Characterizations and Structure of Sequential Graphs

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Abstract A sequential labeling of a simple graph G (non-tree) with m edges is an injective labeling f such that the vertex labels $f(x)$ are from $\{0,1,\ldots,m-1\}$ and the edge labels induced by $f(x)+f(y)$ for each edge xy are distinct consecutive positive integers. A graph is sequential if it has a sequential labeling. We give some properties of sequential labeling and the criterion to verify sequential labeling. Necessary and sufficient conditions are obtained for every case of sequential graphs. A complete characterization of non-tree sequential graphs is obtained by vertex closure. Also, characterizations of sequential trees are given. The structure of sequential graphs is revealed.

Keywords *sequential graph; sequential matrix; edge label matrix; vertex closure*

1 INTRODUCTION

Sequential graphs relate directly to additive bases problems stemming from error-correcting code [1]. Chang, Hsu, Rogers [2] and Grace [3] have investigated the sequential graphs. The study of sequential graphs has been focusing on special classes of graphs (See [4]). The systematic theory of sequential graphs has not been founded up to the present [5]. In this paper, we will give some properties and characterizations of sequential graphs.

Throughout this article, all our graphs are simple, non-empty ([6]) and finite. $V(G)$ and $E(G)$ are the vertex set and the edge set of a graph G, respectively. The order of graph G is denoted by $n(G)$ and the size of G is denoted by $\varepsilon(G)$. $[x]$ is the greatest integer $\leq x$. The undefined symbols and terminologies from graph theory can be found in [6] and [7].

Definition 1 ([3]). A *sequential labeling* of a simple graph G with ε edges is an injection $f: V(G) \to \{0,1,\ldots,\varepsilon-1\}$ (ε is also allowed if G is a tree) such that

$$\{f(u)+f(v)|uv \in E(G)\} = \{c, c+1,\ldots,c+\varepsilon-1\}$$

where c is a positive integer. *A graph is sequential if it has a sequential labeling.*

The bijection $f': E(G) \to \{c, c+1, \ldots, c+\varepsilon-1\}, f'(uv) = f(u)+f(v), uv \in E(G)$, is called the *induced edge labeling* of G by f. $f(u)+f(v)$ is the *induced label* of edge uv.

Definition 2. A *base matrix* E_{ij} $(i, j \in \{1, 2, \ldots, n\})$ is an $n \times n$ matrix in which both of the entries a_{ij}, a_{ji} are 1 and the other entries are all zero.

An $n \times n$ symmetric matrix A is called a *sequential matrix* if there are two positive integers d and m $(d+m \leq 2n-1)$ such that

$$A = \sum_{t=1}^{m} E_{i_t j_t},\ i_t < j_t,\ i_t + j_t = d+t,\ t = 1, 2, \ldots, m,$$

where each $E_{i_t j_t}$ is a base matrix.

Note that a sequential matrix can be decomposed into m base matrices in which the sums $i_t + j_t$ are distinct consecutive positive integers. Obviously, given a sequential matrix the decomposition consisting of base matrices is unique.

On the other hand, to construct a sequential matrix we must present the choices of (i_t, j_t) 's, furthermore, given t and d, the base matrix $E_{i_t j_t}$ which satisfies $i_t + j_t = d+t$ can be selected in $[\frac{d+t-1}{2}]$ ways, if $2 < d+t \leq n+1$.

For example, $A_{4\times 4} = E_{12} + E_{13} + E_{23}$ is a sequential matrix. Replacing E_{23} with E_{14} we also obtain a sequential matrix. If $A(G)$ is the adjacency matrix of a simple graph G and $V(G) = \{v_1, v_2, \ldots, v_n\}$, then

$$A(G) = \sum_{v_i v_j \in E(G), i<j} E_{ij}.$$

2 PROPERTIES OF SEQUENTIAL LABELING

In the following sections we use the symbol $\langle n \rangle$ to denote the set $\{0, 1, \ldots, n\}$, where n is a nonnegative integer. c denotes the minimum edge label induced by a vertex labeling f of graph G. $A(G)$ is always the adjacency matrix of a simple graph G, and let a_{ij} denote the entries of $A(G)$.

