Some basic properties of a class of total transformation digraphs*

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Abstract: Let D be a simple digraph without loops and parallel arcs. Deng and Kelmans [A. Deng, A. Kelmans, Spectra of digraph transformations, Linear Algebra and its Applications, 439(2013)106-132] gave the definition of transformation digraphs by introducing symbol $'0'$ and $'1'$, and investigated the regular and spectra of digraph transformation. In this paper we discuss a class of total transformation digraphs associate with symbol $'0'$. Furthermore, we determine the regularity of these ten new kinds of total transformation digraphs and also give necessary and sufficient conditions for them to be strongly connected.

Key words: Total transformation digraphs; Regularity; Strong connectedness;

1 Introduction

In this paper, we only consider simple digraph D with vertex set $V(D)$ and arc set $A(D)$. For a vertex $v \in V(D)$, we denote the out-degree, the in-degree of v by $d^+(v)$, $d^-(v)$. We denote the minimum out-degree, the minimum in-degree and minimum degree of D by $\delta^+(D)$, $\delta^-(D)$ and $\delta(D) = min\{\delta^+(D), \delta^-(D)\}$. C_n denote the directed cycle of order n. K_n denote the complete digraph of order n. A star denotes by $K_{1,k}$, is a bipartite digraph $D[X,Y]$ with $|X| = 1$ or $|Y| = 1$ and this vertex has only out-neighbours(out-star) or in-neighbours(in-star).

Let $D = (V(D), A(D))$ be a digraph, where $|V(D)| = n$, $|A(D)| = m$ and $V(D) = \{v_1, v_2, \cdots, v_n\}$. The *line digraph* of D, denoted by $L(D)$, is

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the digraph with vertex set $V(L(D)) = \{a_{ij} | (v_i, v_j) \in A(D)\}$, and a vertex a_{ij} is adjacent to a vertex a_{st} in $L(D)$ if and only if $v_j = v_s$ in D.

Wu and Meng[5]introduced a kinds of transformation graphs and investigated some basic properties of them. The authors[2] determined the regularity and spectral radius of transformation graphs. The authors[4] gave the definition of transformation digraphs and discussed some properties of middle digraph. The authors[6] discussed the properties of D^{xyz}, where x, y, z taking values $-$ or $+$. The authors [3]gave the definition of a new class of transformation graphs by introducing symbol '0', and discussed some properties: regularity, connectedness and spectra. Recently, the authors[1] gave the definition of transformation digraphs by introducing symbol '0' and '1', and investigated the regular and spectra of digraph transformation. In this paper, we main consider the case of '0'.

Definition 1.1. *Let $D = (V(D), A(D))$ be a digraph, x, y, z be three variables taking values $-$ or $+$. The transformation digraph of D, denoted by D^{xyz}, is a digraph with vertex set $V(D^{xyz}) = V(D) \cup A(D)$. For any vertex $a, b \in V(D^{xyz})$, $(a, b) \in A(D^{xyz})$ if and only if one of the following four cases holds:*

(i) If $a \in V(D)$ and $b \in V(D)$, then $(a, b) \in A(D)$ in D if $x = +$ and $(a, b) \notin A(D)$ in D if $x = -$.

(ii) If $a \in A(D)$ and $b \in A(D)$, then the head of arc a is the tail of arc b in D if $y = +$ and the head of arc a is not the tail of arc b in D if $y = -$.

(iii) If $a \in V(D)$ and $b \in A(D)$, then a is the tail of arc b in D if $z = +$ and a is not the tail of arc b in D if $z = -$.

(iv) If $a \in A(D)$ and $b \in V(D)$, then b is the head of arc a in D if $z = +$ and b is not the head of arc a in D if $z = -$.

Furthermore, (i') for any vertex $a \in V(D)$ and $b \in V(D)$, then there is no arc between a and b if $x = 0$;

(ii') for any vertex $a \in A(D)$ and $b \in A(D)$, then there is no arc between a and b if $y = 0$;

(iii') for any vertex $a \in V(D)$ and $b \in A(D)$, then there is no arc between a and b if $z = 0$.

According to the definition, we can get twenty-seven kinds of total transformation graphs, in which D^{+++} is total digraph of D, and D^{---} is its complement. Also, D^{--+}, D^{-+-} and D^{-++} are the complement of D^{++-}, D^{+-+} and D^{+--}. Moreover, D^{000}, D^{0+0}, D^{+00}, D^{-00}, D^{--0}, D^{+-0}, D^{0-0}, D^{-+0} and D^{++0} are simple. So, in this paper we investigate basic properties of other ten kinds of new total transformation digraphs associate with symbol '0'.

First, we list the vertex number, arc number and out-degree of new transformation digraphs D^{xyz}. Let $A = \sum_{v \in V(D)} d_D^+(v) d_D^-(v)$.

TD	vertex	$d^+(v), v \in V(D)$	$d^+(a), a = (u, v) \in A(D)$	arc number
D^{+0-}	$m+n$	m	$n-1$	$2mn-m$
D^{0+-}	$m+n$	$m-d_D^+(v)$	$d_D^+(v)+n-1$	$2mn-2m+A$
D^{-0-}	$m+n$	$n+m-1-2d_D^+(v)$	$n-1$	$n^2-n+2mn-3m$
D^{00-}	$m+n$	$m-d_D^+(v)$	$n-1$	$2mn-2m$
D^{0-+}	$m+n$	$d_D^+(v)$	$m-d_D^+(v)$	m^2+m-A
D^{00+}	$m+n$	$d_D^+(v)$	1	$2m$
D^{-0+}	$m+n$	$n-1$	1	n^2-n+m
D^{0++}	$m+n$	$d_D^+(v)$	$d_D^+(v)+1$	$2m+A$
D^{0--}	$m+n$	$m-d_D^+(v)$	$m+n-2-d_D^+(v)$	$m^2+2mn-3m-A$
D^{+0+}	$m+n$	$2d_D^+(v)$	1	$3m$

Table 1

In a digraph D, two vertices u and v are strongly connected if each of u and v is reachable from the other. In this paper, we investigate the regularity of ten kinds of new total transformation digraphs and also give necessary and sufficient conditions for them to be strongly connected. For convenience, we use $V(D) \cup V(L(D))$ to denote the vertex set of D^{xyz}.

2 Regularity of D^{xyz}

In this section, we will study the regular of transformation digraphs. A digraph D is k-regular, if for any $v \in V(D)$, $d^+(v) = d^-(v) = k$.

