EXPONENTIAL OPERATOR DECOMPOSITION FOR CARLITZ TYPE GENERATING FUNCTIONS

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Abstract. In this paper, we give the Hahn polynomials represents by Carlitz's q-operators, then show how to deduce Carlitz type generating functions by the technique of exponential operator decomposition.

1. Introduction

One of the customary ways to define the Hermite polynomials is by the relation [14, p. 193]

$$H_n(x) = (-1)^n \exp(x^2) D^n \exp(-x^2), \quad D = \mathrm{d}/\mathrm{d}x, \tag{1.1}$$

which is equivalent to [22, p. 191]

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}, \quad n \in \mathbf{N}, \tag{1.2}$$

where the symbol $[n/2]$ means the largest integer smaller or equal to $n/2$.

Burchnall [4] deduce many important formulas of the classic Hermite polynomials by employing the operational formula

$$(D-2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}(x) D^k. \tag{1.3}$$

For more information about the classic Hermite polynomial and its operational formula, please refer to [1, 4, 9, 14, 15, 17, 20].

The Rogers-Szegö polynomials [7, 24]

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \quad \text{and} \quad g_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^k = h_n(x|q^{-1}), \tag{1.4}$$

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which are in some respects the analogue of the Hermite polynomial (See [8]), are closely related to the continuous q-Hermite polynomials via [23]

$$H_n(\cos\theta|q) = e^{-in\theta}h_n(e^{2i\theta}|q). \tag{1.5}$$

For more information, please refer to [2, 5, 6, 12, 13, 18, 19, 21, 25].

Carlitz gave a clever q-analogue of Burchnall's method by defining the shifted operator $\mathbb{E}$ and it's formula [8, Eq. (4) and p. 522] as

$$\mathbb{E}^n f(x) = f(xq^n) \quad \text{and} \quad (\mathbb{E}_x + x)^n = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^{n-r}\mathbb{E}_x^r. \tag{1.6}$$

In this paper, we give the auxiliary operator of (1.6)

$$\left(\mathbb{E}_x^{-1} + x\right)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^{n-k}\mathbb{E}_x^{-k} \tag{1.7}$$

by the noncommutative q-analogue of binomial theorem (See Lemma 2.1 below).

The reason why we use Carlitz's q-operator and the auxiliary ones are based on the following facts.

$$h_n(x|q) = (\mathbb{E}_x + x)^n\{1\} \quad \text{and} \quad g_n(x|q) = \left(\mathbb{E}_x^{-1} + x\right)^n\{1\}. \tag{1.8}$$

In [8], Carlitz gave a elegant proof of q-Mehler's formula for $h_n(x|q)$ (See Proposition 3.1 below) by relations among operators $\mathbb{E}_x$, $\mathbb{E}_y$ and $\mathbb{E}_t$. There are many others methods, such as the method of transformation theory [10], the combinatorial method [18], the method of parameter augmentation [12, 13], etc.

In fact, we can deduce q-Mehler's formula for Rogers-Szegö polynomials as one purpose by Carlitz's q-operators directly, the thought is decomposition, so the method may be called "exponential operator decomposition". See details in Sections 3.

In this paper, we first represent the Hahn polynomials [3, 11, 25]

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k x^k \quad \text{and} \quad \psi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^k (aq^{1-k};q)_k \tag{1.9}$$

by Carlitz's q-operators as follows.

Theorem 1.1. *For $k \in \mathbb{N}$, we have*

$$\phi_k^{(t)}(x|q) = (t, xt;q)_\infty \left\{ \frac{(\mathbb{E}_x + x)^k}{((\mathbb{E}_x + x)t;q)_\infty} \{1\} \right\}. \tag{1.10}$$

Theorem 1.2. *For $k \in \mathbb{N}$, we have*

$$\psi_k^{(t)}(x|q) = \frac{1}{(tq, xtq;q)_\infty} \left\{ \left(\mathbb{E}_x^{-1} + x\right)^k \left(\left(\mathbb{E}_x^{-1} + x\right)tq;q\right)_\infty \{1\} \right\}. \tag{1.11}$$

Remark 1. *For $t = 0$,* (1.10) *and* (1.11) *reduce to* (1.8).

Another purpose of this paper is to show how to deduce the following Carlitz type generating functions by the method of exponential operator decomposition.