2.1 Sequential Labeling of Non-Tree Graphs

Theorem 1. *If $f: V(G) \to \langle \varepsilon - 1 \rangle$ is an injection, $m = \max\{f(u) | u \in V(G)\}$, let $g(u) = (m-f)(u) = m - f(u)$, whenever $u \in V(G)$, then we have:*

(i) *f is a sequential labeling of G if and only if $g = m - f$ is a sequential labeling of G.*

(ii) *$m = f(u_m), u_m \in V(G)$ if, and only if, $g(u_m) = (m-f)(u_m) = 0$.*

Proof. (i) For the vertex labeling, it is easy to see that $f: V(G) \to \langle \varepsilon - 1 \rangle$ is an injection if and only if $g: V(G) \to \langle \varepsilon - 1 \rangle$ is an injection.

On the other hand, if f is a sequential labeling of G, then the edge labels $f'(uv) = f(u) + f(v), uv \in E(G)$, and $g'(uv) = g(u) + g(v) = 2m - [f(u) + f(v)]$, thus, the range of f' is $\{c, c+1, \ldots, c+\varepsilon-1\}$ if and only if the range of g' is

$$\{2m - c, 2m - (c+1), \ldots, 2m - (c+\varepsilon-1)\}.$$

It is easy to check that $2m - (c+\varepsilon-1) > 0$. So g is a sequential labeling of G.

Using the same method, we can prove that if g is a sequential labeling of G, then so is f. Conclusion (ii) is clear. □

Given two vectors $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$, let $\alpha \cdot * \beta = (a_1b_1, a_2b_2, \ldots, a_nb_n)$, $\alpha \cdot * \beta$ is called the *product vector* of α and β. The product vector of column vectors is defined similarly.

Suppose that G is a simple graph with vertices $v_1, v_2, \ldots, v_n$, and an injection f assigns the vertex labels $f(v_1), f(v_2), \ldots, f(v_n), v_i \in V(G)$. Let $A(G)$ denote the adjacency matrix of G corresponding to the vertex ordering $v_1, v_2, \ldots, v_n$, write $L = (f(v_1), f(v_2), \ldots, f(v_n))$, and α_i and β_j denote the ith row vector and the jth column vector of $A(G)$, respectively. Using product vectors $\alpha_i \cdot * L$ and $\beta_j \cdot * L^T$, we define a matrix B_L as follows:

$$B_r = \begin{pmatrix} \alpha_1 \cdot * L \\ \alpha_2 \cdot * L \\ \vdots \\ \alpha_n \cdot * L \end{pmatrix}, B_c = (\beta_1 \cdot * L^T, \beta_2 \cdot * L^T, \ldots, \beta_n \cdot * L^T), B_L = B_r + B_c. \quad (1)$$

B_L is called the *edge label matrix* of G corresponding to the vertex labels vector L. The induced edge labels of G by f can be obtained from B_L. The concept of edge label matrix can be used to give a description of sequential labeling.

Theorem 2. *If G is a non-tree simple graph with ε edges and $f : V(G) \to \langle \varepsilon - 1 \rangle$ is an injection, then we have the conclusions:*

(i) *The edge label matrix B_L induced by f and $A(G)$ is a symmetric matrix.*

(ii) *The entry b_{ij} of B_L has the expression $b_{ij} = a_{ij}[f(v_i) + f(v_j)], 1 \le i, j \le n$.*

(iii) *f is a sequential labeling of G if, and only if, the entries of B_L satisfy*

$$\{b_{ij} \in B_L | i < j, a_{ij} = 1\} = \{c, c+1, \ldots, c+\varepsilon-1\},$$

where c is a positive integer.