Theorem 2.1. *For a digraph D of order n, then D^{00+} is regular if and only if $D \cong C_n$.*

Proof. By Table 1, for any vertex $v \in V(D)$, $d^+_{D^{00+}}(v) = d_D{}^+(v)$, $d^-_{D^{00+}}(v) = d^-_D(v)$. For any arc $a \in A(D)$, $d^+_{D^{00+}}(a) = 1$, $d^-_{D^{00+}}(a) = 1$. If D^{00+} is regular, then $d_D^+(v) = d_D^-(v) = 1$ for every vertex $v \in V(D)$, hence $D \cong C_n$. If $D \cong C_n$, then D^{00+} is regular. □

By the similar argument, we have the following theorem:

Theorem 2.2. *For a digraph D of order n. Then*

(1) D^{0-+} and D^{-0-} are regular if and only if D is an $m/2$-regular diagraph.

(2) D^{+0+} and D^{0++} are not regular.

(3) D^{0--} and D^{-0+} are regular if and only if $n = 2$.

(4) D^{0+-} is regular if and only if D is an $(m-n+1)/2$-regular diagraph.

(5) D^{00-} *is regular if and only if* D *is an* $m-n+1$*-regular diagraph.*
(6) D^{+0-} *is regular if and only if* $m=n-1$.

3 Connectedness of D^{xyz}

Theorem 3.1. *For a digraph* D, D^{00+} *is strongly connected if and only if* D *is strongly connected.*

Proof. Suppose that D is not strongly connected, then there are two subsets X_1, X_2 of $V(D)$, such that there is no directed path from X_1 to X_2. That is, there is no directed path in D^{00+} from X_1 to X_2, hence, D^{00+} is strongly connected, then D is strongly connected.

Conversely, if D is strongly connected, for any $u,v \in V(D)$, there is a directed path in D from u to v, then it is also a directed path in D^{00+}.

For any $a,b \in V(L(D))$, let $a=(u,v), b=(x,y)$. If $v \neq x$, there is a directed path P_1 from v to x in D^{00+}, and $(a,v) \cup P_1 \cup (x,b)$ is a directed path from a to b in D^{00+}; if $v=x$, then (a,v,b) is a directed path in D^{00+} from a to b.

For any $u \in V(D)$, $b=(x,y) \in V(L(D))$, if $u=x$, then $(u,b) \in A(D^{00+})$; if $u \neq x$, then there exists a directed path P_2 from u to x in D^{00+}, and $P_2 \cup (x,b)$ is a directed path from u to b in D^{00+}. If $u=y$, then $(b,u) \in A(D^{00+})$; if $u \neq y$, then there is a directed path P_3 from y to u in D^{00+}, and $(b,y) \cup P_3$ is a directed path from b to u in D^{00+}. □

Since D^{00+} is spanning subdigraph of D^{+0+} and D^{0++}, then we can get the following theorem:

Theorem 3.2. *For a digraph* D, D^{+0+} *and* D^{0++} *are strongly connected if and only if* D *is strongly connected.*

Theorem 3.3. *For a digraph* D, D^{0-+} *is strongly connected if and only if* $\delta(D) \geq 1$.

Proof. The 'only if' part is obvious. We now show the 'if' part. For any vertices $u,v \in V(D)$, if there is a directed path in D from u to v, then there is a directed path in D^{0-+}. If there is no a directed path in D from u to v, then there are two arcs $a=(u,x), b=(y,v) \in A(D)(x \neq y)$ since $\delta(D) \geq 1$, hence $(a,b) \in A(D^{0-+})$. Thus, (u,a,b,v) is a directed path in D^{0-+} from u to v.

For any two vertices $a = (u, x), c = (w, z) \in V(L(D))$, if $x = w$, then (a, x, c) is a directed path in D^{0-+} from a to c; if $x \neq w$, then $(a, c) \in A(D^{0-+})$.

For any two vertices $u \in V(D), c = (w, z) \in V(L(D))$, if $u = w$, then $(u, c) \in A(D^{0-+})$; if $u \neq w$, there is a directed path P_1 from u to w in D^{0-+}, then $P_1 \cup (w, c)$ is a directed path in D^{0-+} from u to c. If $u = z$, then $(c, u) \in A(D^{0-+})$; if $u \neq z$, then there is a directed path P_2 from z to u in D^{0-+}, thus, $(c, w) \cup P_2$ is a directed path in D^{0-+} from c to u. □

Theorem 3.4. *For a digraph D, D^{-0+} is strongly connected for any digraph.*

Proof. If D is empty, then it is obvious. Now we consider D is nonempty. For any two vertices $u, v \in V(D)$, if $(u, v) \notin A(D)$, then $(u, v) \in A(D^{-0+})$; if $a = (u, v) \in A(D)$, then (u, a, v) is a directed path in D^{-0+} from u to v.

For any two vertices $a = (u, v), b = (x, y) \in V(L(D))$, if $v = x$, then (a, v, b) is a directed path in D^{-0+} from a to b; if $v \neq x$, by the above argument, there is a directed path P from v to x in D^{-0+}, then $(a, v) \cup P \cup (x, b)$ is a directed path in D^{-0+} from a to b.

For any two vertices $u \in V(D), b = (x, y) \in V(L(D))$, if $u = x$, then $(u, b) \in A(D^{-0+})$; if $u \neq x$, then $P_1 \cup (x, b)$ is a directed path in D^{-0+} from u to b (P_1 is a directed path from u to x in D^{-0+}). Similarly, if $u = y$, then $(b, u) \in A(D^{-0+})$; if $u \neq y$, then $(b, y) \cup P_2$ is a directed path in D^{-0+} from b to u (P_2 is a directed path from y to u in D^{-0+}). □

Theorem 3.5. *For a digraph D, D^{+0-} is strongly connected if and only if D has at least one arc.*

Proof. If D has no arc, then D^{+0-} is not strongly connected. Therefore, if D^{+0-} is strongly connected, then D has at least one arc.

On the other hand, if D has at least one arc, let $a = (u, v) \in A(D)$, then (u, v, a, u) is a directed cycle of length 3.

For any two vertices $x, y \in V(D)$, if $b = (x, y) \in A(D)$, then $(x, y) \in A(D^{+0-})$. Consider $(x, y) \notin A(D)$, if $x = u$ and $y \neq v$, then (x, v, a, y) is a directed path in D^{+0-} from x to y; if $x \neq u$ and $y = v$, then (x, a, u, y) is a directed path in D^{+0-} from x to y; if $x \neq u$ and $y \neq v$, then (x, a, y) is a directed path in D^{+0-} from x to y.

For any two vertices $b = (x, y), c = (w, z) \in V(L(D))$, if $x = w$ and $y \neq z$, then (b, x, z, c) is a directed path in D^{+0-} from b to c; if $x \neq w$, then (b, x, c) is a directed path in D^{+0-} from b to c.

