The Binet formula for the k-generalized Fibonacci numbers

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Abstract In this paper, using the generating function, we derive Binet formulas and determinant expressions for the k-generalized Fibonacci numbers and Lucas numbers. As applications, we obtain some new recurrence relations for the Stirling numbers of the second kind and power sums.

Keywords: k-generalized Fibonacci sequence; k-generalized Lucas sequence; generating function; Stirling numbers; power sum

1 Introduction

The Fibonacci sequence and its various generalizations have been discussed in so many articles and books (see [3, 12, 15]). The well-known Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$ are defined by the following recurrence relation: for $n > 1$

$$F_n = F_{n-1} + F_{n-2}, \tag{1}$$

where $F_0 = F_1 = 1$, and

$$L_n = L_{n-1} + L_{n-2}, \tag{2}$$

where $L_0 = 2$, $L_1 = 1$. We call F_n and L_n the nth Fibonacci number and nth Lucas number respectively. The Binet formulas are well known in the Fibonacci numbers theory (see [1, 3]). These formulas allow all Fibonacci numbers F_n and Lucas numbers L_n to be represented by the roots $\alpha = \frac{1+\sqrt{5}}{2}$

and $\beta = \frac{1-\sqrt{5}}{2}$ of the characteristic equation $x^2 - x - 1 = 0$:

$$F_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}, \tag{3}$$

$$L_n = \alpha^n + \beta^n. \tag{4}$$

Using the Girard-Waring formulas [3,4]

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (x+y)^{n-2k} (xy)^k = \sum_{k=0}^{n} x^k y^{n-k}, \tag{5}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (x+y)^{n-2k} (xy)^k = x^n + y^n, \tag{6}$$

we have that the Fibonacci numbers F_n and Lucas numbers L_n satisfy the formulas:

$$F_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}, \tag{7}$$

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k}. \tag{8}$$

Furthermore, we have the determinant formulas (see [1,3]):

$$F_n = \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{vmatrix},$$

$$L_n = \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{vmatrix}.$$

We also know that the Lucas number L_n can be expressed in terms of the Fibonacci numbers as

$$L_n = 2F_n - F_{n-1}, \tag{9}$$

$$L_n = F_{n-1} + 2F_{n-2}. \tag{10}$$

There has been much interests in applications of the Fibonacci sequence and related sequences. In [9, 15], the authors have investigated the relationships between the generalized order-k Lucas sequences and Fibonacci sequences. In [1], the author has extended the identities and divisibility properties enjoyed by the Fibonacci numbers to the solutions of other recurrence relations. P. Stanimirović [13] investigate the general second-order non-degenerated sequence by matrix method. E. Kilic [6, 7] obtain the generalized Binet formulas for the generalized Fibonacci p-numbers and generalized order-k Fibonacci-Pell sequence by matrix method. In [16], we derive the determinant formulas for the k-generalized Fibonacci numbers and Lucas numbers by using symmetric functions. More general linear recursive sequences have been studied by Z.-H. Sun [14]. Relationships between the determinants or permanents of certain matrices and the terms of certain linear recurrences have been investigated in [11].

In this paper, we recover the determinantal representations of k-generalized Fibonacci numbers and Lucas numbers by using recurrence relation and the relationship between the k-generalized Fibonacci sequence and k-generalized Lucas sequence. We extend the identities (3), (4), (7) and (8) to the k-generalized Fibonacci numbers and Lucas numbers, which can be seen as the Binet formulas for the k-generalized Fibonacci numbers and Lucas numbers. As application, we obtain some new recurrence relations for the Stirling numbers of the second kind and power sums.

2 Determinant formulas

We consider a generalization of the Fibonacci sequence which is called the k-generalized Fibonacci sequence for positive integer $k \geq 2$. Define the k-generalized Fibonacci sequence $\{f_n\}$ as shown,

$$f_n = a_1 f_{n-1} - a_2 f_{n-2} + a_3 f_{n-3} - \cdots + (-1)^{k-1} a_k f_{n-k}, \quad \text{for } n \geq 1, \tag{11}$$

where the coefficients a_i's are arbitrary real numbers and

$$f_n = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } 1 - k \leq n \leq 0.$$