Proof. Let $A(G)(= (a_{ij}))$ be the adjacency matrix of G corresponding to the vertex ordering $v_1, v_2, \ldots, v_n$, the ith row vector of B_r and the jth column vector of B_c are respectively

$$\alpha_i \cdot * L = (a_{i1} f(v_1), \ldots, a_{ij} f(v_j), \ldots, a_{in} f(v_n)),$$

$$\beta_j \cdot *L^T = (a_{1j}f(v_1),\ldots,a_{ij}f(v_i),\ldots,a_{nj}f(v_n))^T.$$

By the method constructing B_L, we obtain

$$b_{ij} = a_{ij}f(v_j) + a_{ij}f(v_i) = a_{ij}[f(v_j) + f(v_i)] \tag{2}$$

Since $A(G)$ is an adjacency matrix, we have $a_{ij} = a_{ji}$, hence $b_{ij} = b_{ji}$ by (2). Parts (i) and (ii) are completed.

Finally, since $v_iv_j \in E(G) \Leftrightarrow a_{ij} = 1$, applying formula (2) we obtain the expression of entry b_{ij} :

$$b_{ij} = \begin{cases} f(v_j) + f(v_i), & a_{ij} = 1, \\ 0, & a_{ij} = 0. \end{cases} \tag{3}$$

By (3), it is clear that every edge of G corresponds to a unique positive integer in

$$\{b_{ij} \in B_L | i < j, a_{ij} = 1\}.$$

If f is a sequential labeling of G, let $c, c+1, \ldots, c+\varepsilon-1$ be the induced edge labels by f, then

$$\{c, c+1, \ldots, c+\varepsilon-1\} = \{f(v_i) + f(v_j) | i < j, a_{ij} = 1\} = \{b_{ij} \in B_L | i < j, a_{ij} = 1\}.$$

Conversely, now assume that the edge label matrix B_L induced by the injection f satisfies $\{b_{ij} \in B_L | i < j, a_{ij} = 1\} = \{c, c+1, \ldots, c+\varepsilon-1\}, c > 0$. By (3) again, we have the following equations

$$\{f(v_i) + f(v_j) | v_iv_j \in E(G)\} = \{b_{ij} \in B_L | i < j, a_{ij} = 1\} = \{c, c+1, \ldots, c+\varepsilon-1\}.$$

Thus f is a sequential labeling of G. □

2.2 Sequential Labeling of Tree

Theorem 3. *Given a tree T with ε edges, and $f : V(T) \to \langle\varepsilon\rangle$ is an injection, let $g(u) = (\varepsilon - f)(u) = \varepsilon - f(u)$, whenever $u \in V(T)$, then*

(i) *f is a sequential labeling of T if and only if g $(= \varepsilon - f)$ is a sequential labeling of T.*

(ii) *$\varepsilon = f(u_\varepsilon)$ if, and only if, $g(u_\varepsilon) = (\varepsilon - f)(u_\varepsilon) = 0, u_\varepsilon \in V(T)$.*

Proof. Note that the injection $f : V(T) \to \langle\varepsilon\rangle$ is also a bijection. Therefore, $\varepsilon = \max\{f(u) | u \in V(T)\}$. Using the method of proving Theorem 1, we can obtain Theorem 3. □

The following corollary is the immediate result of Theorem 1 and 3. It can be used to decrease the cases which must be discussed to confirm a non-sequential graph.

Corollary 4. *If graph G has a sequential labeling, then G has a sequential labeling which uses vertex label* 0.

Theorem 5. *Given a tree T with ε edges, and $f: V(T) \longrightarrow \langle\varepsilon\rangle$ is an injection, then we have*

(i) *The edge label matrix B_L induced by f and $A(T)(=(a_{ij}))$ is symmetric.*

(ii) *The entry b_{ij} of B_L has the expression $b_{ij} = a_{ij}[f(v_i)+f(v_j)]$.*

(iii) *f is a sequential labeling of T if and only if the entries of B_L satisfy*

$$\{b_{ij} \in B_L | i < j, a_{ij} = 1\} = \{c, c+1, \ldots, c+\varepsilon-1\},$$

where c is a positive integer.

