COMPONENT ORDER EDGE CONNECTIVITY FOR GRAPHS OF FIXED SIZE AND ORDER

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ABSTRACT. Given a graph $G := (V, E)$ and an integer $k \geq 2$, the *component order edge connectivity of* G is the smallest size of an edge set D such that the subgraph induced by $G - D$ has all components of order less than k. Let $G(n, m)$ denote the collection of simple graphs G which have n vertices and m edges. In this paper we consider properties of component order edge connectivity for $G(n, m)$. Particularly we prove properties of the maximum and minimum values of the component order edge connectivity for $G(n, m)$ for specific values of n, m and k.

1. INTRODUCTION

Reliability is a significant characteristic of any network. Depending upon the nature of the network being analyzed, reliability can be determined in a number of ways. As is often the case with networks, nodes or edges may be predisposed to failure apropos the structure being represented.

For models where edges are reliable but nodes are prone to failure, one reliability measure that has been extensively studied is vertex connectivity. For a connected graph the *vertex connectivity* is the minimum number of vertices that must be deleted so that the resulting graph is disconnected. A recent generalization of vertex connectivity is the component order connectivity of a graph studied in [1], [4] and [5]. The *component order connectivity* of a graph is the minimum number of vertices that must be deleted so that the resulting graph has all components of order less than a predetermined value k. If we consider a network to be operational, or in an *operating state*, if there exists at least one component of order greater than or equal to k,

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then the component order connectivity of the graph is the minimum number of nodes that must be deleted to reach a *failure state*. This parameter can be used to measure a network's reliability and fault-tolerance.

In many communication networks, however, it is the edges rather than the nodes that are susceptible to failure. For networks in which the edges are prone to failure, a measure of reliability to consider, *edge connectivity*, is the minimum number of edges that must be deleted to disconnect the graph. Similar to vertex connectivity described above, we can generalize the notion of edge connectivity to another measure of network reliability: component order edge connectivity. The *component order edge connectivity* of a graph is the minimum number of edges that must be deleted so that the resulting graph has all components of order less than a predetermined value k. If we consider a network to be operational, or in an operating state, if it contains at least one component of order k or more, then the component order edge connectivity of a graph is the smallest number of edges that must be deleted to reach a failure state. This edge-focused parameter can be used as a measure of a network's reliability and failure-tolerance. For a review, history and applications of the component order edge connectivity of a graph see [2] and [3].

Assume we are to construct a network using n nodes and m edges, and we know the network is operational if there is at least one component of order k or more. Two important questions to consider are: (1) What would be the most reliable network structure we can create under these restrictions? and (2) What would be the least reliable network structure we can create under these restrictions? The answer to these questions can be found by studying the maximum and minimum value of the component order edge connectivity parameter for all graphs for given values of n, m and k.

Much attention has been given to finding similar bounds for vertex connectivity and edge connectivity. In [7], Harary et al. found a lower bound for the vertex connectivity of a graph based on the order and the minimum degree of the graph. In [10], Hellwig et al. found similar bounds for graphs and digraphs with particular restrictions on the degree, minimum, maximum, or degree sequence. Similar results have been found for edge connectivity. See for example [8] and [11].

In this paper, we focus our attention on the collection $G(n, m)$, which consists of *all* graphs with n vertices and m edges. We study the component order edge connectivity of graphs in $G(n, m)$, particularly the minimum and maximum values of the parameter for different values of n, m and k.

2. Background and definitions

Throughout this paper, let $G := (V, E)$ denote a simple graph with vertex or node set V and edge set E. Let $|V|$ and $|E|$ denote the cardinality of V

and E respectively. For any edge set $D \subseteq E$, let $G - D$ denote the subgraph of G containing the vertex set V and the edge set $E - D$.

Definition 2.1. For any graph G, and any integer $k \geq 2$, the *component order edge connectivity* of G, denoted $\lambda_k(G)$, is the minimum $|D|$ where $D \subseteq E$ and $G - D$ has no components of order k or larger.