By usual computation, we obtain the generating function for the k-generalized Fibonacci sequence

$$F(t) = \sum_{n=0}^{\infty} f_n t^n = \frac{1}{1 - a_1 t + a_2 t^2 - \cdots + (-1)^k a_k t^k}. \tag{12}$$

We define the k-generalized Lucas sequence $\{l_n\}$ satisfies the same recurrence equation as k-generalized Fibonacci sequence $\{f_n\}$

$$l_n = a_1 l_{n-1} - a_2 l_{n-2} + a_3 l_{n-3} - \cdots + (-1)^{k-1} a_k l_{n-k}, \quad \text{for } n \geq k,$$

but its initial values l_0, l_1, $\cdots$, l_{k-1} are determined by the following equations:

$$\sum_{i=0}^{r} (-1)^i a_i l_{r-i} = (-1)^r (k-r) a_r, r = 0, 1, 2, \cdots, k-1, \tag{13}$$

where $a_1, a_2, \cdots, a_k$ are same with those in (11) and $a_0 = 1$. Generalizing (9) and (10), we have following result.

Theorem 1. The k-generalized Fibonacci sequence $\{f_n\}$ and k-generalized Lucas sequence $\{l_n\}$ are connected by the following relations

$$l_n = \sum_{r=0}^{k} (-1)^r (k-r) a_r f_{n-r}, \tag{14}$$

$$l_n = \sum_{r=1}^{k} (-1)^{r-1} r a_r f_{n-r}. \tag{15}$$

Proof. By straightforward computation, we see that the k-generalized Lucas sequence $\{l_n\}$ have a closed-form ordinary generating function

$$L(t) = \sum_{n=0}^{\infty} l_n t^n = \frac{\sum_{r=0}^{k} (-1)^r (k-r) a_r t^r}{1 - a_1 t + a_2 t^2 - \cdots + (-1)^k a_k t^k}. \tag{16}$$

Hence $\sum\limits_{n=0}^{\infty} l_n t^n = (\sum\limits_{r=0}^{k} (-1)^r (k-r) a_r t^r) \frac{1}{1 - a_1 t + a_2 t^2 - \cdots + (-1)^k a_k t^k}$
$= (\sum\limits_{r=0}^{k} (-1)^r (k-r) a_r t^r)(\sum\limits_{n=0}^{\infty} f_n t^n) = \sum\limits_{n=0}^{\infty} (\sum\limits_{r=0}^{k} (-1)^r (k-r) a_r f_{n-r}) t^n$,
and $l_n = \sum\limits_{r=0}^{k} (-1)^r (k-r) a_r f_{n-r}$. Moreover, $l_n = \sum\limits_{r=0}^{k} (-1)^r (k-r) a_r f_{n-r} =$
$\sum\limits_{r=0}^{k} (-1)^r k a_r f_{n-r} - \sum\limits_{r=0}^{k} (-1)^r r a_r f_{n-r} = k \sum\limits_{r=0}^{k} (-1)^r a_r f_{n-r} - \sum\limits_{r=0}^{k} (-1)^r r a_r f_{n-r} =$
$\sum\limits_{r=1}^{k} (-1)^{r-1} r a_r f_{n-r}$. □

In [16], we obtain that the nth k-generalized Fibonacci number f_n and the nth k-generalized Lucas number l_n can be expressed by two order-n determinants with entries given by the coefficients of their recurrence equation, here we give another elementary proof. It is worth noting that there is a determinantal formula involving the Toeplitz matrices for the inverse of an infinite power series (see [2,8]). Applying [2, Eq. (9.1)] or [8, Theorem 4], we can also deduce determinantal representation of k-generalized Fibonacci number.

Theorem 2. For any integer $n > 0$, the following determinant formulas are valid

$$f_n = \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_2 & a_1 & 1 \\ a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}, \qquad (17)$$

where $a_r = 0$ for all $r > k$.

Proof. Denote the determinant on the right hand of Eq. (17) by D_n. We use induction on n to show $D_n = f_n$. When $n = 1$, $D_1 = a_1 = f_1$; When $n = 2$, $D_2 = \begin{vmatrix} a_1 & 1 \\ a_2 & a_1 \end{vmatrix} = a_1^2 - a_2 = f_2$. Let $n \geq 3$ and suppose $D_k = f_k$ for all positive integers $k < n$. If we expand the determinant D_n by Laplace expansion with respect to the first column, we get $D_n = a_1 D_{n-1} - a_2 D_{n-2} + a_3 D_{n-3} - \cdots + (-1)^{k-1} a_k D_{n-k} = a_1 f_{n-1} - a_2 f_{n-2} + a_3 f_{n-3} - \cdots + (-1)^{k-1} a_k f_{n-k} = f_n$. By induction, we have the conclusion.