Proof. The proof of Theorem 5 is almost the same as that of Theorem 2. The only difference between them is that the vertex labeling of tree can use ε. □

3 SEQUENTIAL GRAPHS WITH ORDER EQUAL TO SIZE

The next theorem shows a strong connection between sequential graph and sequential matrix. Let $A(v_1, v_2, \ldots, v_n)$ denote the adjacency matrix corresponding to the vertex ordering $v_1, v_2, \ldots, v_n$ of a graph G.

Theorem 6. *Let G be a simple graph with $n(G) = \varepsilon(G)$, then G is a sequential graph if and only if there is a suitable vertex ordering $v_1, v_2, \ldots, v_n$ of G such that the adjacency matrix $A(v_1, v_2, \ldots, v_n)$ is a sequential matrix.*

Proof. Necessity. Given a simple graph G with $n(G) = \varepsilon(G)$, if G is a sequential graph, then there is a sequential labeling $f: V(G) \longrightarrow \langle\varepsilon-1\rangle$ such that the induced edge labels are $c, c+1, \ldots, c+\varepsilon-1$. We rename the vertices in G such that $f(v_1) < f(v_2) < \ldots < f(v_n)$. A vertex ordering $v_1, v_2, \ldots, v_n$ of G is obtained.

Because $n = \varepsilon$, f is a bijection, hence the vertex label vector

$$(f(v_1), f(v_2), \ldots, f(v_n)) = (0, 1, \ldots, n-1), \text{ namely, } f(v_i) = i-1.$$

For convenience, let $A_G(a_{ij})$ denote the adjacency matrix $A(v_1, \ldots, v_n)$ that result from the vertex ordering $v_1, v_2, \ldots, v_n$. Let $B_L(b_{ij})$ be the edge labels matrix corresponding to A_G and the vertex labels $f(v_i) = i-1, i = 1, 2, \ldots, n$.

Because c is the minimum edge label, the entry a_{ij} must be zero whenever the subscripts i, j satisfy $i+j-2 < c$. Otherwise $a_{ij} = 1$, by Theorem 2 (ii), we could obtain an induced edge label $l < c$.

Since $c+\varepsilon-1$ is the maximum edge label, by the similar reason, we also obtain $a_{ij} = 0$ whenever the subscripts satisfy $i+j-2 > c+\varepsilon-1$. Hence, the subscripts i, j of ε pairs $(a_{ij}, a_{ji}) = (1, 1)$ in A_G satisfy $c \leq i+j-2 \leq c+\varepsilon-1$.

By Theorem 2 (iii), it is clear that there is exactly one pair of entries $(a_{ij}, a_{ji}) = (1, 1)$ in A_G satisfying $i+j-2 = l$, for every induced edge label l.

Let $l_t = (c-1)+t, t = 1,2,\ldots,\varepsilon$, the unique pair $(a_{i_t j_t}, a_{j_t i_t}) = (1,1)$ satisfying $i_t + j_t - 2 = l_t$ determines a unique base matrix $E_{i_t j_t}$. This means that

$$A_G = \sum_{t=1}^{\varepsilon} E_{i_t j_t}, i_t < j_t, i_t + j_t = (c+1)+t,\ t = 1,2,\ldots,\varepsilon.$$

So A_G is a sequential matrix.

Sufficiency. We now assume that there is a vertex ordering $v_1, v_2, \ldots, v_n$ of G such that the adjacency matrix $A_G (= A(v_1, \ldots, v_n))$ is a sequential matrix. Let us define the injection f as follows:

$$f(v_i) = i-1,\ i = 1,2,\ldots,n \tag{4}$$

By the definition of sequential matrix, we have

$$A_G = E_{i_1 j_1} + \cdots + E_{i_t j_t} + \cdots + E_{i_\varepsilon j_\varepsilon},\ i_t < j_t,\ i_t + j_t = d+t,\ t = 1,2,\ldots,\varepsilon.$$

and $i_t + j_t > 2$, it follows that $d \geq 2$.

