Cubic symmetric graphs of order $4p^3$

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Abstract

A graph is said to be *symmetric* if its automorphism group acts transitively on its arcs. Let p be a prime. In [J. Combin. Theory B 97 (2007) 627–646], Feng and Kwak classified connected cubic symmetric graphs of order $4p$ or $4p^2$. In this article, all connected cubic symmetric graphs of order $4p^3$ are classified. It is shown that up to isomorphism there is one and only one connected cubic symmetric graph of order $4p^3$ for each prime p, and all such graphs are normal Cayley graphs on some groups.

Key Words: Symmetric graphs; s-Regular graphs; Cayley graphs
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1 Introduction

In this paper, we consider undirected finite connected graphs without loops and multiple edges. For a graph X, we use $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set and full automorphism group, respectively. For $u, v \in V(X)$, $\{u, v\}$ is the edge incident to u and v in X. An *s-arc* in a graph X for some positive integer s is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A symmetric graph is said to be *s-regular* if its full automorphism group acts regularly on its s-arcs, that is, only the trivial automorphism fixes an s-arc.

In 1947, Tutte [19] initiated the investigation of cubic symmetric graphs by proving that there exist no cubic s-regular graphs for $s \geq 6$. Following this pioneering article, cubic symmetric graphs have been extensively studied over decades by many authors. For instance, for a cubic s-regular graph

X, by Djoković and Miller [4, Propositions 2–5], the stabilizer of $v \in V(X)$ in $\mathrm{Aut}(X)$ is isomorphic to $\mathbb{Z}_3$, S_3, $S_3 \times \mathbb{Z}_2$, S_4 or $S_4 \times \mathbb{Z}_2$ for $s = 1, 2, 3, 4$ or 5, respectively. Based on this result, an exhaustive computer search by Conder and Dobcsányi [3] resulted in a complete list of cubic symmetric graphs on up to 768 vertices. Let p be a prime. The classification of cubic symmetric graphs of order $2p$ can be obtained from [2]. By analyzing automorphism groups of graphs, a classification of cubic symmetric graphs of order $2p^2$ was given by Feng et al. [6]. Using covering techniques developed in [12, 13, 14, 15], Feng et al. [5, 8, 9] classified cubic symmetric graphs of order np or np^2 with $4 \leq n \leq 10$, and Oh [16, 17] classified cubic symmetric graphs of order $14p$ or $16p$. For the further classification of cubic symmetric graphs with given orders, Feng et al. [5, Problems 6.1-6.3] posed three problems, of which the third is the following.

Problem: For each natural number m, classify all connected cubic symmetric graphs of order $2mp^3$ for each prime p.

When $m = 1$, an answer for this problem was given in [7]. In this paper, we completely answer this problem for the case when $m = 2$. It is shown that for each prime p, there is one and only one connected cubic symmetric graph of order $4p^3$, and all such graphs are 2-regular and can be constructed as normal Cayley graphs on some groups.

2 Preliminaries

An epimorphism $\wp : \widetilde{X} \to X$ of graphs is called a *regular covering projection* if there is a semiregular subgroup $\mathrm{CT}(\wp)$ of the automorphism group $\mathrm{Aut}(\widetilde{X})$ of $\widetilde{X}$ whose orbits in $V(\widetilde{X})$ coincide with the *vertex fibres* $\wp^{-1}(v)$, $v \in V(X)$, and the arc and edge orbits of $\mathrm{CT}(\wp)$ coincide with the *arc fibres* $\wp^{-1}(u, v)$, $(u, v) \in A(X)$, and the *edge fibres* $\wp^{-1}\{u, v\}$, $\{u, v\} \in E(X)$, respectively. In particular, we call the graph $\widetilde{X}$ a *regular cover* of the graph X. The semiregular group $\mathrm{CT}(\wp)$ is called the *covering transformation group*.

For a graph X, let $G \leq \mathrm{Aut}(X)$ act vertex-transitively on X. Let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N in $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. If X is a symmetric cubic graph and G acts arc-transitively on X, then by [11, Theorem 9], we have

Proposition 2.1 *If N has more than two orbits in $V(X)$, then N is semiregular on $V(X)$, X_N is a cubic symmetric graph with G/N as an s-regular group of automorphisms, and X is a regular cover of X_N with the covering transformation group N.*

For a finite group G, and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G relative to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. It is known that $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Given $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg, x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G, which acts regularly on G. Thus the Cayley graph $\mathrm{Cay}(G, S)$ is vertex-transitive. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. It is easy to see that the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Xu [20, Proposition 1.5] proved the following.