Theorem 3. For any integer $n > 0$, the following determinant formulas are valid for the k-generalized Lucas numbers

$$l_n = \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2a_2 & a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 3a_3 & a_2 & a_1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (n-1)a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_2 & a_1 & 1 \\ na_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}, \qquad (18)$$

where $a_r = 0$ for all $r > k$.

Proof. Denote the determinant on the right hand of Eq. (18) by C_n.

Expanding the determinant C_n by Laplace expansion with respect to the first column and applying Theorem 1, we get $C_n = a_1 f_{n-1} - 2a_2 f_{n-2} + \cdots + (-1)^{k-1} k a_k f_{n-k}$. By using (15) we obtain $l_n = \sum_{r=1}^{k} (-1)^{r-1} r a_r f_{n-r} = C_n$. This completes the proof. □

3 Binet formulas

The characteristic equation of the recurrence relation of the k-generalized Fibonacci sequence $\{f_n\}$ given in (11) is

$$x^k - a_1 x^{k-1} + a_2 x^{k-2} - \cdots + (-1)^{k-1} a_{k-1} x + (-1)^k a_k = 0. \qquad (19)$$

All roots of the equation (19) are supposed to be distinct from each other. Suppose $x^k - a_1 x^{k-1} + a_2 x^{k-2} - \cdots + (-1)^{k-1} a_{k-1} x + (-1)^k a_k = (x - x_1)(x - x_2) \cdots (x - x_k)$, then $(x - x_1)(x - x_2) \cdots (x - x_k) = \sum_{j=0}^{k} (-1)^j e_j(x_1, x_2, \ldots, x_k) x^{k-j}$. Thus, a_j is equal to the jth elementary symmetric function on $x_1, x_2, \ldots, x_k$, i.e., $a_j = e_j(x_1, x_2, \ldots, x_k) = \sum_{1 \leq m_1 < m_2 < \cdots < m_k \leq k} x_{m_1} x_{m_2} \cdots x_{m_j}$, $j = 1, 2, \ldots, k$. It is easy to show that $1 - a_1 t + a_2 t^2 - \cdots + (-1)^k a_k t^k = (1 - x_1 t)(1 - x_2 t) \cdots (1 - x_k t)$, where $x_1, x_2, \cdots, x_k$ are roots of the characteristic equation (19). Therefore,

$$\sum_{n=0}^{\infty} f_n t^n = \frac{1}{1 - a_1 t + a_2 t^2 - \cdots + (-1)^k a_k t^k} = \frac{1}{(1 - x_1 t)} \frac{1}{(1 - x_2 t)} \cdots \frac{1}{(1 - x_k t)}$$

$$= \left(\sum_{n=0}^{\infty} x_1^n t^n\right)\left(\sum_{n=0}^{\infty} x_2^n t^n\right) \cdots \left(\sum_{n=0}^{\infty} x_k^n t^n\right) = \sum_{n=0}^{\infty} \left(\sum_{n_1 + n_2 + \cdots + n_k = n} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}\right) t^n,$$

and

$$\sum_{n=0}^{\infty} l_n t^n = \frac{\sum_{r=0}^{k} (-1)^r (k-r) a_r t^r}{1 - a_1 t + a_2 t^2 - \cdots + (-1)^k a_k t^k} = \frac{\sum_{r=0}^{k} (-1)^r (k-r) a_r t^r}{(1 - x_1 t)(1 - x_2 t) \cdots (1 - x_k t)}$$

$$= \frac{1}{(1 - x_1 t)} + \frac{1}{(1 - x_2 t)} + \cdots + \frac{1}{(1 - x_k t)} = \sum_{n=0}^{\infty} x_1^n t^n + \sum_{n=0}^{\infty} x_2^n t^n + \cdots + \sum_{n=0}^{\infty} x_k^n t^n$$

$$= \sum_{n=0}^{\infty} (x_1^n + x_2^n + \cdots + x_k^n)\, t^n.$$

Thus we obtain the Binet formulas by which the k-generalized Fibonacci number f_n and Lucas number l_n can be expressed in terms of the roots of the characteristic equation of the recurrence relation.

