$, so we have

$$v = 2(n+2l+3) \quad , \quad e = 2n+4l+5$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, ..., v+e\}$ as follows:

$$\lambda(c) = 2n+2l+3$$

$$\text{For odd } j, \quad \lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & \text{for } i=1 \\ n+\frac{1-j}{2}, & \text{for } i=2 \\ n+\frac{j+1}{2}, & \text{for } i=3 \\ n+l+1+\frac{1-j}{2}, & \text{for } i=4 \\ n+2l+3+\frac{1-j}{2}, & \text{for } i=5 \end{cases}$$

$$\text{For even } j, \qquad \lambda(x_{ij}) = \begin{cases} n+2l+3+\frac{j}{2}, & \text{for } i=1 \\ 2n+2l+3-\frac{j}{2}, & \text{for } i=2 \\ 2n+2l+3+\frac{j}{2}, & \text{for } i=3 \\ 2n+3l+5-\frac{j}{2}, & \text{for } i=4 \\ 2n+4l+7-\frac{j}{2}, & \text{for } i=5 \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s = n+2l+5, n+2l+6, \cdots, 3n+6l+8, 3n+6l+9$. Therefore, by Lemma [1] λ can be extended to a super edge-magic total labeling and we obtain the magic constant $a = v + e + s = 5(n + 2l) + 16$.

Theorem 3 *The graph $G \cong T(n, n-1, l, l+2, 2l+4, 4l+8)$ admits a super edge-magic total labeling with magic constant $a = 5n + 20l + 36$, where n, l both are even and $l > n$.*

Proof. Let us denote $v = |V(G)|$ and $e = |E(G)|$, so we have

$$v = 2n + 8l + 14, \qquad , \qquad e = 2n + 8l + 13$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, ..., v + e\}$ as follows:

$$\lambda(c) = 2n + 4l + 7$$

$$\text{For } j \text{ odd}, \qquad \lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & \text{for } i=1 \\ n+\frac{1-j}{2}, & \text{for } i=2 \\ n+\frac{1+j}{2}, & \text{for } i=3 \\ n+l+1+\frac{1-j}{2}, & \text{for } i=4 \\ n+2l+3+\frac{1-j}{2}, & \text{for } i=5 \\ n+4l+7+\frac{1-j}{2}, & \text{for } i=6 \end{cases}$$

$$\text{For } j \text{ even}, \qquad \lambda(x_{ij}) = \begin{cases} n+4l+7+\frac{j}{2}, & \text{for } i=1 \\ 2n+4l+7-\frac{j}{2}, & \text{for } i=2 \\ 2n+4l+7+\frac{j}{2}, & \text{for } i=3 \\ 2n+5l+9-\frac{j}{2}, & \text{for } i=4 \\ 2n+6l+11-\frac{j}{2}, & \text{for } i=5 \\ 2n+8l+15-\frac{j}{2}, & \text{for } i=6 \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s = n+4l+9, n+4l+10, \cdots, 3n+12l+20, 3n+12l+21$. Therefore, by Lemma [1] λ can be extended to a super edge-magic total labeling and we obtain the magic constant $a = v + e + s = 5n + 20l + 36$.

Theorem 4 *The graph $G \cong T(n, n-1, l, l+2, l_5, \cdots, l_p)$ admits a super edge-magic total labeling with magic constant $a = 5(n + 2^{p-4}(l+2)) - 4$, where both n and l are even, $l > n$ and $l_r = 2^{r-4}(l+2)$ for $5 \le r \le p$.*

Proof. Let us denote $v = |V(G)|$ and $e = |E(G)|$, so we have

$$v = 2n + 2^{p-3}(l+2) - 2, \qquad , \qquad e = 2n + 2^{p-3}(l+2) - 3$$

Now, we define the labeling $\lambda : V \cup E \to \{1, 2, ..., v+e\}$ as follows:

$$\lambda(c) = 2n + 2^{p-4}(l+2) - 1$$

For j odd,

$$\lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & \text{for } i = 1 \\ n - \frac{j-1}{2} & \text{for } i = 2 \\ n + \frac{j+1}{2} & \text{for } i = 3 \\ n + l + 1 - \frac{j-1}{2} & \text{for } i = 4 \\ n + 2^{i-4}(l+2) - 1 - \frac{j-1}{2} & \text{for } i \ge 5 \end{cases}$$