Theorem 1.3. *For $m,n,k\in\mathbb{N}$ and* $\max\{|u|,|xu|,|v|,|yv|,|w|\}<1$*, we have*

$$\sum_{m,n,k=0}^{\infty}\phi_{m+k}^{(a)}(x|q)\phi_{n+k}^{(b)}(y|q)\frac{u^m v^n w^k}{(q;q)_m(q;q)_n(q;q)_k}=\frac{(aux,byv;q)_\infty}{(u,xu,v,yv,w;q)_\infty}\times\sum_{r,s,k=0}^{\infty}\frac{(a;q)_{r+k}(u;q)_{r+k}(b;q)_{s+k}(v;q)_{s+k}(w;q)_k w^{r+s+k}x^{k+r}y^{k+s}}{(q;q)_r(q;q)_s(q;q)_k(axu;q)_{k+r}(byv;q)_{k+s}}.\tag{1.12}$$

Remark 2. *Carlitz* [11, Eq. (3.5)] *gave the result of Theorem 1.3, but it's missing* $x^{k+r}y^{k+s}/(axu;q)_{k+r}(byv;q)_{k+s}$ *on the right of* (1.12). *We can just taking $x=y=0$ for checking.*

Theorem 1.4. *For $m,n,k\in\mathbb{N}$ and* $\max\{|axuq|,|byvq|\}<1$*, we have*

$$\sum_{m,n,k=0}^{\infty}\psi_{m+k}^{(a)}(x|q)\psi_{n+k}^{(b)}(y|q)\frac{(-1)^{m+n+k}q^{\binom{m+1}{2}+\binom{n+1}{2}+\binom{k+1}{2}}u^m v^n w^k}{(q;q)_m(q;q)_n(q;q)_k}$$
$$=\frac{(uq,xuq,vq,yvq,wq;q)_\infty}{(axuq,byvq;q)_\infty}\tag{1.13}$$
$$\times\sum_{k,r,s=0}^{\infty}\frac{(1/u,1/a;q)_{k+r}(1/v,1/b;q)_{k+s}(1/w;q)_k w^{r+s+2k}q^{(1-k)(r+s)+k(2-k)}}{(q;q)_k(q;q)_r(q;q)_s(1/(axu);q)_{k+r}(1/(byv);q)_{k+s}}.$$

Remark 3. *For $u=v=0$, Theorems 1.3 and 1.4 reduce to q-Mehler's formula for Hahn polynomials* [3, Eq. (1.17) and (1.18)] *respectively. For $a=b=0$, Theorems 1.3 and 1.4 reduce to Carlitz type generating functions for Rogers-Szegö polynomials (See Lemma 3.3 and 3.4 below), respectively.*

2. Some lemmas and proof of Theorems 1.1 and 1.2

In this paper, we follow the notations and terminology in [16] and suppose that $0<q<1$. The q-shifted and its compact factorials are defined by

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)$$

and $(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\ldots(a_m;q)_n$, respectively, where $n\in\mathbb{N}=\{0,1,2,\cdots\}$. The operator $\mathbb{E}$ acting on the variable x will be denoted by $\mathbb{E}_x$.

The basic hypergeometric series ${}_r\phi_s$ is given by

$${}_r\phi_s\left[\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}z^n\left[(-1)^nq^{\binom{n}{2}}\right]^{s+1-r},\tag{2.1}$$

for convergence of the infinite series in (2.1), $|q|<1$ and $|z|<\infty$ when $r\le s$, or $|q|<1$ and $|z|<1$ when $r=s+1$, provided that no zeros appear in the denominator.

The q-difference operator D_q and $\theta = \eta^{-1}D_q$, and the q-shifted operator η are defined by

$$D_q\{f(a)\} = \frac{1}{a}(f(a) - f(aq)) \quad \text{and} \quad \eta\{f(a)\} = f(aq),$$

the q-difference operator D_q and θ acting on the variable a will be denoted by D_a and θ_a, respectively.