Theorem 4. For any integer $n \geq 0$, the nth k-generalized Fibonacci number f_n and nth k-generalized Lucas number l_n can be represented in the form

$$f_n = \sum_{\substack{n=n_1+n_2+\cdots+n_k \\ n_1,n_2,\cdots,n_k \geq 0}} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}, \tag{20}$$

$$l_n = x_1^n + x_2^n + \cdots + x_k^n. \tag{21}$$

Generalizing Eq.(7) and (8), we can derive the following theorem.

Theorem 5. For any integer $n \geq 0$, the nth k-generalized Fibonacci number f_n and nth k-generalized Lucas number l_n can be written as

$$f_n = \sum_{\substack{n=s_1+2s_2+\cdots+ks_k \\ s=s_1+s_2+\cdots+s_k}} (-1)^{n+s} \binom{s}{s_1, s_2, \ldots, s_k} a_1^{s_1} a_2^{s_2} \cdots a_k^{s_k}, \tag{22}$$

$$l_n = \sum_{\substack{n=s_1+2s_2+\cdots+ks_k \\ s=s_1+s_2+\cdots+s_k}} (-1)^{n+s} \frac{n}{s} \binom{s}{s_1, s_2, \ldots, s_k} a_1^{s_1} a_2^{s_2} \cdots a_k^{s_k}. \tag{23}$$

Proof. Let C_k be the $k \times k$ companion matrix associated with the k-generalized Fibonacci sequence $\{f_n\}$, i.e.,

$$C_k = \begin{pmatrix} a_1 & -a_2 & a_3 & -a_4 & \cdots & (-1)^{k-2}a_{k-1} & (-1)^{k-1}a_k \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{24}$$

It follows that the linear homogeneous recurrence relation (11) can be rewritten as:

$$\begin{pmatrix} f_n \\ f_{n-1} \\ f_{n-2} \\ \vdots \\ f_{n-k+1} \end{pmatrix} = C_k \begin{pmatrix} f_{n-1} \\ f_{n-2} \\ f_{n-3} \\ \vdots \\ f_{n-k} \end{pmatrix}. \tag{25}$$

By iteration one obtains

$$\begin{pmatrix} f_n \\ f_{n-1} \\ f_{n-2} \\ \vdots \\ f_{n-k+1} \end{pmatrix} = C_k^n \begin{pmatrix} f_0 \\ f_{-1} \\ f_{-2} \\ \vdots \\ f_{-k+1} \end{pmatrix}, \quad \text{for } n \geq 0. \tag{26}$$

Since $\begin{pmatrix} f_0 \\ f_{-1} \\ f_{-2} \\ \vdots \\ f_{-k+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, we have that f_n equals the $(1,1)$-entry of the matrix C_k^n. Thus, from [2, Theorem 3.1], we can derive

$$f_n = C_k^n(1,1) = \sum_{\substack{n=s_1+2s_2+\cdots+ks_k \\ s=s_1+s_2+\cdots+s_k}} (-1)^{n+s} \binom{s}{s_1, s_2, \ldots, s_k} a_1^{s_1} a_2^{s_2} \cdots a_k^{s_k}.$$

The formula (23) follows directly from Waring's formula (see [2,12]) which express the power sum symmetric function $p_n = x_1^n + x_2^n + \cdots + x_k^n$ in terms of the elementary symmetric functions of $x_1, x_2, \cdots, x_k$. □

Example 1. For $k = 3$, the 3-generalized Fibonacci sequence $\{f_n\}$ is

$$f_n = a_1 f_{n-1} - a_2 f_{n-2} + a_3 f_{n-3},\ n \geq 1, \tag{27}$$

where $f_0 = 1$, $f_{-1} = f_{-2} = 0$, and a_1, a_2, a_3 are arbitrary real numbers. The generating function is $F(t) = \sum\limits_{n=0}^{\infty} f_n t^n = \frac{1}{1-a_1t+a_2t^2-a_3t^3}$, and

$$f_n = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & 1 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ 0 & a_3 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & 1 \\ 0 & 0 & 0 & \cdots & a_2 & a_1 \end{vmatrix} = \sum_{\substack{n=s_1+2s_2+3s_3 \\ s=s_1+s_2+s_3}} (-1)^{n+s} \binom{s}{s_1, s_2, s_3} a_1^{s_1} a_2^{s_2} a_3^{s_3}.$$