For any finite collection of graphs $\mathcal{G}$, let

$$\lambda_k^-(\mathcal{G}) := \min\{\lambda_k(G) : G \in \mathcal{G}\}$$

and

$$\lambda_k^+(\mathcal{G}) := \max\{\lambda_k(G) : G \in \mathcal{G}\}.$$

Assume n and m are positive integers such that $m \leq \binom{n}{2}$. Let $G(n,m)$ denote the collection of graphs G where $|V| = n$ and $|E| = m$. Thus $G(n,m)$ consists of all graphs that have n vertices and m edges. For a review of $G(n,m)$ see for example [6].

In order to find the most (and least) reliable network structures as described above (for a network with n vertices and m nodes) we will consider $\lambda_k^-(G(n,m))$ and $\lambda_k^+(G(n,m))$ for different values of n, m, and k.

3. Results

First we consider the parameters λ_k^+ and λ_k^- when $k = 2$. In order to be in a failure state when $k = 2$, we must have no edges remaining in the graph. Hence we can determine the parameter of $\lambda_2(G)$ for any graph G by knowing the size of the edge set of G. This observation proves the following theorem.

Theorem 3.1. *For any $G \in G(n,m)$,*

$$\lambda_2(G) = m$$

and

$$\lambda_2^+(G(n,m)) = \lambda_2^-(G(n,m)) = m.$$

Now we will consider case when $k \geq 3$. The following theorem shows that λ_k^+ and λ_k^- are non-decreasing functions of m. Also if we increase m by 1, then λ_k^+ and λ_k^- can increase by at most 1. In Proposition 2.6 of [3], Boesch et al. proved a similar result for trees.

Theorem 3.2. *For any n and $m < \binom{n}{2}$,*

$$0 \leq \lambda_k^+(G(n,m+1)) - \lambda_k^+(G(n,m)) \leq 1$$

and

$$0 \leq \lambda_k^-(G(n,m+1)) - \lambda_k^-(G(n,m)) \leq 1.$$

Proof. We will show the result for λ_k^+. A similar argument holds for λ_k^- as well.

For all graphs $G_{m+1} \in G(n, m+1)$, there exists a subgraph $G_m \in G(n,m)$ of G_{m+1} and an edge e such that $G_{m+1} = G_m \cup e$. Lemma 3.2 of [3] states if H is a subgraph of G, then $\lambda_k(H) \leq \lambda_k(G)$. Since G_m is a subgraph of G_{m+1}, $\lambda_k(G_m) \leq \lambda_k(G_{m+1})$. This holds for all graphs $G_{m+1} \in G(n, m+1)$. Thus

$$\lambda_k^+(G(n,m)) \leq \lambda_k^+(G(n,m+1)). \tag{3.1}$$

Also because $G_{m+1} = G_m \cup e$ we know $\lambda_k(G_{m+1}) \leq \lambda_k(G_m) + 1$. This holds for all graphs $G_{m+1} \in G(n, m+1)$, thus

$$\lambda_k^+(G(n,m+1)) \leq \lambda_k^+(G(n,m)) + 1. \tag{3.2}$$

From equations 3.1 and 3.2 we obtain our result. □

Iteration of Theorem 3.2 produces the following corollary. It shows if the size of the edge set and $\lambda_k^-(G(n,m))$ increase by the same amount, then the value of $\lambda_k^-(G(n,m))$ can be determined for all intermediary values of m. The same holds for $\lambda_k^+(G(n,m))$.

Corollary 3.3. *Assume a and b are positive integers such that $b < a < \binom{n}{2}$. If*

$$\lambda_k^-(G(n,a)) - \lambda_k^-(G(n,b)) = a - b,$$

then for all $b \leq m \leq a$

$$\lambda_k^-(G(n,m)) = \lambda_k^-(G(n,b)) + m - b.$$

Using Corollary 3.3 we will find $\lambda_k^-(G(n,m))$ for all values of n, m, and $k \geq 3$. We will consider two cases, depending on m. If m is sufficiently small we can construct a graph that will have all components of order less than k. It follows that $\lambda_k^-(G(n,m)) = 0$ for small values of m. However for large values of m we *must* remove edges in order to have only components of order less than k.