The q-binomial theorem reads that [16, Eq. (1.3.2)]

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1. \tag{2.2}$$

The noncommutative q-analogue of binomial theorem states that

Lemma 2.1 ([16, p. 28] or [13, Lem. 2.2]). *Let A and B be two noncommutative identerminates satisfying $BA = qAB$, then we have*

$$(A + B)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} A^k B^{n-k}. \tag{2.3}$$

Lemma 2.2. *For $k \in \mathbb{N}$, we have*

$$\frac{1}{\big((\mathbb{E}_x + x)a;q\big)_\infty} \left\{ \frac{x^k(xuwq^k;q)_\infty}{(xu,xw;q)_\infty} \right\} = \frac{(axu;q)_\infty}{(a,ax,xu;q)_\infty} \sum_{r=0}^{\infty} \frac{(uq^k;q)_r(a;q)_{k+r}w^r x^{k+r}}{(q;q)_r(axu;q)_{k+r}}. \tag{2.4}$$

Proof. The left hand side of (2.4) is equal to

$$\begin{aligned}
&\sum_{r,s=0}^{\infty} \frac{(uq^k;q)_r w^r u^s}{(q;q)_r(q;q)_s} \frac{1}{\big((\mathbb{E}_x + x)a;q\big)_\infty} \left\{x^{k+r+s}\right\} \\
&= \sum_{r,s=0}^{\infty} \frac{(uq^k;q)_r w^r u^s}{(q;q)_r(q;q)_s} \sum_{j=0}^{\infty} \frac{a^j}{(q;q)_j} (\mathbb{E}_x + x)^j \left\{x^{k+r+s}\right\} \\
&= \sum_{r,s=0}^{\infty} \frac{(uq^k;q)_r w^r u^s}{(q;q)_r(q;q)_s} \sum_{j=0}^{\infty} \frac{a^j}{(q;q)_j} \sum_{l=0}^{j} \begin{bmatrix} j \\ l \end{bmatrix} x^{j-l} x^{k+r+s} q^{l(k+r+s)} \\
&= \sum_{r,s=0}^{\infty} \frac{(uq^k;q)_r w^r u^s}{(q;q)_r(q;q)_s} \frac{x^{k+r+s}}{(ax, aq^{k+r+s};q)_\infty} \\
&= \frac{1}{(a,ax;q)_\infty} \sum_{r=0}^{\infty} \frac{(uq^k;q)_r(a;q)_{k+r}w^r x^{k+r}}{(q;q)_r} \sum_{s=0}^{\infty} \frac{u^s x^s (aq^{k+r};q)_s}{(q;q)_s},
\end{aligned}$$

which is the right hand side of (2.4) by q-binomial theorem and simplification. The proof is complete. □

Lemma 2.3. *For $k \in \mathbb{N}$, we have*

$$\left(\left(E_x^{-1}+x\right)aq;q\right)_\infty \left\{x^k \frac{(xuq,xwq;q)_\infty}{(xuwq^{1-k};q)_\infty}\right\}$$
$$= \frac{(aq,axq,xuq;q)_\infty}{(axuq;q)_\infty} \sum_{r=0}^{\infty} \frac{(q^k/u;q)_r (1/a;q)_{k+r} w^r q^{(1-k)r}}{(q;q)_r (1/(axu);q)_{k+r} u^k}. \quad (2.5)$$

Proof. The left hand side of (2.5) equals

$$\sum_{r,s=0}^{\infty} \frac{(q^u/u;q)_r (uwq^{1-k})^r u^s (-1)^s q^{\binom{s+1}{2}}}{(q;q)_r (q;q)_s} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} a^j q^j}{(q;q)_j} \left(\mathbb{E}_x^{-1}+x\right)^j \left\{x^{k+r+s}\right\}$$
$$= \sum_{r,s=0}^{\infty} \frac{(q^k/u;q)_r (uwq^{1-k})^r u^s (-1)^s q^{\binom{s+1}{2}}}{(q;q)_r (q;q)_s}$$
$$\times \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} a^j q^j}{(q;q)_j} \sum_{l=0}^{j} \begin{bmatrix} j \\ l \end{bmatrix} q^{l(l-j)} x^{j-l} x^{k+r+s} q^{-l(k+r+s)}$$
$$= \sum_{r,s=0}^{\infty} \frac{(q^k/u;q)_r (uwq^{1-k})^r u^s (-1)^s q^{\binom{s+1}{2}} x^{k+r+s}}{(q;q)_r (q;q)_s} (axq, aq^{1-k-r-s};q)_\infty$$
$$= (aq,axq;q)_\infty \sum_{r=0}^{\infty} \frac{(q^k/u;q)_r (uwq^{1-k})^r x^{k+r}}{(q;q)_r} (-1)^{k+r} q^{-\binom{k+r}{2}} a^{k+r} \left(\frac{1}{a};q\right)_{k+r}$$
$$\times \sum_{s=0}^{\infty} \frac{q^{-s(k+r)} (q^{k+r}/a;q)_s (axuq)^s}{(q;q)_s},$$