The 3-generalized Lucas sequence $\{l_n\}$ is

$$l_n = a_1 l_{n-1} - a_2 l_{n-2} + a_3 l_{n-3},\ n \geq 3, \tag{28}$$

with $l_0 = 3$, $l_1 = a_1$, $l_2 = a_1^2 - 2a_2$, and a_1, a_2, a_3 are arbitrary real numbers.

Its generating function is $L(t) = \sum_{n=0}^{\infty} l_n t^n = \frac{3-2a_1t+a_2t^2}{1-a_1t+a_2t^2-a_3t^3}$, and

$$l_n = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 \\ 2a_2 & a_1 & 1 & \cdots & 0 & 0 \\ 3a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ 0 & a_3 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & 1 \\ 0 & 0 & 0 & \cdots & a_2 & a_1 \end{vmatrix} = \sum_{\substack{n=s_1+2s_2+3s_3 \\ s=s_1+s_2+s_3}} (-1)^{n+s}\frac{n}{s}\binom{s}{s_1,s_2,s_3} a_1^{s_1}a_2^{s_2}a_3^{s_3}.$$

The Binet formulas for the 3-generalized Fibonacci numbers and Lucas numbers are

$$f_n = \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3\geq 0}} x_1^{n_1}x_2^{n_2}x_3^{n_3},$$

$$l_n = x_1^n + x_2^n + x_3^n,$$

where x_1, x_2, x_3 are roots of the characteristic equation $x^3-a_1x^2+a_2x-a_3 = 0$.

4 Applications

The Stirling numbers are often defined as the coefficients in an expansion of positive integral powers of a variable in terms of falling factorial powers, or vice-versa [5]: $x^{\underline{n}} = \sum_{k=0}^{n}(-1)^{n-k}\left[{n \atop k}\right]x^k$, $x^n = \sum_{k=0}^{n}\left\{{n \atop k}\right\}x^{\underline{k}}$, $n \geq 0$, where $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$ for $n \geq 1$ and $x^{\underline{0}} = 1$. The numbers $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ are the Stirling number of the first kind and the second kind, respectively. Z.-H. Sun [14] have given a expression for the Stirling number of the second kind in terms of a general linear recursive sequence. In this section, we give some new recurrence relations for the Stirling numbers of the second kind and power sums.

Let $S_n(k) = 1^n + 2^n + \cdots + k^n$ be sum of nth power of natural numbers from 1 to k. $S_n(k)$ is usually called sum of powers, or simply power sum. It is well known that $S_n(k)$ is a polynomial in k of degree $n+1$ (see [3,5]): $S_n(k) = \frac{1}{n+1}\sum_{j=0}^{n}(-1)^j B_j\binom{n+1}{j}k^{n+1-j}$, where B_j are Bernoulli numbers defined by generating function $\sum_{n=0}^{\infty} B_n t^n/n! = \frac{t}{e^t-1}$. We also know that power sum $S_n(k)$ can be expressed in terms of the Stirling numbers of the second kind and binomial coefficients as [3, 10] $S_n(k) = \sum_{j=1}^{n}(-1)^{n+j}\left\{{n \atop j}\right\}j!\binom{k+j}{k-1}$.

Now if we select $x_1 = 1, x_2 = 2, \cdots, x_k = k$ in (20) and (21), then $F(t) = \sum_{n=0}^{\infty} f_n t^n = \frac{1}{(1-t)(1-2t)\cdots(1-kt)} = \sum_{n=0}^{\infty} \left\{{n+k \atop k}\right\} t^n$, and $L(t) = \sum_{n=0}^{\infty} l_n t^n = \frac{1}{1-t} + \frac{1}{1-2t} + \cdots + \frac{1}{1-kt} = \sum_{n=0}^{\infty}(1^n + 2^n + \cdots + k^n)t^n$. Hence $f_n = \left\{{n+k \atop k}\right\}$ is the Stirling number of the second kind and $l_n = S_n(k) = 1^n + 2^n + \cdots + k^n$ is power sum. Since $1 - a_1 t + a_2 t^2 - \cdots + (-1)^k a_k t^k = (1-t)(1-2t)\cdots(1-kt) = \sum_{n=0}^{k}(-1)^n \left[{k+1 \atop k+1-n}\right] t^n$, thus $a_n = \left[{k+1 \atop k+1-n}\right]$ is the Stirling number of the first kind. According to our theory above, we obtain the following results involving the Stirling numbers and power sums:
Recurrence relation for the Stirling number of the second kind