For any two vertices $w \in V(D), b = (x, y) \in V(L(D))$, if $w = x$, then (w, y, b) is a directed path in D^{+0-} from w to b; if $w \neq x$, then $(w, b) \in A(D^{+0-})$. If $w = y$, then (b, x, w) is a directed path in D^{+0-} from b to w; if $w \neq y$, then $(b, w) \in A(D^{+0-})$. □

Theorem 3.6. *For a digraph D, D^{-0-} is strongly connected if and only if D is not a star.(out-star or in-star).*

Proof. If D is a star, then D^{-0-} is not strongly connected. Therefore, D^{-0-} is strongly connected, then D is not a star.

Let D is not a star, without loss of generality, we may assume that D is not an in-star. For any two vertices $u, v \in V(D)$, if $(u, v) \notin A(D)$, then $(u, v) \in A(D^{-0-})$. Now we consider the case that $a = (u, v) \in A(D)$. There is a vertex $w(\neq v) \in V(D)$ such that $(u, w) \notin A(D)$, then $(u, w) \in A(D^{-0-})$, if $(w, v) \notin A(D)$, then $(w, v) \in A(D^{-0-})$, thus (u, w, v) is a directed path in D^{-0-}; if $b = (w, v) \in A(D)$, then there is an isolated vertex z or an arc c such that v is not the head of c since D is not an in-star, thus $(u, z, v), (u, c, v)$ or (u, w, c, v) is a directed path in D^{-0-}.

For any two vertices $a = (u, v), e = (x, y) \in V(L(D))$, by the above argument, there is a directed path P from u to y in D^{-0-}, then $(a, u) \cup P \cup (y, e)$ is a directed path in D^{-0-} from a to e.

For any two vertices $u \in V(D), e = (x, y) \in V(L(D))$, if $u \neq x$, then $(u, e) \in A(D^{-0-})$; if $u = x$, there is a directed path P_1 from u to y, then (u, P_1, y, e) is a directed path from u to e. Similarly, if $u \neq y$, then $(e, u) \in A(D^{-0-})$; if $u = y$, then $(e, x, P_2, u)(P_2$ is a directed path from x to u) is a directed path in D^{-0-} from e to u.

Similarly, we can show that if D is not an out-star, then D^{-0-} is strongly connected. □

Theorem 3.7. *For a digraph D, D^{00-} is strongly connected if and only if $D \ncong K_{1,k} \cup (n-k-1)K_1$, where $0 \leq k \leq n-1$.*

Proof. If $D \cong K_{1,k} \cup (n-k-1)K_1$, where $0 \leq k \leq n-1$, then D^{00-} is not strongly connected. Therefore, D^{00-} is strongly connected, then $D \ncong K_{1,k} \cup (n-k-1)K_1$, where $0 \leq k \leq n-1$.

On the other hand, if $D \ncong K_{1,k} \cup (n-k-1)K_1$, for any two vertices $u, v \in V(D)$, if $a = (v, u) \in A(D)$, then (u, a, v) is a directed path in D^{00-} from u to v. Now, we consider the case that $(v, u) \notin A(D)$, there are at least two arcs with u is not as the tail of a, v is not as the head of b, let $a = (x, y), b = (w, z)$. If $y \neq z$, then (u, a, z, b, v) is a directed path in D^{00-} from u to v; if $y = z$, then there is an arc c such that y is not the head of c since $D \ncong K_{1,k} \cup (n-k-1)K_1$, let $c = (i,j)$. If $i \neq w$, then (u,a,w,c,y,b,v) is a directed path in D^{00-} from u to v; if $i = w$, then (u,a,j,c,y,b,v) is a directed path in D^{00-} from u to v.

For any two vertices $a = (v,u), b = (w,z) \in V(L(D))$, by the above argument, there is a directed path P from v to z, then $(a,v) \cup P \cup (z,b)$ is a directed path in D^{00-} from a to b.

For any two vertices $w \in V(D), a = (v,u) \in V(L(D))$, if $w \neq v$, then $(w,a) \in A(D^{00-})$; if $w = v$, there is a directed path P_1 from w to u, then (w, P_1, u, a) is a directed path in D^{00-} from w to a. Similarly, if $w \neq u$, then $(a,w) \in A(D^{00-})$; if $w = u$, then (a, v, P_2, w) (P_2 is a directed path from v to w) is a directed path in D^{00-} from a to w. □

By the similar argument, we have the following theorem:

Theorem 3.8. *For a digraph D, D^{0--} and D^{0+-} are strongly connected if and only if $D \ncong K_{1,k} \cup (n-k-1)K_1$, where $0 \leq k \leq n-1$.*

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Fault-free cycles passing through prescribed a linear forest in a hypercube with faulty edges *

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Abstract

Chen considered the problem of fault-free cycles passing through prescribed a linear forest in an n-dimensional hypercube Q_n with some faulty edges and obtained the following result: Let $n > h \geq 2$, $F \subset E(Q_n)$ with $|F| < n - h$, and $E_0 \subset E(Q_n) \backslash F$ with $|E_0| = h$. If the subgraph induced by E_0 is a linear forest, then in the graph $Q_n - F$ all edges of E_0 lie on a cycle of every even length l with $2^{h-1}(n+1-h) + 2(h-1) \leq l \leq 2^n$. In this paper, above result is improved as follows: under the same condition in $Q_n - F$ all edges of E_0 lie on a cycle of every even length l with $2(h-1)n - 6(h-2) \leq l \leq 2^n$.

Keywords: Hypercube; Cycle embedding; Hamiltonian cycle; Fault tolerance; Interconnection network

1 Introduction

It is well known that the n-dimensional hypercube, denoted by Q_n, is one of the most popular and efficient interconnection networks. It possesses many excellent properties such as recursive structure, symmetry, small diameter, low degree, popular structure embedding, and easy routing. There is a large amount of literature on graph-theoretical properties of hypercubes and their applications in parallel computing (e.g., see [7,9]).

In this paper, we follow [2] for graph-theoretical terminology and notation, and a graph $G = (V, E)$ means a simple graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set of the graph G. The Hamiltonian property is one of the major requirements in designing network topologies since a topology structure containing Hamiltonian paths or cycles can efficiently simulate many algorithms designed on linear arrays or rings. It is well known that the n-dimensional hypercube has Hamiltonian cycles.