Because $E_{i_t j_t}$ determines the unique pair $(a_{i_t j_t}, a_{j_t i_t}) = (1,1)$, we have exactly ε pairs of nonzero entries of A_G satisfying

$$(a_{i_t j_t}, a_{j_t i_t}) = (1,1),\ i_t < j_t,\ i_t + j_t = d+t,\ t = 1,2,\ldots,\varepsilon.$$

By Theorem 2 (ii), for every t we have

$$b_{i_t j_t} = a_{i_t j_t}[f(v_{i_t}) + f(v_{j_t})] = (i_t - 1) + (j_t - 1) = d + t - 2,\ i_t < j_t.$$

It follows that

$$\{b_{ij} \in B_L | i < j, a_{ij} = 1\} = \{d + t - 2 | t = 1,2,\ldots,\varepsilon\}.$$

By Theorem 2(iii), G is a sequential graph. □

Corollary 7. *Let G be a simple graph with $n(G) = \varepsilon(G)$, then G is a sequential graph if and only if G has a sequential adjacency matrix $A(G)$.*

Proof. It is immediate from Theorem 6. □

4 SEQUENTIAL GRAPHS WITH ORDER GREATER THAN SIZE

For sequential trees we have the following results.

Theorem 8. *A tree T is a sequential graph if and only if T has a sequential adjacency matrix $A(T)$.*

Proof. The proof of Theorem 8 is almost the same as that of Theorem 6. To prove the necessity, note that the sequential labeling $f : V(T) \to \{0, 1, \ldots, \varepsilon\}$ is a bijection as well. Conversely, if $A(T)$ is a sequential matrix, we can get a vertex ordering $v_1, v_2, \ldots, v_n$ by $A(T)$. Define the vertex labeling of T

$$f(v_i) = i - 1,\ t = 1, 2, \ldots, n.$$

We can see that f is sequential. □

Theorem 9. *If G is not a tree and $n(G) > \varepsilon(G)$, then G is not a sequential graph.*

Proof. If graph G is not a tree and $n(G) > \varepsilon(G)$, then every function $f : V(G) \to \{0, 1, \ldots, \varepsilon - 1\}$ cannot be an injection, thus it is impossible that G was a sequential graph. □

5 SEQUENTIAL GRAPHS WITH ORDER LESS THAN SIZE

Definition 3. Given a graph G with $n(G) \leq \varepsilon(G)$, the *vertex closure* of G, written G°, is a graph obtained by appending $\varepsilon - n$ isolated vertices to G.

It is trivial that $n(G^\circ) = \varepsilon(G^\circ)$, and if $n(G) = \varepsilon(G)$, then $G^\circ = G$.

Theorem 10. *If G is a simple graph with $n(G) \leq \varepsilon(G)$, then G is a sequential graph if and only if the vertex closure G° is a sequential graph.*

Proof. For $n(G) = \varepsilon(G)$, the assertion follows from the fact that $G^\circ = G$. Now assume that $n(G) < \varepsilon(G)$, let $f : V(G) \to \langle \varepsilon - 1 \rangle$ be a sequential labeling of G, we shall construct a sequential labeling of G° by f. Using the numbers in $\langle \varepsilon - 1 \rangle - f(V(G))$, we assign different labels to $\varepsilon - n$ newly appended vertices in G°; for the remaining vertices in G° we assign the same labels just as f does in G, then we obtain a vertex labeling g of G°. By Theorem 2 (iii) g is sequential, because

$$\{b_{ij}^g \in B_L^g | i < j, a_{ij}(G^\circ) = 1\} = \{b_{ij}^f \in B_L^f | i < j, a_{ij}(G) = 1\}.$$

Conversely, let g be a sequential labeling of G°, then the restriction of g to $V(G)$, written $g|_{V(G)}$, is a sequential labeling of G. □

Combining Theorem 10 and Corollary 7, we can get the next result.