$$\left\{{n+k \atop k}\right\} = \left[{k+1 \atop k}\right]\left\{{n+k-1 \atop k}\right\} - \left[{k+1 \atop k-1}\right]\left\{{n+k-2 \atop k}\right\} + \cdots + (-1)^{k-1}\left[{k+1 \atop 1}\right]\left\{{n \atop k}\right\}. \tag{29}$$

Recurrence relation for the power sums

$$S_n(k) = \left[{k+1 \atop k}\right] S_{n-1}(k) - \left[{k+1 \atop k-1}\right] S_{n-2}(k) + \cdots + (-1)^{k-1}\left[{k+1 \atop 1}\right] S_{n-k}(k). \tag{30}$$

Relation between $\left\{{n+k \atop k}\right\}$ and $S_n(k)$

$$S_n(k) = \left[{k+1 \atop k}\right]\left\{{n+k-1 \atop k}\right\} - 2\left[{k+1 \atop k-1}\right]\left\{{n+k-2 \atop k}\right\} + \cdots + (-1)^{k-1}k\left[{k+1 \atop 1}\right]\left\{{n+ \atop k}\right\}; \tag{31}$$

$$S_n(k) = k\left[{k+1 \atop k+1}\right]\left\{{n+k \atop k}\right\} - (k-1)\left[{k+1 \atop k}\right]\left\{{n+k-1 \atop k}\right\} + \cdots + (-1)^{k-1}\left[{k+1 \atop 2}\right]\left\{{n+1 \atop k}\right\}. \tag{32}$$

Determinant formulas

$$\left\{ {n+k \atop k} \right\} = \begin{vmatrix} \left[{k+1 \atop k}\right] & 1 & 0 & \cdots & 0 & 0 & 0 \\ \left[{k+1 \atop k-1}\right] & \left[{k+1 \atop k}\right] & 1 & \cdots & 0 & 0 & 0 \\ \left[{k+1 \atop k-2}\right] & \left[{k+1 \atop k-1}\right] & \left[{k+1 \atop k}\right] & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \left[{k+1 \atop k-n+2}\right] & \left[{k+1 \atop k-n+3}\right] & \left[{k+1 \atop k-n+4}\right] & \cdots & \left[{k+1 \atop k-1}\right] & \left[{k+1 \atop k}\right] & 1 \\ \left[{k+1 \atop k-n+1}\right] & \left[{k+1 \atop k-n+2}\right] & \left[{k+1 \atop k-n+3}\right] & \cdots & \left[{k+1 \atop k-2}\right] & \left[{k+1 \atop k-1}\right] & \left[{k+1 \atop k}\right] \end{vmatrix};$$

$$S_n(k) = \begin{vmatrix} \left[{k+1 \atop k}\right] & 1 & 0 & \cdots & 0 & 0 & 0 \\ 2\left[{k+1 \atop k-1}\right] & \left[{k+1 \atop k}\right] & 1 & \cdots & 0 & 0 & 0 \\ 3\left[{k+1 \atop k-2}\right] & \left[{k+1 \atop k-1}\right] & \left[{k+1 \atop k}\right] & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (n-1)\left[{k+1 \atop k-n+2}\right] & \left[{k+1 \atop k-n+3}\right] & \left[{k+1 \atop k-n+4}\right] & \cdots & \left[{k+1 \atop k-1}\right] & \left[{k+1 \atop k}\right] & 1 \\ n\left[{k+1 \atop k-n+1}\right] & \left[{k+1 \atop k-n+2}\right] & \left[{k+1 \atop k-n+3}\right] & \cdots & \left[{k+1 \atop k-2}\right] & \left[{k+1 \atop k-1}\right] & \left[{k+1 \atop k}\right] \end{vmatrix}.$$

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