which is equivalent to the right hand side of (2.5). The proof is complete. □

Proof of Theorem 1.1. The right hand side of (1.10) is equivalent to

$$(t,xt;q)_\infty \left\{\frac{1}{((\mathbb{E}_x+x)t;q)_\infty}\{h_k(x|q)\}\right\} = (t,xt;q)_\infty \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{1}{((\mathbb{E}_x+x)t;q)_\infty}\left\{x^j\right\}$$
$$= (t,xt;q)_\infty \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \sum_{s=0}^{\infty} \frac{t^s}{(q;q)_s} (\mathbb{E}_x+x)^s \left\{x^j\right\}$$
$$= (t,xt;q)_\infty \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} x^j \sum_{s=0}^{\infty} \frac{t^s}{(q;q)_s} \sum_{l=0}^{s} \begin{bmatrix} s \\ l \end{bmatrix} x^{s-l} q^{lj}$$
$$= (t,xt;q)_\infty \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{x^j}{(xt,tq^j;q)_\infty},$$

which is the right hand side of (1.10) by the definition (1.9). The proof is complete. □

Proof of Theorem 1.2. The right hand side of (1.11) is equal to

$$\frac{1}{(tq,xtq;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}q^{j(j-k)}\Big(\big(\mathbb{E}_x^{-1}+x\big)tq;q\Big)_\infty\{x^j\}$$

$$=\frac{1}{(tq,xtq;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}q^{j(j-k)}\sum_{s=0}^{\infty}\frac{(-1)^s q^{\binom{s+1}{2}}t^s}{(q;q)_s}\sum_{l=0}^{s}\begin{bmatrix}s\\ l\end{bmatrix}q^{l(l-s)}x^{s-l+j}q^{-lj}$$

$$=\frac{1}{(tq,xtq;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}q^{j(j-k)}x^j\big(tq^{1-j},xtq;q\big)_\infty,$$

which is the right hand side of (1.11) by (1.9). The proof is complete. □

3. Carlitz type generating functions for Rogers-Szegö polynomials

Carlitz [7] deduced the following q-Mehler's formulas by using the recurrence relations of Rogers-Szegö polynomials.

Proposition 3.1 ([7, Eq. (3.9)]). *For* $\max\{|t|,|xt|,|yt|,|xyt|\}<1$, *we have*

$$\sum_{n=0}^{\infty}h_n(x|q)h_n(y|q)\frac{t^n}{(q;q)_n}=\frac{(xyt^2;q)_\infty}{(t,xt,yt,xyt;q)_\infty}. \tag{3.1}$$

Proposition 3.2 ([7, Eq. (3.13)]). *For* $|xyt^2/q|<1$, *we have*

$$\sum_{n=0}^{\infty}g_n(x|q)g_n(y|q)\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}=\frac{(t,xt,yt,xyt;q)_\infty}{(xyt^2/q;q)_\infty}. \tag{3.2}$$

In this section, we deduce Propositions 3.1 and 3.2 directly by the method of exponential operator decomposition. Then we continue to deduce the following Propositions.

Proposition 3.3 ([11, Eq. (1.3)]). *For* $\max\{|u|,|v|,|w|,|xu|,|yv|,|xw|,|yw|\}<1$, *we have*

$$\sum_{m,n,k=0}^{\infty}h_{m+k}(x|q)h_{n+k}(y|q)\frac{u^m v^n w^k}{(q;q)_m(q;q)_n(q;q)_k}$$
$$=\frac{(xuw,yvw;q)_\infty}{(u,v,w,xu,yv,xw,yw;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}u,v,w\\ xuw,yvw\end{matrix};q,xyw\right]. \tag{3.3}$$

Proposition 3.4 ([5, Eq. (4.4)]). *For* $\max\{|xuwq|,|yvwq|\}<1$, *we have*

$$\sum_{m,n,k=0}^{\infty}(-1)^{m+n+k}q^{\binom{m+1}{2}+\binom{n+1}{2}+\binom{k+1}{2}}g_{m+k}(x|q)g_{n+k}(y|q)\frac{u^m v^n w^k}{(q;q)_m(q;q)_n(q;q)_k}$$
$$=\frac{(uq,vq,wq,xuq,yvq,xwq,ywq;q)_\infty}{(xuwq,yvwq;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}1/u,1/v,1/w\\ 1/(xuw),1/(yvw)\end{matrix};q,q\right]. \tag{3.4}$$

Remark 4. *The form of the right hand side of* (3.4) *is different from that of* [5, Eq. (4.4)], *but they are equivalent, see details the proof in Section* 5.