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A graph G is called edge-pancyclic if its every edge lies on a cycle of every length from 3 to $|V(G)|$ (see, e.g., [1]). Similarly, a bipartite graph (a graph containing no cycle of odd length) G is called edge-bipancyclic if its every edge lies on a cycle of every even length from 4 to $|V(G)|$. Fault tolerant ability is an important factor for interconnection networks. A (bipartite) graph G is called $k-$edge-fault-tolerant edge-(bi)pancyclic if every graph obtained by deleting any up to k edges from G remains edge-(bi)pancyclic. Many results on (edge-fault-tolerant) edge-(bi)pancyclicity were obtained for hypercubes and their variants [6,8,10-17].

On the other hand, a few papers [3,5] investigated the following problem: Given a set of prescribed edges in a hypercube without faulty edges, which conditions guarantee the existence of a Hamiltonian path or cycle passing through all edges of this set in the hypercube? In hypercube-like interconnection networks with faulty vertices and/or edges, the problem of constructing a Hamiltonian cycle or path passing through prescribed edge list in the order given was considered in [11]. Recently, Chen [4] studied the problem of fault-free cycles passing through prescribed edges in a hypercube with faulty edges and obtained a result, see Lemma 1 below.

In this paper, the same problem is considered, and the result of [4] is improved as follows:

Theorem 1. Let $n > h \geq 2$, $F \subset E(Q_n)$ with $|F| < n - h$, and $E_0 \subset E(Q_n) \backslash F$ with $|E_0| = h$. If the subgraph induced by E_0 is a linear forest (i.e., pairwise vertex-disjoint paths), then in the graph $Q_n - F$ all edges of E_0 lie on a cycle of every even length l with $2(h-1)n - 6(h-2) \leq l \leq 2^n$. Moreover, if $h = 2$, then the result is optimal in the sense that Q_n contains (1) two edges such that any cycle in Q_n passing through them is of length at least $2n$, and (2) edge subsets E_0 and F with $|E_0| = 2, |F| = n - 2$ such that no Hamiltonian cycle passes through the two edges of E_0 in $Q_n - F$.

The proof of Theorem 1 is in Section 3.

2 Preliminaries

In this section, we present some preliminaries on hypercubes.

The n-dimensional (binary) hypercube Q_n is a bipartite graph with 2^n, its any vertex v is denoted by an n-bit binary string $v = \lambda_1\lambda_2...\lambda_{n-1}\lambda_n$ where $\lambda_i \in \{0,1\}$ for all $i, 1 \leq i \leq n$. Two vertices of Q_n are adjacent if and only if their binary strings differ in exactly one bit position, so Q_n is an n regular graph. Assume $e = (u, v)$ is any edge of Q_n. If the two binary strings of u and v differ in the ith bit position, where $1 \leq i \leq n$, then the edge e is called a dimension i edge of Q_n. The set of all dimension i edges of Q_n is denoted by E_i. It is clear that any two edges of E_i are not adjacent (that is, E_i is a matching in Q_n) and $|E_i| = 2^{n-1}$ for all $i, 1 \leq i \leq n$. For any given $j \in \{1, 2, ...n\}$, let Q^0_{n-1} and Q^1_{n-1} be two $(n-1)$-dimensional subcubes of Q_n induced by all the vertices with the jth bit positions being 0 and 1, respectively. It is clear that $Q_n - E_j = Q^0_{n-1} \cup Q^1_{n-1}$, we say that Q_n is decomposed into two $(n-1)$-dimensional subcubes Q^0_{n-1} and Q^1_{n-1} by E_j. For a given $\lambda \in \{0,1\}$, if v_λ is a vertex of Q^λ_{n-1}, then there is exactly one corresponding vertex $v_{1-\lambda}$ in $Q^{1-\lambda}_{n-1}$ such that the edge $(v_\lambda, v_{1-\lambda}) \in E_j$, and if (u_λ, v_λ) is an edge of Q^λ_{n-1}, then there is exactly one corresponding edge $(u_{1-\lambda}, v_{1-\lambda})$ in $Q^{1-\lambda}_{n-1}$ such that $(u_\lambda, u_{1-\lambda}) \in E_j$ and $(v_\lambda, v_{1-\lambda}) \in E_j$, and $u_\lambda u_{1-\lambda} v_{1-\lambda} v_\lambda$ is a cycle with length four in Q_n.

Lemma 1.[4] *Let $n > h \geq 2$, $F \subset E(Q_n)$ with $|F| < n-h$, and $E_0 \subset E(Q_n)\backslash F$ with $|E_0| = h$. If the subgraph induced by E_0 is a linear forest, then in the graph $Q_n - F$ all edges of E_0 lie on a cycle of every even length l with $2^{h-1}(n+1-h)+2(h-1) \leq l \leq 2^n$. Moreover, if $h = 2$, then the result is optimal in the sense that Q_n contains (1) two edges such that any cycle in Q_n passing through them is of length at least $2n$, and (2) edge subsets E_0 and F with $|E_0| = 2, |F| = n-2$ such that no Hamiltonian cycle passes through the two edges of E_0 in $Q_n - F$.*

Lemma 2.[5] *Given a set E_0 of at most $2n-3$ edges in an $n-$dimensional hypercube $Q_n (n \geq 2)$. Then Q_n contains a Hamiltonian cycle passing through all edges of E_0 if and only if the subgraph induced by E_0 is a linear forest.*

Lemma 3.[10] *The n-dimensional hypercube Q_n is $(n-2)$-edge-fault-tolerant edge-bipancyclic for every $n \geq 3$.*

Lemma 4.[16] *If Q_n $(n \geq 2)$ has at most $n-2$ faulty edges, then for any two different vertices u and v there exists a fault-free uv-path of length l with $d(u,v)+2 \leq l \leq 2^n - 1$ and $2|(l-d(u,v))$, where $d(u,v)$ is the distance of vertices u and v.*

Lemma 5. *Let $e = (u_1, v_1)$ and $f = (u_2, v_2)$ be two edges in the n-dimensional hypercube Q_n. Then edges e and f are not contained in any $(n-1)$-dimensional subcube of Q_n if and only if there are two vertices belonging to $\{u_1, v_1, u_2, v_2\}$ with the distance between them being n.*

Proof. Sufficiency. Without loss of generality, we assume $d(u_1, u_2) = n$. Since the distance between any two different vertices in an $(n-1)$-dimensional hypercube is at most $n-1$, the two vertices u_1 and u_2 are not contained in any $(n-1)$-dimensional subcube of Q_n, and so do the two edges e and f.

Necessity. By contradiction, assume the distance between any two vertices belonging to $\{u_1, v_1, u_2, v_2\}$ is at most $n-1$.

