Universal Graphs for Two Graph Properties

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ABSTRACT. The graphs we consider are all countable. A graph U is universal in a given set $\mathcal{P}$ of graphs if every graph in $\mathcal{P}$ is an induced subgraph of U and $U \in \mathcal{P}$. In this paper we show the existence of a universal graph in the set of all countable graphs with block order bounded by a fixed positive integer. We also investigate some classes of interval graphs and work towards finding universal graphs for them. The sets of graphs we consider are all examples of induced-hereditary graph properties.

1 Introduction

All graphs considered are simple, undirected, unlabelled, and have countable vertex sets. The symbol $\mathcal{I}$ denotes the set of all such graphs. A *(graph) property* is a class of graphs, closed under isomorphisms.

We follow the notation of [3] on graphs in general. The order of a graph is the number (or cardinality) of its vertex set and a *block of a graph* is a maximal non-separable subgraph, i.e., a maximal non-trivial connected subgraph containing no cut-vertices. We further follow [1] on graph properties in particular. Hence a property $\mathcal{P}$ is called an *(induced-)hereditary graph property* if, whenever $G \in \mathcal{P}$ and H is an (induced) subgraph of G, then $H \in \mathcal{P}$ too.

If $\mathcal{P}$ is a set of graphs, then (following [6]) we define a graph U to be a *universal graph for* $\mathcal{P}$ if every graph in $\mathcal{P}$ is an induced subgraph of U; it is a *universal graph in* $\mathcal{P}$ if $U \in \mathcal{P}$ too. The Rado graph [12] is the best known example of a universal graph, it is universal in the set of all countable graphs.

Universal graphs do not exist in every property – in [2] it is shown that, in fact, the overwhelming majority of induced-hereditary properties do not *contain* universal graphs. Concrete examples of specific induced-hereditary properties of this kind can be found for example in [4], [5], [8] and [10].

Let c be any fixed positive integer. Then we define the property $\mathcal{B}_c$ of graphs by

$$\mathcal{B}_c := \{G \in \mathcal{I} \mid \text{every block of } G \text{ has order at most } c\}$$

and will refer to it as *the property of graphs with block order bounded by c*. It is easy to see that $\mathcal{B}_c$ is an induced-hereditary property of graphs.

Hajós introduced (finite) interval graphs in [9] and they were characterised by Lekkerkerker and Boland in [11]. This concept finds interesting applications in, amongst others, biology and computer science [7]. For our purposes, the *interval graph* $G_{\mathcal{D}}$ of a countable set $\mathcal{D}$ of intervals on the real line $\mathbf{R}$ is the *intersection graph* of $\mathcal{D}$, i.e., the vertex set of $G_{\mathcal{D}}$ is $\mathcal{D}$ while two vertices I and J of $G_{\mathcal{D}}$ are adjacent if and only if $I \cap J \neq \emptyset$. It is (again) easy to see that the set

$$\mathcal{I}nt := \{G_{\mathcal{D}} \mid \mathcal{D} \text{ is a countable set of intervals}\}$$

is an induced-hereditary property of graphs.

In this paper we show that, for each positive integer c, there exists a universal graph in the set $\mathcal{B}_c$. We also investigate some subclasses of $\mathcal{I}nt$ and work towards finding universal graphs for them.

2 Graphs with bounded block order

In order to prove that there is a universal graph U in the property $\mathcal{B}_c$ of countable graphs with block order bounded by c, we need the following construction: Let G be a graph with a prescribed vertex v and let $\mathcal{H}$ be any set of graphs with X a set of prescribed vertices, one from each graph $H \in \mathcal{H}$, and suppose that all these graphs are pairwise vertex disjoint. Then the graph formed from all these graphs by identifying v with every vertex from X will be called the *fused graph* of the given situation, and the process will be called *fusing*.

Theorem 1. *For every positive integer c there is a universal graph U in the property $\mathcal{B}_c$ of countable graphs with block order bounded by c.*

Proof. The result is clear if $c = 1$ since the graph with $\aleph_0$ vertices and no edges satisfies the requirements of U in this case. Hence we assume henceforth that $c \geq 2$.