Proof of Proposition 3.1. The left hand side of (3.1) equals

$$\sum_{n=0}^{\infty} h_n(y|q)\frac{t^n}{(q;q)_n}(\mathbb{E}_x + x)^n\{1\} = \frac{1}{\big((\mathbb{E}_x + x)(\mathbb{E}_y + y)t; q\big)_\infty}\{1\}$$

$$= \frac{1}{((\mathbb{E}_x + x)t, (\mathbb{E}_x + x)yt; q)_\infty}\{1\}$$

$$= \frac{1}{((\mathbb{E}_x + x)t; q)_\infty}\left\{\sum_{k=0}^{\infty}\frac{(yt)^k}{(q;q)_k}(\mathbb{E}_x + x)^k\{1\}\right\}$$

$$= \frac{1}{((\mathbb{E}_x + x)t; q)_\infty}\left\{\frac{1}{(yt, xyt; q)_\infty}\right\}$$

$$= \frac{1}{(yt;q)_\infty}\frac{1}{((\mathbb{E}_x + x)t; q)_\infty}\left\{\frac{1}{(xyt;q)_\infty}\right\}.$$

Taking $(a, k, u, w) = (t, 0, yt, 0)$ in Lemma 2.2, we achieve the proof of Proposition 3.1. □

Proof of Proposition 3.2. The left hand side of (3.2) is equal to

$$\Big((\mathbb{E}_x^{-1} + x)(\mathbb{E}_y^{-1} + y)t; q\Big)_\infty\{1\}$$

$$= \Big((\mathbb{E}_x^{-1} + x)t, (\mathbb{E}_x^{-1} + x)yt; q\Big)_\infty\{1\}$$

$$= \Big((\mathbb{E}_x^{-1} + x)t; q\Big)_\infty\{(yt, xyt; q)_\infty\}$$

$$= (yt;q)_\infty\Big((\mathbb{E}_x^{-1} + x)t; q\Big)_\infty\{(xyt;q)_\infty\}.$$

Letting $(a, k, u, w) = (t/q, 0, yt/q, 0)$ in Lemma 2.3, we conclude the proof of Proposition 3.2. □

Proof of Proposition 3.3. The left hand side of (3.3) is equal to

$$\frac{1}{\big((\mathbb{E}_x + x)u, (\mathbb{E}_y + y)v, (\mathbb{E}_x + x)(\mathbb{E}_y + y)w; q\big)_\infty}\{1\}$$

$$= \frac{1}{\big((\mathbb{E}_x + x)u, (\mathbb{E}_y + y)v; q\big)_\infty}\left\{\sum_{k=0}^{\infty}\frac{(w;q)_k w^k}{(q;q)_\infty}\frac{x^k}{(xw;q)_\infty}\frac{y^k}{(yw;q)_\infty}\right\}$$

$$= \sum_{k=0}^{\infty}\frac{(w;q)_k w^k}{(q;q)_\infty}\frac{1}{\big((\mathbb{E}_x + x)u; q\big)_\infty}\left\{\frac{x^k}{(xw;q)_\infty}\right\}\frac{1}{\big((\mathbb{E}_y + y)v; q\big)_\infty}\left\{\frac{y^k}{(yw;q)_\infty}\right\}$$

$$= \frac{(xuw, yvw; q)_\infty}{(u, xu, xw, v, yv, yw; q)_\infty} \sum_{k=0}^{\infty} \frac{(w; q)_k w^k}{(w; q)_\infty (q; q)_k} \frac{(u; q)_k x^k}{(xuw; q)_k} \frac{(v; q)_k y^k}{(yvw; q)_k},$$

which is the right hand side of (3.3). This completes the proof. □

Proof of Proposition 3.4. The left hand side of (3.4) is equivalent to

$$\left(\left(\mathbb{E}_x^{-1} + x\right)uq, \left(\mathbb{E}_y^{-1} + y\right)vq, \left(\mathbb{E}_x^{-1} + x\right)\left(\mathbb{E}_y^{-1} + x\right)wq; q\right)_\infty \{1\}$$

$$= \sum_{k=0}^{\infty} \frac{(wq; q)_\infty (1/w; q)_k (w^2 q)^k}{(q; q)_k} \left(\left(\mathbb{E}_x^{-1} + x\right)uq; q\right)_\infty \left\{(xwq; q)_\infty x^k\right\}$$

$$\times \left(\left(\mathbb{E}_y^{-1} + y\right)vq; q\right)_\infty \left\{(ywq; q)_\infty y^k\right\},$$

which is the right hand side of (3.4) after simplification. This leads to the proof. □

4. Proof of Theorems 1.3 and 1.4

In this section, we deduce the Carlitz type generating functions directly.