For j even,

$$\lambda(x_{ij}) = \begin{cases} n + 2^{p-4}(l+2) - 1 + \frac{j}{2} & \text{for } i = 1 \\ 2n + 2^{p-4}(l+2) - 1 - \frac{j}{2} & \text{for } i = 2 \\ 2n + 2^{p-4}(l+2) - 1 + \frac{j}{2} & \text{for } i = 3 \\ 2n + 2^{p-4}(l+2) + l + 1 - \frac{j}{2} & \text{for } i = 4 \\ 2n + 2^{p-4}(l+2) + l + 1 + (l+2)(2^{i-4} - 1) - \frac{j}{2} & \text{for } i \ge 5 \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s = n + 2^{p-4}(l+2) + 1,\ n + 2^{p-4}(l+2) + 2,\ n + 2^{p-4}(l+2) + 3, \cdots, 3(n + 2^{p-4}(l+2) - 1)$. Therefore, by Lemma [1] λ can be extended to a super edge-magic total labeling and we obtain the magic constant $a = v + e + s = 5(n + 2^{p-4}(l+2)) - 4$.

Theorem 5 *The graph $G \cong T(n,n,m,m+1,l_5,\cdots,l_p)$ admits a super edge-magic total labeling with magic constant $a = 5(2^{p-4}m+n)+3p-5$, where n and m both odd , $m \geq n$ and $l_r = 2^{r-4}m+1$ for $5 \leq r \leq p$.*

Proof. Let us denote $v = |V(G)|$ and $e = |E(G)|$, so we have

$$v = 2(2^{p-4}m+n)+p-2, \qquad , \qquad e = 2(2^{p-4}m+n)+p-3$$

Now, we define the labeling $\lambda : V \cup E \rightarrow \{1,2,...,v+e\}$ as follows:

$$\lambda(c) = 2n + 2^{p-4}m + p - 2$$

For j odd,

$$\lambda(x_{ij}) = \begin{cases} \frac{j+1}{2}, & \text{for } i=1 \\ n+1-\frac{j-1}{2} & \text{for } i=2 \\ n+1+\frac{j+1}{2} & \text{for } i=3 \\ n+m+2-\frac{j-1}{2} & \text{for } i=4 \\ n+2^{i-4}m+i-2-\frac{j-1}{2} & \text{for } i \geq 5 \end{cases}$$

For j even,

$$\lambda(x_{ij}) = \begin{cases} n+2^{p-4}m+p-2+\frac{j}{2} & \text{for } i=1 \\ 2n+2^{p-4}m+p-2-\frac{j}{2} & \text{for } i=2 \\ 2n+2^{p-4}m+p-2+\frac{j}{2} & \text{for } i=3 \\ 2n+m(2^{p-4}+1)+p-1-\frac{j}{2} & \text{for } i=4 \\ 2n+2^{p-4}m+p-2+m2^{i-4}-\frac{j-2}{2} & \text{for } i \geq 5 \end{cases}$$

The set of all edge-sums generated by the above formula forms a consecutive integer sequence $s = 2^{p-4}m+n+p+1,\ 2^{p-4}m+n+p+2,\cdots,3(2^{p-4}m+n)+2p-4$. Therefore, by Lemma [1] λ can be extended to a super edge-magic total labeling and we obtain the magic constant $a = v+e+s = 5(2^{p-4}m+n)+3p-5$.

3 Conclusion

In section 2, we have established a super edge-magic total labeling on $T(n, n-1, l, l+2, l_5, \cdots, l_p)$ and $T(n, n, m, m+1, l_5, \cdots, l_p)$ with some certain conditions on n, l, m and l_p. The general combination of $(l_1, l_2, \cdots, l_p)$ is still open.

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