By symmetry of Q_n, without loss of generality, assume $u_1 = 000...000$ and $v_1 = 000...001$, then u_2 and v_2 are neither $111...111$ nor $111...110$. Since edges e and f are not contained in any $(n-1)$-dimensional subcube of Q_n, the 1st position of u_2 or v_2 is 1. Assume $u_2 = 1\lambda_2...\lambda_{n-1}\lambda_n$, then $v_2 = 0\lambda_2...\lambda_{n-1}\lambda_n$ or $v_2 = 1\lambda_2...\overline{\lambda_i}...\lambda_{n-1}\lambda_n$, where $\overline{\lambda_i} = 1 - \lambda_i$, $\lambda_i \in \{0,1\}$, $2 \leq i \leq n$.

If $v_2 = 0\lambda_2...\lambda_{n-1}\lambda_n$, since u_2 is neither $111...111$ nor $111...110$, there exists j, $2 \leq j \leq n-1$, such that $\lambda_j = 0$, then the jth position of the four vertices is 0, so they are contained in an $(n-1)$-dimensional subcube of Q_n, it is a contradiction.

If $v_2 = 1\lambda_2...\overline{\lambda_i}...\lambda_{n-1}\lambda_n$, then $\lambda_j = 1$ for every j $(2 \leq j \leq n-1)$ and $j \neq i$. Since $\lambda_i = 1$ or $\overline{\lambda_i} = 1$, and u_2 and v_2 are neither $111...111$ nor $111...110$, it is a contradiction.

So Lemma 5 holds. []

Lemma 6. *Given an edge e in Q_n. Then there are exactly $2n-1$ edges such that any one of these edges and the edge e are not contained in any*

$(n-1)$-dimensional subcube of Q_n. Denote by E' the set of these $2n-1$ edges. Let H be a linear forest in Q_n, then $|E(H) \cap E'| \leq 4$.

Proof. By symmetry of Q_n, assume $e = (u_0, v_0)$, where $u_0 = 000...000$ and $v_0 = 000...001$. By Lemma 5, if an end-vertex of an edge is $u_1 = 111...111$ or $v_1 = 111...110$, then the edge and e are not contained in any $(n-1)$-dimensional subcube of Q_n. Since Q_n is an n-regular graph, there are exactly n edges incident with u_1 and v_1 respectively, notice that u_1 and v_1 are two end-vertices of the edge (u_1, v_1), so these $2n-1$ edges consist of E'. It is easy to see that the latter half of the lemma is also true. []

Lemma 7. *Given an edge e in Q_n, then there are exactly $n-1$ cycles with length four such that any two of these cycles have the edge e in common.*

Proof. Assume $e = (u_0, v_0)$, where $u_0 = 000...000$ and $v_0 = 000...001$. In Q_n all vertices adjacent to u_0 are $u_1 = 100...000$, $u_2 = 010...000$, ... , $u_{n-1} = 000...010$ and v_0; all vertices adjacent to v_0 are $v_1 = 100...001$, $v_2 = 010...001$, ... , $v_{n-1} = 000...011$ and u_0. Clearly, $n-1$ cycles $u_0u_iv_iv_0$ with length four for every $i = 1, ..., n-1$, meet the lemma. []

3 Proof of Theorem 1

Since $|E_0 \cup F| \leq h + n - 1 - h = n - 1$, there exists $j, 1 \leq j \leq n$, such that $(E_0 \cup F) \cap E_j = \emptyset$, where E_j is the set of all dimension j edges of Q_n. Suppose Q_n is decomposed into two $(n-1)$-dimensional subcubes Q^0_{n-1} and Q^1_{n-1} by E_j. Let $F_\lambda = F \cap E(Q^\lambda_{n-1})$ and $E^\lambda_0 = E_0 \cap E(Q^\lambda_{n-1})$ for $\lambda \in \{0,1\}$.

By Lemma 1, Theorem 1 holds if $h = 2$. Assume $3 \leq h < n$, we now prove Theorem 1 by induction on n.

If $n = 4$, then $h = 3$ and $F = \emptyset$. By Lemma 1, it is sufficient to prove that all edges of E_0 lie on a cycle of length 10 in Q_4.

Case 1. $E_0 \subset E(Q^0_3)$ or $E_0 \subset E(Q^1_3)$.

Without loss of generality, assume $E_0 \subset E(Q^0_3)$.

By Lemma 2, in Q^0_3 all edges of E_0 lie on a cycle C_0 of length $2^3 = 8$. Since $8 > 3$, there is an edge $(u_0, v_0) \in E(C_0) \backslash E_0$, such that the corresponding edge $(u_1, v_1) \in E(Q^1_3)$. Hence in Q_4 all edges of E_0 lie on the cycle

$C^* := (C_0 - (u_0, v_0)) \cup (u_0, u_1) \cup (u_1, v_1) \cup (v_1, v_0)$

of length 10.

Case 2. $E_0 = E^0_0 \cup E^1_0$ and $\emptyset \neq E^\lambda_0 \subset E(Q^\lambda_3)$, $\lambda \in \{0,1\}$. Assume $|E^0_0| \geq |E^1_0|$, then $|E^0_0| = 2$ and $|E^1_0| = 1$. Let $E^0_0 = \{a, b\}$ and $E^1_0 = \{c\}$.

Subcase 2.1. Edges a and b are adjacent.

It is easy to see that in Q^0_3 edges a and b lie on a cycle C_0 of length 4. Since $4 - 2 > 1$, there is an edge $(u_0, v_0) \in E(C_0) \backslash \{a, b\}$ such that the corresponding edge $(u_1, v_1) \in E(Q^1_3) \backslash c$. By Lemma 1, in Q^1_3 edges (u_1, v_1) and c lie on a cycle C_1 of length 6. Hence in Q_4 all edges of E_0 lie on the cycle

$C^{**} := (C_0 - (u_0, v_0)) \cup (C_1 - (u_1, v_1)) \cup (u_0, u_1) \cup (v_0, v_1)$

of length 10 (see Fig. 1).

Subcase 2.2. Edges a and b are not adjacent.

Subsubcase 2.2.1. The corresponding edge of the edge c is the edge a or b. Assume edge $a = (u_0, x_0)$ is the corresponding edge of $c = (u_1, x_1)$.

By Lemma 1, in Q_3^0 edges a and b lie on a cycle C_0 of length 6, let $(u_0, v_0) \in E(C_0)\backslash a$, then $(u_0, v_0) \neq b$. Clearly, the edge c and the corresponding edge (u_1, v_1) of the edge (u_0, v_0) lie on a cycle C_1 of length 4 in Q_3^1. Hence in Q_4 all edges of E_0 lie on the cycle C^{**} as before of length 10.

Subsubcase 2.2.2. The corresponding edge of edge c is neither edge a nor edge b.