Proof of Theorem 1.3. The left hand side of (1.12) is equivalent to

$$\sum_{m,n,k=0}^{\infty} (a, ax, b, by; q)_\infty \frac{(\mathbb{E}_x + x)^{m+k} (\mathbb{E}_y + y)^{n+k}}{\left((\mathbb{E}_x + x)a, (\mathbb{E}_y + y)b; q\right)_\infty} \frac{u^m v^n w^k}{(q; q)_m (q; q)_n (q; q)_k} \{1\}$$

$$= (a, ax, b, by; q)_\infty \frac{1}{\left((\mathbb{E}_x + x)a, (\mathbb{E}_y + y)b; q\right)_\infty}$$

$$\times \frac{1}{\left((\mathbb{E}_x + x)u, (\mathbb{E}_x + x)v, (\mathbb{E}_x + x)(\mathbb{E}_x + x)w; q\right)_\infty} \{1\}$$

$$= (a, ax, b, by; q)_\infty \frac{1}{\left((\mathbb{E}_x + x)a, (\mathbb{E}_y + y)b; q\right)_\infty}$$

$$\times \left\{ \frac{(xuw, yvw; q)_\infty}{(u, xu, v, yv, w, xw, yw; q)_\infty} \sum_{k=0}^{\infty} \frac{(u, v, w; q)_k}{(q, xuw, yvw; q)_k} (xyw)^k \right\}$$

$$= \frac{(a, ax, b, by; q)_\infty}{(u, v, w; q)_\infty} \sum_{k=0}^{\infty} \frac{(u, v, w; q)_k w^k}{(q; q)_k} \frac{1}{\left((\mathbb{E}_x + x)a; q\right)_\infty} \left\{ \frac{x^k (xuwq^k; q)_\infty}{(xu, xw; q)_\infty} \right\}$$

$$\times \frac{1}{\left((\mathbb{E}_x + x)a; q\right)_\infty} \left\{ \frac{y^k (yvwq^k; q)_\infty}{(yv, yw; q)_\infty} \right\}$$

$$= \frac{(axu, byv; q)_\infty}{(u, v, w, xu, yv; q)_\infty} \sum_{k,r,s=0}^{\infty} \frac{(u, v, w; q)_k w^k}{(q; q)_k} \frac{(uq^k; q)_r (a; q)_{k+r} w^r x^{k+r}}{(q; q)_r (axu; q)_{k+r}}$$

$$\times \frac{(vq^k;q)_s(b;q)_{k+s}w^sy^{k+s}}{(q;q)_s(byv;q)_{k+s}},$$

which equals the right hand side of (1.12). The proof is complete. □

Proof of Theorem 1.4. The left hand side of (1.13) is equal to

$$\frac{1}{(aq,axq,bq,byq;q)_\infty}\sum_{m,n,k=0}^{\infty}\left(\mathbb{E}_x^{-1}+x\right)^{m+k}\left(\mathbb{E}_y^{-1}+y\right)^{n+k}\left(\left(\mathbb{E}_x^{-1}+x\right)aq;q\right)_\infty$$

$$\times\left(\left(\mathbb{E}_y^{-1}+y\right)bq;q\right)_\infty\frac{(-1)^{m+n+k}q^{\binom{m+1}{2}+\binom{n+1}{2}+\binom{k+1}{2}}u^mv^nw^k}{(q;q)_m(q;q)_n(q;q)_k}\{1\}$$

$$=\frac{1}{(aq,axq,bq,byq;q)_\infty}\left(\left(\mathbb{E}_x^{-1}+x\right)aq,\left(\mathbb{E}_y^{-1}+y\right)bq;q\right)_\infty$$

$$\times\left(\left(\mathbb{E}_x^{-1}+x\right)uq,\left(\mathbb{E}_y^{-1}+y\right)vq,\left(\mathbb{E}_x^{-1}+x\right)\left(\mathbb{E}_y^{-1}+y\right)wq;q\right)_\infty\{1\}$$