Proposition 2.2 *The Cayley graph* $\mathrm{Cay}(G, S)$ *is normal if and only if* $\mathrm{Aut}(\mathrm{Cay}(G, S))_1 = \mathrm{Aut}(G, S)$.

Combining [8, Theorem 6.2] and [21, Theorem 2.3], we have the following.

Proposition 2.3 *Let X be a connected cubic symmetric graph of order $4p$ or $4p^2$ for a prime p. Then X is isomorphic to the 2-regular hypercube Q_3 of order 8, the 2-regular generalized Petersen graphs $P(8,3)$ or $P(10,7)$ of order 16 or 20 respectively, the 3-regular Dodecahedron of order 20 or the 3-regular Coxeter graph of order 28. In particular, X is Cayley if and only if $X \cong Q_3$ or $P(8,3)$.*

3 Classification

Denote by $\mathbb{Z}_n$ the cyclic group of order n, and by $\mathbb{Z}_p^m$ the elementary abelian group $\underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p}_{m \text{ times}}$. In this section, we shall give a classification of connected cubic symmetric graphs of order $4p^3$ for each prime p. To do this, we need the following lemma.

Lemma 3.1 *Let X be a connected cubic graph of order $4p^n$ with p an odd prime and n a positive integer. Suppose that $A \leq \mathrm{Aut}(X)$ acts transitively on the arc set of X. Then, we have the following statements.*

(1) *If $n \geq 2$, then A has a non-trivial normal p-subgroup.*

(2) *If $p > 7$, then A has a normal Sylow p-subgroup, say P, such that $P/\Phi(P) \cong \mathbb{Z}_p^3$, where $\Phi(P)$, called the Frattini subgroup of P, is the intersection of all maximal subgroups of P.*

Proof. To show (1), let A_v be the stabilizer of a vertex $v \in V(X)$ in A. By Tutte [19], $|A_v| \mid 48$ and hence $|A| \mid 2^6 \cdot 3 \cdot p^n$. Take a minimal normal subgroup, say N, of A. Suppose that N is non-solvable. Then N is a product of isomorphic non-abelian simple groups. Since $|N| \mid 2^{4+\ell} \cdot 3 \cdot p^n$, by [10, p.12–14], $N \cong A_5$ or $\mathrm{PSL}(2,7)$. Since $n > 1$, N has more than two orbits in $V(X)$. By Proposition 2.1, N is semiregular on $V(X)$. This forces that $|N| \mid 4p^n$ and by [18, Theorem 8.5.3], N is solvable, a contradiction. Thus, N is solvable. Then N is an elementary abelian p- or 2-group. To complete the proof of (1), we may assume that N is a 2-group. Then N has more than two orbits in $V(X)$. Again by Proposition 2.1, N is semiregular, and the quotient graph X_N of X relative to N is still a cubic symmetric graph with A/N as an arc-transitive group of automorphisms. It follows that $N \cong \mathbb{Z}_2$ and X_N is a cubic symmetric graph of order $2p^n$. Since $p > 2$ and $n > 1$, by [6, Lemma 3.1], $\mathrm{Aut}(X_N)$ has a normal p-subgroup, say $\overline{M}$. Then $\overline{M}$ is contained in every Sylow p-subgroup of $\mathrm{Aut}(X_M)$. It follows that $\overline{M} \trianglelefteq A/N$ because every Sylow p-subgroup of A/N is also a Sylow p-subgroup of $\mathrm{Aut}(X_N)$. Set $\overline{M} = M/N$. Since $p > 2$, one has $M = P_1 \times N$, where P_1 is a Sylow p-subgroup of M. Then P_1 is characteristic in M, and so it is normal in A because $M \trianglelefteq A$. Thus, (1) holds.