A key aspect in determining λ_k^- will be to consider a graph $\tilde{G} \in G(n,m)$ where

$$m = A(n,k) := \left\lfloor \frac{n}{k-1} \right\rfloor \binom{k-1}{2} + \binom{n-(k-1)\left\lfloor \frac{n}{k-1} \right\rfloor}{2},$$

such that all components have order less than k, making $\tilde{G}$ a failure state. Construct $\tilde{G}$ to be the disjoint union of $\left\lfloor \frac{n}{k-1} \right\rfloor$ complete components of order $k-1$ and all other $n-(k-1)\left\lfloor \frac{n}{k-1} \right\rfloor$ vertices (if nonzero) form a completely connected component. Thus if K_v denotes a complete graph on v vertices, then

$$\tilde{G} = \left(\bigcup_{i=1}^{\lfloor \frac{n}{k-1} \rfloor} K_{k-1} \right) \cup K_r,$$

where $r := n - (k-1) \left\lfloor \frac{n}{k-1} \right\rfloor$. By construction $\tilde{G} \in G(n, A(n,k))$ and $\lambda_k(\tilde{G}) = 0$. Obviously if $\lambda_k(\tilde{G}) = 0$, then $\lambda_k^-(G(n, A(n,k)) = 0$.

The following theorem uses this observation, Corollary 3.3 and the value of $\lambda_k(K_n)$ to find λ_k^- for all values of n, m, and $k \geq 3$.

Theorem 3.4. *For any n, m and $k \geq 3$,*

$$\lambda_k^-(G(n,m)) = \begin{cases} 0 & \textit{if} \quad m \leq A(n,k) \\ m - A(n,k) & \textit{otherwise.} \end{cases}$$

Proof. Let $\tilde{G} \in G(n, A(n,k))$ be defined as above. If $m \leq A(n,k)$, there exists a graph $G_m \in G(n,m)$ which is a subgraph of $\tilde{G}$ and therefore G_m has no components of order k or larger. Thus $\lambda_k(G_m) = 0$, which implies $\lambda_k^-(G(n,m)) = 0$.

In [2] the authors showed that

$$\lambda_k(K_n) = \lambda_k^-\left(G\left(n, \binom{n}{2}\right)\right) = \binom{n}{2} - A(n,k).$$

Note that K_n is the only graph in $G(n, \binom{n}{2})$. Hence

$$\begin{aligned} \lambda_k(K_n) &= \lambda_k^-\left(G\left(n, \binom{n}{2}\right)\right) \\ &= \lambda_k^+\left(G\left(n, \binom{n}{2}\right)\right) \\ &= \binom{n}{2} - A(n,k). \end{aligned}$$

Recall $\tilde{G} \in G(n, A(n,k))$ where $\lambda_k(\tilde{G}) = \lambda_k^-(G(n, A(n,k))) = 0$. Comparing graphs K_n and $\tilde{G}$ we can see that

$$\begin{aligned} \lambda_k(K_n) - \lambda_k(\tilde{G}) &= \lambda_k^-\left(G\left(n, \binom{n}{2}\right)\right) - \lambda_k^- G(n, A(n,k)) \\ &= \binom{n}{2} - A(n,k). \end{aligned}$$

By Corollary 3.3, for all $A(n,k) \le m \le \binom{n}{2}$, we have

$$\begin{aligned}\lambda_k^-(G(n,m)) &= \lambda_k^-(G(n,A(n,k))) + m - A(n,k)\\ &= m - A(n,k).\end{aligned}$$

□

Combining Theorems 3.1 and 3.4 we have found $\lambda_k^-(G(n,m))$ for all n, m and k as well as values for $\lambda_k^+(G(n,m))$ for $k = 2$. For λ_k^+ when $k = 3$, we need to remove the minimum number of edges to have a subgraph which has all components of order two or less. This would imply that the resulting components include a set of independent edges and a set of isolated vertices.

A collection of edges E is said to be *independent* if no two edges have a common vertex. An edge set E is said to be a *maximal independent edge set* if there does not exist an independent edge set of size $|E| + 1$. This implies if E is a maximal independent edge set then $|E|$ is the maximum size of all independent edge sets. The size of a maximal independent edge set is called the *independent edge number* of the graph. For a given graph G let $I(G)$ denote the independent edge number of G.

Given a graph G, we can find $\lambda_3^+(G)$ by removing edges until we have a maximal independent edge set remaining. Thus $\lambda_3^+(G) = |E| - I(G)$. In order to generalize this to a collection of graphs, we will fix the value of $I(G)$ and maximize $|E|$ over all graphs with a fixed order n.