$$=\frac{(uq,vq,wq;q)_\infty}{(aq,axq,bq,byq;q)_\infty}\sum_{k=0}^{\infty}\frac{(1/u,1/v,1/w;q)_kq^{k(2-k)}(uvw^2)^k}{(q;q)_k}$$

$$\times\left(\left(\mathbb{E}_x^{-1}+x\right)aq,\left(\mathbb{E}_y^{-1}+y\right)bq;q\right)_\infty\left\{\frac{(xuq,xwq;q)_\infty x^k}{(xuwq^{1-k};q)_\infty}\cdot\frac{(yvq,ywq;q)_\infty y^k}{(yvwq^{1-k};q)_\infty}\right\}$$

$$=\frac{(uq,xuq,vq,yvq,wq;q)_\infty}{(axuq,byvq;q)_\infty}\sum_{k=0}^{\infty}\frac{(1/u,1/v,1/w;q)_kq^{k(2-k)}(uvw^2)^k}{(q;q)_k}$$

$$\times\sum_{r,s=0}^{\infty}\frac{(q^k/u;q)_r(1/a;q)_{k+r}(q^k/v;q)_s(1/b;q)_{k+s}w^{r+s}q^{(1-k)(r+s)}}{(q;q)_r(q;q)_s(1/(axu);q)_{k+r}(1/(byv);q)_{k+s}},$$

which is equivalent to the right hand side of (1.13). This completes the proof. □

5. Additional proof of $_3\phi_2$ transformation

Zhang [26] utilized the method of parameter augmentation [12, 13] to deduce the following q-exponential operator identity .

Proposition 5.1 ([26, Theorem 2.5]). *For* $\max\{|av|,|adsw/q|\}<1$, *we have*

$$\mathbb{E}(d\theta_a)\left\{\frac{(at,as,aw;q)_\infty}{(av;q)_\infty}\right\}=\frac{(at,as,aw,ds,dw;q)_\infty}{(av,adsw/q;q)_\infty}{}_3\phi_2\left[\begin{matrix}t/v,q/(as),q/(aw)\\q/(av),q^2/(adsw)\end{matrix};q,q\right],\tag{5.1}$$

where $\mathbb{E}(b\theta_a)$ *defined by*

$$\mathbb{E}(b\theta_a)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(q;q)_n}(b\theta_a)^n.\tag{5.2}$$

The author [5] used parameter augmentation to deduce many generating functions of Carlitz, one of which is

Proposition 5.2 ([5, Eq. (4.4)]). *For* $\max\{|xuwq|, |xyvwq|\} < 1$, *we have*

$$\sum_{m,n,k=0}^{\infty} (-1)^{m+n+k} q^{\binom{m+1}{2}+\binom{n+1}{2}+\binom{k+1}{2}} g_{m+k}(x|q) g_{n+k}(y|q) \frac{u^m v^n w^k}{(q;q)_m (q;q)_n (q;q)_k}$$

$$= \frac{(uq, vq, wq, xuq, yvq, xwq, xywq; q)_\infty}{(xuwq, xyvwq; q)_\infty} {}_3\phi_2\left[\begin{matrix} 1/v, 1/(xu), 1/(xw) \\ 1/(xuw), 1/(xyvw) \end{matrix}; q, q\right]. \quad (5.3)$$

In this section, we will prove the equivalence of (3.4) and (5.3).

Proof. The left hand side of (5.1) is symmetric in s and w, and so is the right hand side. Interchanging s and w we obtain the following result.

$$\frac{(ds;q)_\infty}{(adsw/q;q)_\infty} {}_3\phi_2\left[\begin{matrix} t/v, q/(as), q/(aw) \\ q/(av), q^2/(adsw) \end{matrix}; q, q\right]$$

$$= \frac{(dt;q)_\infty}{(adtw;q)_\infty} {}_3\phi_2\left[\begin{matrix} s/v, q/(at), q/(aw) \\ q/(av), q^2/(adtw) \end{matrix}; q, q\right]. \quad (5.4)$$

Taking $(a, d, s, t, v, w) = (uv/(yw), v, xywq/v, ywq/v, xyw^2q/v, ywq/u)$ in (5.4), we obtain

$$\frac{(xywq)_\infty}{(xyvwq)_\infty} {}_3\phi_2\left[\begin{matrix} 1/v, 1/(xu), 1/(xw) \\ 1/(xuw), 1/(xyvw) \end{matrix}; q, q\right] = \frac{(ywq)_\infty}{(yvwq)_\infty} {}_3\phi_2\left[\begin{matrix} 1/u, 1/v, 1/w \\ 1/(xuw), 1/(yvw) \end{matrix}; q, q\right],$$

substituting this equation to the right hand side of (5.3) gives (3.4). The proof is complete. □

References

[1] W. A. Al–Salam, *Operational representations for the Laguerre and other polynomials*, Duke Math. J. **31**(1964), 127–142.