Let $c = (u_1, v_1)$, then its corresponding edge $e = (u_0, v_0) \in E(Q_3^0)\backslash E_0^0$. Since edges a and b are not adjacent, the subgraph induced by $e \cup E_0^0$ is a linear forest. By Lemma 2, in Q_3^0 all edges of $e \cup E_0^0$ lie on a cycle C_0 of length $2^3 = 8$. Hence in Q_4 all edges of E_0 lie on the cycle C^* as before of length 10.

So Theorem 1 holds for $n = 4$. Suppose Theorem 1 holds for $n-1$ (≥ 4), we now prove Theorem 1 for n (≥ 5).

There are two cases to consider.

Case 1. $E_0 \subset E(Q_{n-1}^0)\backslash F_0$ or $E_0 \subset E(Q_{n-1}^1)\backslash F_1$.

Without loss of generality, assume $E_0 \subset E(Q_{n-1}^0)\backslash F_0$.

Subcase 1.1. $h = n-1$.

Then $F = \emptyset$. We now prove that in Q_n all edges of E_0 lie on a cycle of every even length l with $2n^2 - 10n + 18 \leq l \leq 2^n$. Since $n - 1 < 2n - 3$ $(n \geq 5)$, by Lemma 2, it is sufficient to prove that in Q_n all edges of E_0 lie on a cycle of every even length l with $2n^2 - 10n + 18 \leq l \leq 2^n - 2$.

Since the subgraph induced by E_0 is a linear forest, denoted by G, there is an edge e in G such that an end-vertex of edge e is of degree 1. Since $|F_0| = 0 < (n-1) - (n-2)$ and $|E_0 \setminus e| = h - 1 = n - 2$, by the induction hypothesis, in Q_{n-1}^0 all edges of $E_0 \setminus e$ lie on a cycle C_0 of every even length l_0 with $2(n-3)(n-1) - 6(n-4) = 2n^2 - 14n + 30 \leq l_0 \leq 2^{n-1} - 2$.

Subsubcase 1.1.1. $e \in E(C_0)$.

Since $2n^2 - 14n + 30 > n - 1$, there is an edge $(u_0, v_0) \in E(C_0) \setminus E_0^0$, such that its corresponding edge $(u_1, v_1) \in E(Q_{n-1}^1)$. By Lemma 3, in Q_{n-1}^1 the edge (u_1, v_1) lies on a cycle C_1 of every even length l_1 with $4 \leq l_1 \leq 2^{n-1}$. Hence in Q_n all edges of E_0 lie on the cycle C^{**} as before of every even length $l = l_0 + l_1$ with $2n^2 - 14n + 34 \leq l \leq 2^n - 2$ (See Fig. 1). Since $2n^2 - 14n + 34 < 2n^2 - 10n + 18$, Theorem 1 follows.

Subsubcase 1.1.2. $e \notin E(C_0)$. Let $e = (u_0, v_0)$.

There are three cases to consider.

Case (i). $\{u_0, v_0\} \cap V(C_0) = \emptyset$.

Denote by E' the set of all such edges of Q_{n-1}^0 that any one edge of E' and edge e are not contained in any $(n-2)$-dimensional subcube of Q_{n-1}^0. Since the subgraph induced by $E(C_0) \setminus E_0$ is a linear forest, by Lemma 6, then $|(E(C_0)\setminus E_0) \cap E'| \leq 4$. Since $(2n^2 - 14n + 30) - (n-2) > 4$, there is an edge $(x_0, y_0) \in E(C_0) \setminus E_0$ such that edges e and (x_0, y_0) are contained in an $(n-2)$-dimensional subcube of Q_{n-1}^0, and so their corresponding edges (u_1, v_1) and (x_1, y_1) are contained in an $(n-2)$-dimensional subcube of Q_{n-1}^1. By Lemma 1, in Q_{n-1}^1 edges (u_1, v_1) and (x_1, y_1) lie on the cycle C_1

of every even length l_1 with $2(n-2) \le l_1 \le 2^{n-2}$ and $2(n-1) \le l_1 \le 2^{n-1}$. Hence in Q_n all edges of E_0 lie on the cycle

$C := (C_0 - (x_0, y_0)) \cup (C_1 - (x_1, y_1) - (u_1, v_1)) \cup (x_0, x_1) \cup (y_0, y_1) \cup (u_0, u_1) \cup (v_0, v_1) \cup e$

of every even length $l = l_0 + l_1 + 2$ with $2n^2 - 12n + 28 \le l \le 2^n - 2$ (see Fig. 2). Since $2n^2 - 12n + 28 \le 2n^2 - 10n + 18$, Theorem 1 follows.

Case (ii). $|\{u_0, v_0\} \cap V(C_0)| = 1$.

Without loss of generality, assume $\{u_0, v_0\} \cap V(C_0) = \{u_0\}$.

Since G is a linear forest, there exists an edge $(u_0, x_0) \in E(C_0) \setminus E_0$. Let the corresponding vertices of v_0 and x_0 be v_1 and x_1 respectively. Clearly, $d(v_0, x_0) = 2$ and $d(v_1, x_1) = 2$. Since $F = \emptyset$, by Lemma 4, in Q^1_{n-1} there is a fault-free v_1x_1-path P of every even length l_1 with $2 \le l_1 \le 2^{n-1} - 2$. Hence in Q_n all edges of E_0 lie on the cycle

$C := (C_0 - (u_0, x_0)) \cup P \cup (x_0, x_1) \cup (v_0, v_1) \cup e$

of every even length $l = l_0 + l_1 + 2$ with $2n^2 - 14n + 34 \le l \le 2^n - 2$ (see Fig. 3), then Theorem 1 follows.

Case (iii). $\{u_0, v_0\} \subset V(C_0)$.

Since G is a linear forest, and an end-vertex of edge e is of degree 1, in $E(C_0) \setminus E_0$ there are at least three edges adjacent to e. Without loss of generality, assume $(u_0, x_0), (v_0, y_0) \in E(C_0) \setminus E_0$ such that the two u_0v_0-paths on cycle C_0 pass through x_0 and y_0, respectively. Clearly, the distance of x_0 and y_0 is 1 or 3, so the distance of their corresponding vertices x_1 and y_1 is also 1 or 3. By Lemma 4, in Q^1_{n-1} there is a fault-free x_1y_1-path P of every odd length l_1 with $3 \le l_1 \le 2^{n-1} - 1$. Hence in Q_n all edges of E_0 lie on the cycle

$C := (C_0 - (u_0, x_0) - (v_0, y_0)) \cup P \cup (x_0, x_1) \cup (y_0, y_1) \cup e$

of every even length $l = l_0 + l_1 + 1$ with $2n^2 - 14n + 34 \le l \le 2^n - 2$ (see Fig. 4), then Theorem 1 follows.

Subcase 1.2. $3 \le h \le n - 2$.