For (2), since $p > 7$, one has $n \geq 3$ by Proposition 2.3. It follows form (1) that A has a non-trivial normal p-subgroup. Let P be the maximal normal p-subgroup of A. Consider the quotient graph X_P of X relative to P. Clearly, P has more than two orbits in $V(X)$. By Proposition 2.1, X_P is a cubic symmetric graph of order $4p^\ell$ with $\ell = p^n/|P|$. The maximality of P gives $\ell \leq 1$. Since $p > 7$, there are no cubic symmetric graphs of order $4p$ by Proposition 2.3. Thus $\ell = 0$ and P is a Sylow p-subgroup of A. Clearly, $\Phi(P)$ is characteristic in P. Since $P \trianglelefteq A$, one has $\Phi(P) \trianglelefteq A$. Consider the quotient graph $X_{\Phi(P)}$ of X relative to $\Phi(P)$. By Proposition 2.1, $X_{\Phi(P)}$ is a cubic symmetric graph of order $4|P/\Phi(P)|$ with $A/\Phi(P)$ as an arc-transitive group of automorphisms. Since $P/\Phi(P) \trianglelefteq A/\Phi(P)$, by Proposition 2.1, $X_{\Phi(P)}$ is a covering graph of K_4 with covering transformation group $P/\Phi(P)$. By [18, Theorem 5.3.2], $P/\Phi(P)$ is an elementary abelian p-group. It follows from [8, Theorem 6.1] that $P/\Phi(P) \cong \mathbb{Z}_p^3$. □

Below, we introduce a family of cubic symmetric graphs of order $4p^3$ which was constructed by Feng and Kwak in [8, Example 3.2].

Example 3.2 *Let p be a prime and let $\mathbb{Z}_p^3$ be the 3-dimensional row vector space over the field $\mathbb{Z}_p$. Take the standard basis vectors: $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. The graph EC_{4p^3} is defined to have vertex set*

$V(EC_{4p^3}) = V(K_4) \times \mathbb{Z}_p^3$ *and edge set*

$$\begin{aligned} E(EC_{4p^3}) = \; & \{(a,x)(b,x),(a,x)(c,x),(a,x)(d,x), \\ & (b,x)(c,x+e_1),(c,x)(d,x+e_2),(c,x)(d,x+e_2), \\ & (d,x)(b,x+e_3) \mid x \in \mathbb{Z}_p^3\}. \end{aligned}$$

By [8, Theorem 6.1], the cubic graph EC_{4p^3} is connected and 2-regular.

Theorem 3.3 *Let X be a connected cubic s-regular graph of order $4p^3$ with p a prime. Then $s = 2$ and X is isomorphic to the graph EC_{4p^3}.*

Proof. Let $p \leq 7$. By [3], up to isomorphism, there is one and only one connected cubic symmetric graph of order $4p^3$ for each p. It follows that $X \cong EC_{4p^3}$.

Let $p > 7$. Set $A = \mathrm{Aut}(X)$ and let P be a Sylow p-subgroup. It follows from Tutte [19] that $|A| \mid 2^6 \cdot 3 \cdot p^3$. Then $|P| = p^3$. By Lemma 3.1, $P \trianglelefteq A$ and $P = P/\Phi(P) \cong \mathbb{Z}_p^3$. By Proposition 2.1, the quotient graph X_P of X relative to P is isomorphic to K_4, and X is a regular cover of X_P with covering transformation group P. By [8, Theorem 6.1], $X \cong EC_{4p^3}$. □

4 Cayley property

Let p be a prime. In this section, we shall show that all connected cubic symmetric graphs of order $4p^3$ are normal Cayley graphs on some groups. We first introduce an infinite family of cubic 2-regular Cayley graphs.

Let n be a positive integer. Set $E = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$. Let d and e be two automorphisms of E induced by $a \mapsto c$, $b \mapsto (abc)^{-1}$, $c \mapsto a$ and $a \mapsto b$, $b \mapsto a$, $c \mapsto (abc)^{-1}$, respectively. Set $F = \langle d, e \rangle$. It is easy to see that $F \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 4.1 *Let $G(4p^{3n})$ be the semidirect product of E by F. Define*

$$SC_{4p^{3n}} = \mathrm{Cay}(G(4p^{3n}), \{da^{-1}c, eab^{-1}, de(bc^{-1})\}).$$

For the name of the graph $SC_{4p^{3n}}$, the letter S stands for symmetric, C stands for Cayley graph and suffix stands for the order of the graph.