In [9], Erdös et al. showed the following theorem which gives bounds for the number of edges, m, a graph can have if it has an independent edge number i.

Theorem 3.5 (Erdös et al. [9]). *If a graph G has n vertices and $I(G) = i$ then*

$$|E| = m \le \max\left[\binom{2i+1}{2},\ i(n-i) + \binom{i}{2}\right],$$

and equality occurs only when the graph is: 1) isolated vertices and K_{2i+1} or 2) $n - i$ vertices each connected to all i vertices in K_i.

The previous result tells us that if we want to construct a graph on n vertices with independent edge number equal to i, then there is an upper bound on the number of additional edges the graph can contain. This theorem will help us find $\lambda_3^+(G(n,m))$ for all values of n and m.

Theorem 3.6. *Fix n, let $0 \le i \le \frac{n}{2}$ be an integer, and define $f_n(i) := \max\left[\binom{2i+1}{2}, i(n-i) + \binom{i}{2}\right]$. Then for any $f_n(i-1) < m \le \min\left[\binom{n}{2}, f_n(i)\right]$*

$$\lambda_3^+(G(n,m)) = m - i.$$

Proof. To prove the result, we will show that for any $0 \le i < \frac{n}{2}$

$$\lambda_3^+(G(n, f_n(i))) = f_n(i) - i \tag{3.3}$$

and for any $0 \le i \le \frac{n}{2} - 1$

$$\lambda_3^+(G(n, f_n(i) + 1)) = f_n(i) - i; \tag{3.4}$$

then we will apply Corollary 3.3.

Proof of equation (3.3):

By Theorem 3.5 there exists a graph $\tilde{G}_i \in G(n, f_n(i))$ such that $I(\tilde{G}_i) = i$. This implies that $\lambda_3(\tilde{G}_i) = f_n(i) - i$ and $\lambda_3^+(G(n, f_n(i))) \ge f_n(i) - i$. Also there are no graphs in $G(n, f_n(i))$ that have an independent edge number less than i. If there was a graph $G \in G(n, f_n(i))$ with edge independence number $i - 1$, then $|E| = m = f_n(i) > f_n(i-1)$ which contradicts Theorem 3.5. Thus $\lambda_3^+(G(n, f_n(i))) \le f_n(i) - i$. Combining these two statements, we see $\lambda_3^+(G(n, f_n(i))) = f_n(i) - i$.

Proof of equation (3.4):

Notice by Theorem 3.5, a graph G with $I(G) = i$ can have at most $f_n(i)$ edges. This implies that every graph $G \in G(n, f_n(i) + 1)$ has independent edge number at least $i + 1$. Therefore $\lambda_3^+(G(n, f_n(i) + 1)) \le f_n(i) + 1 - (i + 1) = f_n(i) - i$. Also by Theorem 3.5 there exists a graph $H \in G(n, f_n(i) + 1)$ with $I(H) = i + 1$. Therefore $\lambda_3^+(H) = f_n(i) - i$, which implies $\lambda_3^+(G(n, f_n(i) + 1)) \ge f_n(i) - i$. Hence we have shown that $\lambda_3^+(G(n, f_n(i) + 1)) = f_n(i) - i$.

Combining equations (3.3) and (3.4) for $1 \le i < \frac{n}{2}$, we see that

$$\begin{aligned}
&\lambda_3^+(G(n, f_n(i))) - \lambda_3^+(G(n, f_n(i-1) + 1)) \\
&\quad = f_n(i) - i - (f_n(i-1) - (i-1)) \\
&\quad = f_n(i) - (f_n(i-1) + 1).
\end{aligned}$$

Hence by Corollary 3.3 we know that for any $1 \le i < \frac{n}{2}$ and $f_n(i-1) < m \le f_n(i)$

$$\begin{aligned}
\lambda_3^+(G(n, m)) &= \lambda_3^+(G(n, f_n(i-1) + 1)) + m - (f_n(i-1) + 1) \\
&= f_n(i-1) - (i-1) + m - (f_n(i-1) + 1) \\
&= m - i.
\end{aligned}$$