Subsubcase 1.2.1. $F_0 \ne \emptyset$.

Let $w \in F_0$, then $|F_0 \setminus w| \le |F| - 1 < (n-1) - h$. By the induction hypothesis, in $Q^0_{n-1} - (F_0 \setminus w)$ all edges of E_0 lie on a cycle C_0 of every even length l_0 with $2(h-1)(n-1) - 6(h-2) \le l_0 \le 2^{n-1}$. If $w \in E(C_0)$, then let $w = (u_0, v_0)$; otherwise let $e = (u_0, v_0)$, where e is any edge of $E(C_0) \setminus E_0$. Let the corresponding edge of edge (u_0, v_0) be (u_1, v_1). Since $|F_1| < |F| \le (n-1) - h < (n-1) - 2$, by Lemma 3, in $Q^1_{n-1} - (F_1 \setminus (u_1, v_1))$ the edge (u_1, v_1) lies on a cycle C_1 of every even length l_1 with $4 \le l_1 \le 2^{n-1}$. Hence in $Q_n - F$ all edges of E_0 lie on the cycle C^{**} as before of every even length $l = l_0 + l_1$ with $2(h-1)(n-1) - 6(h-2) + 4 \le l \le 2^{n-1}$. Since $2(h-1)(n-1) - 6(h-2) + 4 \le 2(h-1)n - 6(h-2)$, Theorem 1 follows.

Subsubcase 1.2.2. $F_0 = \emptyset$.

Since $|F_0| = 0 < (n-1) - h$, $|F_1| \le (n-1) - h < (n-1) - 2$, by the induction hypothesis, in Q^0_{n-1} all edges of E_0 lie on a cycle C_0 of every even length l_0 with $2(h-1)(n-1) - 6(h-2) \le l_0 \le 2^{n-1}$. Since $n > h \ge 3$, then $2(h-1)(n-1) - 6(h-2) - h = 2(h-2)(n-4) + 2(n-1) - h > n - 1 - h$, and therefore, $|E(C_0) \setminus E_0| > |F_1|$. So there is an edge $(u_0, v_0) \in E(C_0) \setminus E_0$ such that its corresponding edge $(u_1, v_1) \in E(Q^1_{n-1}) \setminus F_1$, by Lemma 3, edge (u_1, v_1) lies on a cycle C_1 of every even length l_1 with $4 \le l_1 \le 2^{n-1}$. Similar to subsubcase 1.2.1, Theorem 1 follows.

Case 2. Let $E_0 = E_0^0 \cup E_0^1$ and $\emptyset \neq E_0^\lambda \subset E(Q_{n-1}^\lambda) \setminus F_\lambda$ for $\lambda \in \{0, 1\}$, and $|E_0^0| = k \ge 1$ and $|E_0^1| = h - k \ge 1$.

Without loss of generality, suppose $k \ge h-k$, then $k \ge 2$ and $h \ge k+1$.

Subcase 2.1. $h = 3$. Then $k = 2$ and $h - k = 1$. Let $E_0^0 = \{a, b\}$ and $E_0^1 = \{c\}$.

By Lemma 1, it is sufficient to prove that in $Q_n - F$ all edges of E_0 lie on a cycle of length $4n - 6$.

Subsubcase 2.1.1. $|F_0| < n - 4 = (n-1) - 3$.

By Lemma 7, in Q_{n-1}^1 there are exactly $n-2$ cycles of length 4 such that any two of these cycles have edge c in common. Since $|F_1| \le n - 4$, then in $Q_{n-1}^1 - F_1$ there are two cycles of length 4, denoted by C_1 and C_2, that contain edge c and no faulty edges. It is easy to see that $(C_1 \cup C_2) - c$ is a cycle of length 6, so in Q_{n-1}^0 its corresponding cycle is also of length 6, denoted by C_3. Clearly, the cycle C_3 contains at most two edges of E_0^0, and if edges a and b have a common vertex v, then v is adjacent to at most two edges of C_3. Since $6 > 2$, there is an edge $e = (u_0, v_0) \in E(C_3) \setminus E_0^0$ such that the subgraph induced by $e \cup E_0^0$ is a linear forest. By the induction hypothesis, in $Q_{n-1}^0 - (F_0 \setminus e)$ all edges of $e \cup E_0^0$ lie on a cycle C_0 of length $4(n-1) - 6$, and in $Q_{n-1}^1 - F_1$ edge c and the corresponding edge $e_1 = (u_1, v_1)$ of edge e lie on the cycle C_1 or C_2 of length 4, without loss of generality, assume c and e_1 lie on a cycle C_1 of length 4. Hence in $Q_n - F$ all edges of E_0 lie on the cycle C^{**} of length $4n - 6$ as before.

Subsubcase 2.1.2. $|F_0| = n - 4 < (n-1) - 2$.

Then $F_1 = \emptyset$. By Lemma 1, in $Q_{n-1}^0 - F_0$ edges a and b lie on a cycle C_0 of length $2(n-1)$. By Lemma 5, in Q_{n-1}^1 there are exactly $2(n-1) - 1 = 2n - 3$ edges such that any one of these edges and edge c are not contained in any $(n-2)$-dimensional subcube of Q_{n-1}^1, denote by E' the set of these $2n-3$ edges. Let E'' be the set of $2n-3$ edges in Q_{n-1}^0 corresponding to E'. Since $C_0 - E_0^0$ is a linear forest, by Lemma 6, $|E(C_0 - E_0^0) \cap E''| \le 4$. Since $2(n-1) - 2 - 4 > 1$, i.e., $|E(C_0 - E_0^0)| - 4 > |\{c\}|$, there is an edge $e = (u_0, v_0) \in E(C_0) \setminus E_0^0$ such that its corresponding edge $e_1 = (u_1, v_1) \in E(Q_{n-1}^1) \setminus (E' \cup c)$, hence edges c and e_1 are contained in an $(n-2)$-dimensional subcube Q_{n-2} of Q_{n-1}^1. Since $|F_1| = 0 < (n-2) - 2$ ($n \ge 5$), by Lemma 1, in Q_{n-2} edges c and e_1 lie on a cycle C_1 of length $2(n-2)$. Hence in $Q_n - F$ all edges of E_0 lie on the cycle C^{**} as before of length $2(n-1) + 2(n-2) = 4n - 6$.

Subcase 2.2. $4 \le h < n$.

Since $|F_0| \le |F| < n - h \le (n-1) - k$, by the induction hypothesis, in $Q_{n-1}^0 - F_0$ all edges of E_0^0 lie on a cycle C_0 of every even length l_0 with $2(k-1)(n-1) - 6(k-2) \le l_0 \le 2^{n-1}$.