[2] W. A. Al–Salam and M. E. H. Ismail, *q-beta integrals and the q-hermite polynomials*, Pacific J. Math. **135**(1988), 209–221.

[3] W. A. Al–Salam and L. Carlitz, *Some orthogonal q-polynomials*, Math. Nachr. **30**(1965), 47–61.

[4] J. L. Burchnall, *A note on the polynomials of Hermite*, Quart. J. Math. Oxford Ser. **12**(1941), 9–11.

[5] J. Cao, *New proofs of generating functions for Rogers-Szegö polynomials*, Appl. Math. Comp. **207**(2009), 486–492.

[6] J. Cao, *New proofs of generating functions for Rogers-Szegö polynomials II*, submitted.

[7] L. Carlitz, *Some polynomials related to theta functions*, Annali di Matematica Pura ed Applicata **41**(1956), 359–373.

[8] L. Carlitz, *Some polynomials related to theta functions*, Duke Math. Jour. **24**(1957), 521–527.

[9] L. Carlitz, *A note on the Laguerre polynomials*, Michigan Math. Jour. 7(1960), 219–223.

[10] L. Carlitz, *A q-identity*, Monatsh. für. Math. **67**(1963), 305–310.

[11] L. Carlitz, *Generating functions for certain q-orthogonal polynomials*, Collectanea Math. **23**(1972), 91–104.

[12] W. Y.C. Chen, Z.-G. Liu, *Parameter augmentation for basic hypergeometric series*, II, J. Combin. Theory Ser. A **80**(1997), 175–195.

[13] W. Y.C. Chen, Z.-G. Liu, *Parameter augmentation for basic hypergeometric series*, I, in: B. E. Sagan, R. P. Stanley (Eds.), Mathematical Essays in honor of Gian-Carlo Rota, Birkäuser, Basel, 1998, pp. 111–129.

[14] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, II, McGraw-Hill, New York (1953).
[15] G. Dattoli, A. Torre and M. Carpanese, *Operational rules and arbitrary order Hermite generating functions*, J. Math. Anal. Appl. **227**(1998), 98–111.
[16] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge Univ. Press, Cambridge, MA, 1990.
[17] H. W. Gould and A. T. Hopper, *Operational formulas connected with two generalizations of Hermite polynomials*, Duke Math. J. **29**(1962), 51–64.
[18] M. E. H. Ismail, D. Stanton and G.Viennot, *The combinatorics of q-Hermite polynomials and the Askey-Wilson integral*, Europ. J. Combinatorics **8**(1987), 379–392.
[19] Z.-G. Liu, *q-Hermite polynomials and a q-Beta integral*, Northeast Math. J. **13**(1997), 361–366.
[20] Z.-G. Liu, *Some identiites of differential operator and Hermite polynomials*, J. Math. Res. Exp. **18**(1998), 412–416 (in Chinese).
[21] Z.-G. Liu, *Some operator identities and q-series transformation formulas*, Discrete Math. **265**(2003), 119–139.
[22] E. D. Rainville, *Special functions*. New York, Macmillan, 1960.
[23] L. J. Rogers, *On the expansion of some infinite products*, Proc. London Math. Soc. **24**(1893), 337–352.
[24] G. Szegö, *Ein Beitrag zur Theorie der Thetafunktionen*, Sitz. Preuss. Akad. Wiss., Phys. Math. Klasse **19**(1926), 242–252.
[25] H. M. Srivastava and V. K. Jain, *Some multilinear generating functions for q-hermite polynomials*, J. Math. Anal. Appl. **144**(1989), 147–157.
[26] Z.Z. Zhang, J. Wang, *Two operator identities and their applications to terminating basic hypergeometric series and q-integrals*, J. Math. Anal. Appl. **312**(2005), 653–665.

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