In the case where $i = \frac{n}{2}$ is an integer, we have $\binom{n}{2} = \min\left[\binom{n}{2}, f_n(i)\right]$. Again from [2], we know $\lambda_3(K_n) = \lambda_3^+\left(G\left(n, \binom{n}{2}\right)\right) = \binom{n}{2} - A(n, 3)$. By the definition of $A(n,3)$ we have

$$\lambda_3^+\left(G\left(n,\binom{n}{2}\right)\right)=\binom{n}{2}-\frac{n}{2}. \tag{3.5}$$

Combining (3.4) and (3.5), we see that

$$\begin{aligned}\lambda_3^+\left(G\left(n,\binom{n}{2}\right)\right)&-\lambda_3^+\left(G\left(n,f_n\left(\tfrac{n}{2}-1\right)+1\right)\right)\\ &=\left(\binom{n}{2}-\frac{n}{2}\right)-\left(f_n\left(\tfrac{n}{2}-1\right)-\left(\tfrac{n}{2}-1\right)\right)\\ &=\binom{n}{2}-\left(f_n\left(\tfrac{n}{2}-1\right)+1\right).\end{aligned}$$

Thus by Corollary 3.3 we know that for any $f_n\left(\frac{n}{2}-1\right)<m\le\binom{n}{2}$

$$\begin{aligned}\lambda_3^+(G(n,m))&=\lambda_3^+\left(G\left(n,f_n\left(\tfrac{n}{2}-1\right)+1\right)\right)+m-\left(f_n\left(\tfrac{n}{2}-1\right)+1\right)\\ &=f_n\left(\tfrac{n}{2}-1\right)-\left(\tfrac{n}{2}-1\right)+m-\left(f_n\left(\tfrac{n}{2}-1\right)+1\right)\\ &=m-\frac{n}{2}\\ &=m-i.\end{aligned}$$

□

Our previous results are mainly concerned with values of $\lambda_k^-(G(n,m))$ for all values of n, m and k, and $\lambda_k^+(G(n,m))$ for all values of n, m and for $k=2$ or 3. The following theorem explores $\lambda_k^+(G(n,m))$ given a restriction on m with respect to n and with no restrictions on k.

Let $K_{1,\nu-1}$ denote the star graph on ν vertices, so $K_{1,\nu-1}\in G(\nu,\nu-1)$. Let $K_{1,\nu-1}^c$ be a graph consisting of $c+1$ components; one component is isomorphic to $K_{1,\nu-1}$ and the remaining c components are isolated vertices. Thus $K_{1,\nu-1}^c\in G(\nu+c,\nu-1)$. We will use $K_{1,\nu-1}^c$ to find $\lambda_k^+(G(n,m))$ when $m\le n-1$.

Theorem 3.7. *Fix n. For any $2\le k$ and $m\le n-1$*

$$\lambda_k^+(G(n,m))=\begin{cases}0 & \text{if } k\ge m+2\\ m-(k-2) & \text{if } k<m+2\end{cases}$$

Proof. Notice that if a graph has m edges or less, the order of the largest component must be less than $m+2$. This implies that if $k\ge m+2$ then $\lambda_k^+(G(n,m))=0$.

Assume $k<m+2$, so we *may* have a component of order k or larger. By deleting at least $m-(k-2)$ edges, the remaining subgraph will have

size $k-2$ or less. Thus the subgraph must have all components of order $k-1$ or less. This implies that for $m+2>k$,

$$\lambda_k^+(G(n,m)) \leq m-(k-2).$$

It remains to show that $\lambda_k^+(G(n,m)) \geq m-(k-2)$. Theorem 2.2 of [3] states that $\lambda_k(K_{1,\nu-1}) = \nu - k + 1$. By construction we know that $K_{1,\nu-1}^c \in G(\nu+c, \nu-1)$ and $\lambda_k(K_{1,\nu-1}^c) = \nu - k + 1$. Setting $\nu = m+1$ and $c = n-(m+1)$, we have $K_{1,m}^{n-(m+1)} \in G(n,m)$ and $\lambda_k(K_{1,m}^{n-(m+1)}) = m-k+2$. Thus $\lambda_k^+(G(n,m)) \geq m-(k-2)$, which completes the proof. □

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