Lemma 4.2 *For each odd prime p, $SC_{4p^{3n}}$ is a connected cubic 2-regular normal Cayley graph on $G(4p^{3n})$.*

Proof. Let $S = \{x, y, z\}$ with $x = da^{-1}c, y = eab^{-1}$ and $z = de(bc^{-1})$. A short computation gives that $(xy)^2 = (bc)^{-4}$, $(yz)^2 = (ac)^{-4}$ and $(zx)^2 = (ab)^{-4}$. Since p is odd,

$$\langle (ab)^{-4}, (bc)^{-4}, (ac)^{-4} \rangle = \langle a, b, c \rangle.$$

This implies that $\langle S\rangle = G(4p^{3n})$ and hence $SC_{4p^{3n}}$ is connected. It is easy to check that the following two maps

$$\begin{array}{rl} \alpha: & a \mapsto b, b \mapsto c, c \mapsto a, d \mapsto e, e \mapsto de \\ \beta: & a \mapsto a, b \mapsto c, c \mapsto b, d \mapsto e, e \mapsto d \end{array}$$

can induce two automorphisms of $G(4p^{3n})$. Furthermore, $\langle\alpha, \beta\rangle$ acts 2-transitively on S. It follows that $\mathrm{Aut}(G(4p^{3n}), S) = \langle\alpha, \beta\rangle$ and hence $SC_{4p^{3n}}$ is at least 2-regular.

Set $X = SC_{4p^{3n}}$ and $A = \mathrm{Aut}(X)$. Since $3n > 1$, by Lemma 3.1, A has a normal p-subgroup. Let N be the maximal normal p-subgroup of A. Clearly, N has more than two orbits in $V(X)$. By Proposition 2.1, the quotient graph X_N of X relative to N is still a cubic symmetric graph of order $4p^m$ with $m < 3n$ and A/N is an arc-transitive group of automorphisms of X_N. By the maximality of N, $m \leq 1$ and hence X_N is a cubic symmetric graph of order 4 or $4p$. Suppose that X_N has order $4p$. By Proposition 2.3, $p = 5$ or 7 and X_N is non-Cayley. However, since $p > 3$, $\langle R(a), R(b), R(c)\rangle$ is a Sylow p-subgroup of A. Therefore, $N \leq \langle R(a), R(b), R(c)\rangle \leq R(G(4p^{3n}))$ and hence $R(G(4p^{3n}))/N$ acts regularly on $V(X_N)$. It follows that X_N is a Cayley graph on $G(4p^{3n})/N$, a contradiction. Thus, $X_N \cong K_4$ and $A/N \leq S_4$. As a result, X is 2-regular. □

With the help of computer software package MAGMA [1], we can obtain that $SC_{4\cdot 2^3}$ is disconnected. This implies that Lemma 4.2 is not true for the case of $p = 2$.

The following theorem shows that all connected cubic symmetric graphs of order $4p^3$ are normal Cayley graphs on some groups.

Theorem 4.3 *If $p > 2$, then $EC_{4p^3} \cong SC_{4p^3}$, which is a normal Cayley graph on $G(4p^3)$. If $p = 2$, then $EC_{4\cdot 2^3}$ is isomorphic to the normal Cayley graph* $\mathrm{Cay}(H, \{c, ac, bc\})$ *where* $H = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, ab = ba, cac = a^{-1}, cbc = b^{-1}\rangle$.

Proof. By Theorem 3.3, up to isomorphism, there is one and only one connected cubic symmetric graph of order $4p^3$ for each prime p. If $p > 2$, then by Lemma 4.2, SC_{4p^3} is a connected cubic symmetric normal Cayley graph on $G(4p^3)$. It follows that $EC_{4p^3} \cong SC_{4p^3}$.

Let $p = 2$. Since $\{c, ac, bc\}$ generates H, $\mathrm{Cay}(H, \{c, ac, bc\})$ is connected. It is easy to see that the automorphism of H induced by $c \mapsto ac, a \mapsto ba^{-1}, b \mapsto a^{-1}$ permutes $\{a, ab, ac\}$ cyclicly, and that the automorphism induced by $c \mapsto c, a \mapsto b, b \mapsto a$ fixes c and interchanges ac and bc. Therefore, $\mathrm{Cay}(H, \{c, ac, bc\})$ is at least 2-regular. By Theorem 3.3, $\mathrm{Cay}(G, \{c, ac, bc\}) \cong EC_{4\cdot 2^3}$ is 2-regular. Thus, $\mathrm{Aut}(H, \{c, ac, bc\}) = \langle\alpha, \beta\rangle$ and by Proposition 2.2, $\mathrm{Cay}(G, \{c, ac, bc\})$ is normal. □

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