We construct the required graph U in a recursive manner using repeated fusing and then prove that it has the required properties, i.e., we then prove that $G \leq U$ for every countable graph G with block order bounded by c and that $U \in \mathcal{B}_c$.

Let, for an integer $b \in \{2, 3, \ldots, c\}$, $\mathcal{H}_b$ denote the set of all the different, i.e., pairwise non-isomorphic, connected graphs of order b: We assume that the vertices of each graph in each $\mathcal{H}_b$ are labelled using a single symbol with subscripts taken from $\{1, 2, \ldots, b\}$. (By this we mean that the vertices could be $v_1, v_2, \ldots, v_b$ or $w_1, w_2, \ldots, w_b$ etc.) Using this convention to name the vertices, we can consider the graphs in $\mathcal{H}_b$ to be *labelled* *graphs*, i.e., two graphs in $\mathcal{H}_b$ on the vertex sets $X := \{x_1, x_2, \ldots, x_b\}$ and $Y := \{y_1, y_2, \ldots, y_b\}$ are considered to be the same if and only if the bijection $f : X \to Y$ defined by $f(x_i) = y_i$ for each i is an isomorphism. We are now ready to construct a sequence of graphs, $U_1, U_2, \ldots$ of which the limit will be taken to form the required graph U.

Let U_1 be the graph consisting of a single vertex u. Let U_2 be the graph obtained from U_1 by fusing u with the vertex with subscript 1 (e.g. x_1, y_1 etc.). from each of denumerably many copies of every graph H taken from the finite set $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \ldots \cup \mathcal{H}_c$ of graphs. Note that U_2 is a countable graph.

For $i = 2, 3, \ldots$ we now construct the graph U_{i+1} from U_i in a similar fashion: Fuse every vertex of U_i with the vertex with subscript 1 from each of denumerably many copies of every graph H taken from the finite set $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \ldots \cup \mathcal{H}_c$ of graphs.

Finally, let U be the graph obtained as the limit of this process, i.e.,

$$V(U) = \bigcup_{i \geq 1} V(U_i) \text{ and } E(U) = \bigcup_{i \geq 1} E(U_i).$$

Again, it follows that U is countable since the above recursive construction has denumerably many steps, each involving a denumerable graph.

Note that U is vertex-transitive since every vertex of U can play the role of the vertex u we started with. Furthermore, every vertex of U is a vertex of denumerably many copies of every conceivable non-separable graph of order at most c.

Now let G be any countable graph such that every block of G has at most c vertices. We have to show that G is isomorphic to some induced subgraph of U. In order to do so, we are going to build, through a recursive process, an isomorphism β from G to a suitable induced subgraph of U.

Note first that, since G is countable, it has countably many (connected) components; suppose $G_1, G_2, \ldots$ are the components of G. Then we first choose any vertex from each component G_i; suppose we choose $v_i \in V(G_i)$. The initial step of building β is then to map the v_i's to vertices of U which are adjacent to u but are in different blocks of U containing u; this is possible since $c \geq 2$.

The next step in this recursive process is now to remark that β can map all the blocks of each G_i containing v_i to suitable blocks of U: such blocks, and enough of them, exist by the recursive construction of U.

This process can of course be repeated to determine, for each i and each vertex $v \in V(G_i)$ of which $\beta(v)$ has been determined, an image in $V(U)$ under β for each vertex in all the blocks of G_i containing v for which the image under β has not been determined before – there are enough blocks fused to every vertex in the construction of U to make these choices possible.

This recursive process, which allocates images in U to vertices of G in a block-by-block fashion, is clearly sufficient to prove that G is indeed isomorphic to an induced subgraph of U.

Finally, it is immediately clear that $U \in \mathcal{B}_c$, i.e., each block of U has order at most c, since where the fusion process creates new blocks, each such block is of order at most c. □

3 Interval graphs

In the sequel, a real number a will be called an *endpoint* of an interval I, if I is any interval of any of the forms $[a, b]$, $(a, b]$, $[a, b)$, (a, b), $[b, a]$, $(b, a]$, $[b, a)$ or (b, a). We will now show that many of the countable interval graphs $G_{\mathcal{D}}$ are induced subgraphs of the interval graph $G_{\mathcal{E}}$ determined by the set $\mathcal{E}$ of all closed and bounded intervals with rational endpoints. We start with a lemma.