Now we are to pick up an edge $e \in E(C_0) \setminus E_0^0$ such that $E_0^1 \cup e_1$ is a linear forest in Q_{n-1}^1. We define three subsets of $E(C_0)$ as follows.

If $e \in E(C_0)$ such that its corresponding edge $e_1 \in E_0^1$, then the edge e

is said to be of class 1. Denote by E_1 the set of all edges being of class 1, then $|E_1| \le |E_0^1| = h - k$.

Suppose G is the subgraph of Q_{n-1}^1 induced by E_0^1 and G consists of t (≥ 1) pairwise vertex-disjoint paths. Denote by V_2 the set of all vertices of degree 2 in G. It is easy to see that $|V_2| = |E_0^1| - t = h - k - t \le h - k - 1$.

If $v \in V(C_0) \subset V(Q_{n-1}^0)$ such that its corresponding vertex $v_1 \in V_2 \subset V(G)$, then any one of the two edges of $E(C_0)$ incident with vertex v is said to be of class 2. Denote by E_2 the set of all edges being of class 2. then $|E_2| \le 2|V_2| \le 2(h-k) - 2$.

If $e \in E(C_0)$ and its corresponding edge is e_1 such that $e_1 \cup P$ is a cycle, where P is a path of length at least three of the graph G, that is, the two end-vertices of edge e is exactly the end-vertices of the path P, then edge e is said to be of class 3. Denote by E_3 the set of all edges being of class 3. Since in Q_n the length of a cycle is at least 4, it is easy to see that $|E_3| \le \frac{1}{3}|E_0^1| = \frac{1}{3}(h-k)$.

Hence $|E_1 \cup E_2 \cup E_3| \le \frac{10}{3}(h-k) - 2$. If $k \ge 5$, then $2(k-1)(n-1) - 6(k-2) - k \ge 8(n-1) - 6(k-2) - k = 2(n-1) + 6(n+1-k) - k > \frac{10}{3}(h-k) - 2$, then $l_0 - k > \frac{10}{3}(h-k) - 2$. Furthermore, it is easy to verify that the inequality $l_0 - k > \frac{10}{3}(h-k) - 2$ holds for $2 \le k \le 4$. So $|E(C_0)| - |E_0^0| > |E_1 \cup E_2 \cup E_3|$. It follows that there exists an edge $e = (u_0, v_0) \in E(C_0) \setminus (E_0^0 \cup E_1 \cup E_2 \cup E_3)$ such that its corresponding edge $e_1 = (u_1, v_1) \in E(Q_{n-1}^1) \setminus E_0^1$.

By the definition of E_1, E_2 and E_3, it is easy to see that the subgraph H induced by $e_1 \cup E_0^1$ is a linear forest. Since $|F_1 \setminus e_1| \le |F_1| \le n - 1 - h < (n-1) - (h-k+1)$ and $|e_1 \cup E_0^1| = h - k + 1 \ge 2$, by the induction hypothesis, in $Q_{n-1}^1 - (F_1 \setminus e_1)$ all edges of $e_1 \cup E_0^1$ lie on a cycle C_1 of every even length l_1 with $2(h-k)(n-1) - 6(h-k-1) \le l_1 \le 2^{n-1}$. So in $Q_n - F$ all edges of E_0 lie on the cycle C^{**} as before of every even length $l = l_0 + l_1$ with $2(h-1)(n-1) - 6(h-3) \le l \le 2^n$. Since $h \ge 4$, then $2(h-1)(n-1) - 6(h-3) \le 2(h-1)n - 6(h-2)$, Theorem 1 follows.

This completes the proof by induction. []

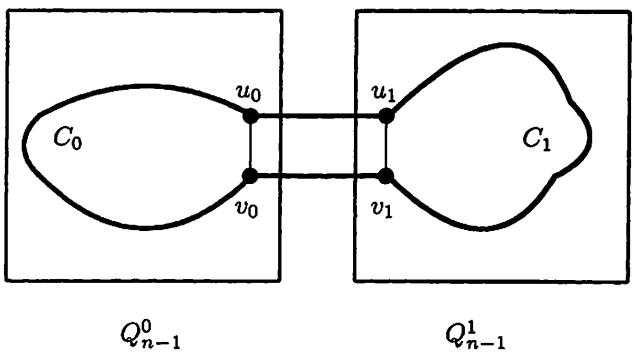

Fig. 1

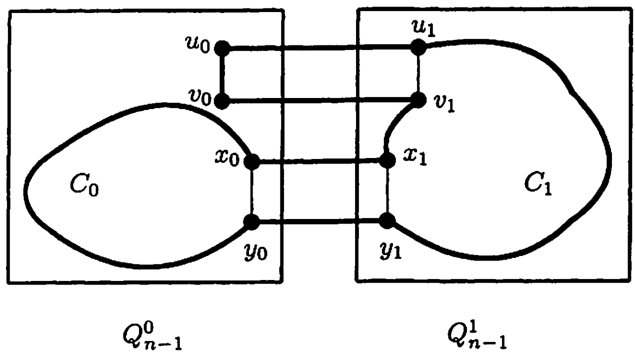

Fig. 2

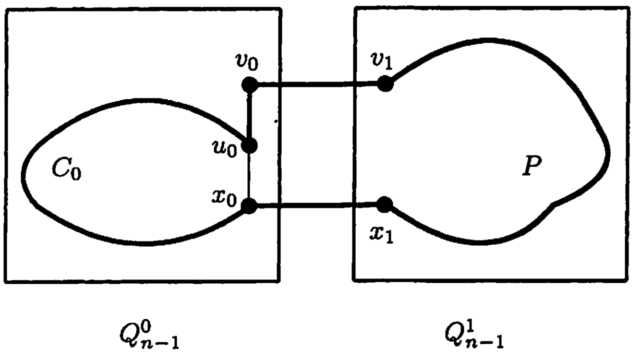

Fig. 3

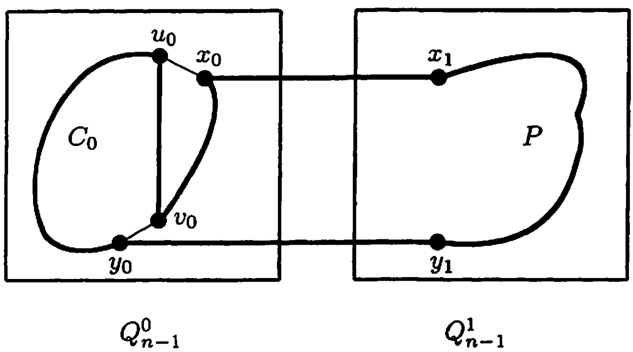

Fig. 4

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