Lemma 1. $G_{\mathcal{E}} \in \mathcal{I}nt$.

Proof. Let $\mathbb{Q}$ denote the set of rational numbers. Then, for any two rational numbers a and b with $a \leq b$, there are at most four intervals with endpoints a and b. Therefore, since $\mathbb{Q}$ is countable, the set $\mathcal{E}$ is countable and hence $G_{\mathcal{E}} \in \mathcal{I}nt$. □

Theorem 2. *Let $\mathcal{D}$ be a countable set of bounded intervals such that for every $a \in \mathbf{R}$ there exists an $\epsilon > 0$ in $\mathbf{R}$ such that there is only a finite number of endpoints of intervals of $\mathcal{D}$ in $[a, a+\epsilon]$. Then $G_{\mathcal{D}}$ is an induced subgraph of $G_{\mathcal{E}}$.*

Proof. Let $\mathcal{D}$ be any fixed countable set of bounded intervals satisfying the given condition on every $a \in \mathbf{R}$. Note that endpoints of the intervals in $\mathcal{D}$ can be rational or irrational numbers and that these intervals are allowed to be closed, half-closed as well as open intervals. Then we show that $G_{\mathcal{D}}$ is isomorphic to an induced subgraph of $G_{\mathcal{E}}$ by replacing, where applicable, the intervals in $\mathcal{D}$ by closed intervals with rational endpoints while preserving the structure of the graph $G_{\mathcal{D}}$.

Hence suppose that a is an endpoint of some interval in $\mathcal{D}$ and suppose that a is irrational or some interval is open at a. If no interval in $\mathcal{D}$ has an endpoint greater that a, let b be any real number such that $b > a$. Otherwise, let b be the least real number which is an endpoint of an interval in $\mathcal{D}$ such that $b > a$. Since there exists an $\epsilon > 0$ in $\mathbf{R}$ such that there is only a finite number of endpoints of intervals from $\mathcal{D}$ in $[a, a+\epsilon]$, there is such a real number. Clearly there exist three rational numbers a_1, a_2, a_3 such that $a < a_1 < a_2 < a_3 < b$.

Now we create a new set of intervals $\mathcal{D}'$ by changing some endpoints of intervals with a as an endpoint. Indeed, for each interval I of $\mathcal{D}$ of positive length with a as an endpoint, we replace

- the right endpoint of I, if it is right-open in a, by a_1 and we make such an interval right-closed in a_1,
- the left endpoint of I, if it is left-open in a, by a_3 and we make such an interval left-closed in a_3, and
- any endpoint a of I for which I is closed in a, by a_2 with the new interval also closed in a_2.

Also, each interval in $\mathcal{D}$ which consists of a single point a will be replaced by the single-point interval containing only a_2.

It is not difficult to check that the intersection graph $G_{\mathcal{D}'}$ of the resulting set $\mathcal{D}'$ of intervals is isomorphic to the given graph $G_{\mathcal{D}}$: By denoting arbitrary intervals in $G_{\mathcal{D}}$ by I and J, and the resulting intervals in $G_{\mathcal{D}'}$ by I' and J', the equivalence $I \cap J \neq \emptyset$ if and only if $I' \cap J' \neq \emptyset$ can be seen to be true.

Clearly, since $G_{\mathcal{D}'}$ is the intersection graph of a set of closed and bounded intervals with rational endpoints, $G_{\mathcal{D}'}$, and hence $G_{\mathcal{D}}$, is an induced subgraph of $G_{\mathcal{E}}$. □

The condition on a in Theorem 2 is not met by the set $\mathcal{E}$ of all closed and bounded intervals with rational endpoints. Hence this theorem does not guarantee the existence of a universal graph which is one of the graphs about which the theorems speaks. This condition, however, is met when any finite set $\mathcal{D}$ of bounded intervals is considered; hence $G_{\mathcal{E}}$ is a denumerable universal graph for the set of all finite interval graphs with real endpoints.

4 Conclusion

The question whether there exists a countable universal graph for the set of all countable interval graphs clearly has a positive answer: the Rado graph [12], being universal in the set of all countable graphs, is an example of such a graph. To ask whether there is a countable universal graph **in** the set of all countable interval graphs remains an open problem.

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