Theorem 11. *Let G be a simple graph with $n(G) \leq \varepsilon(G)$, then G is a sequential graph if and only if the vertex closure G° has a sequential adjacency matrix $A(G^\circ)$.*

Lemma 12. *If G is a simple graph with $n(G) \leq \varepsilon(G)$, then G has a sequential labeling whose vertex label values are distinct consecutive integers if and only if G has a sequential labeling with vertex labels* $0, 1, 2, \ldots, n-1$.

Proof. The sufficiency is obvious. Conversely, let f be a sequential labeling of G with the vertex labels $a, a+1, \ldots, a+n-1$, by Theorem 1, $g = m - f$ is a sequential labeling of G as well, where $m = a+n-1$. Furthermore, the values of g are $n-1, n-2, \ldots, 1, 0$. □

The following theorems give the characterizations of a sequential graph which has a sequential labeling with distinct consecutive vertex labels.

Theorem 13. *If G is a simple graph with $n(G) \leq \varepsilon(G)$, then G has a sequential labeling with vertex labels* $0, 1, \ldots, n-1$ *if and only if G has a sequential adjacency matrix* $A(G)$.

Proof. In the proof of Theorem 6, we replace the bijection $f : V(G) \to \langle \varepsilon - 1 \rangle$ with the restriction bijection $f : V(G) \to \langle n-1 \rangle$. Using the same argument we can prove Theorem 13. □

Corollary 14. *If G is a simple graph with $n(G) \leq \varepsilon(G)$, then G has a sequential labeling whose values are $a, a+1, \ldots, a+n-1$ if and only if G has a sequential adjacency matrix $A(G)$.*

Proof. Combine Lemma 12 and Theorem 13. □

6 STRUCTURE OF SEQUENTIAL GRAPHS

The next result describes the structure of sequential graphs, it implies that sequential graph cannot have "parallel" edges under some ordering of its vertices.

Theorem 15. *Given a simple graph G with $n(G) \leq \varepsilon(G)$, then G is sequential if and only if there is a vertex ordering $v_1, v_2, \ldots, v_\varepsilon$ of G°, and there exists a positive integer d such that the following conditions hold:*

(i) *For every edge $v_i v_j$ of G°, we have always $v_{i+m} v_{j-m} \notin E(G^\circ)$ whenever integer $m \neq 0, j-i$.*

(ii) *For each vertex pair $\{v_i, v_j\}$ of G°, we have always $v_i v_j \notin E(G^\circ)$ whenever $i+j \leq d$ or $i+j > d+\varepsilon$.*

Proof. Sufficiency. If there are a vertex ordering $v_1, v_2, \ldots, v_\varepsilon$ of G° and a positive integer d such that conditions (i) and (ii) hold, let $A(G^\circ) = (a_{ij})$ denote the adjacency matrix of G° corresponding to $v_1, v_2, \ldots, v_\varepsilon$.

It is easy to prove that $A(G^\circ)$ is sequential, thus G is a sequential graph.

Necessity. Let G be a sequential graph. By Theorem 10, G° is also sequential. Thus G° has a sequential adjacency matrix $A(G^\circ)$. $A(G^\circ)$ determines a vertex ordering $v_1, v_2, \ldots, v_\varepsilon$ of G° and also determines a positive integer d. The ordering $v_1, v_2, \ldots, v_\varepsilon$ and d keep the conclusion (i) and (ii) appearing. □

A result similar to Theorem 15 holds.

Theorem 16. *A tree T is sequential if and only if there is a vertex ordering $v_1, v_2, \ldots, v_{\varepsilon+1}$ of $V(T)$ and there exists a positive integer d such that the following conditions hold:*

(i) *For every edge $v_i v_j$ of T, we have $v_{i+k} v_{j-k} \notin E(T)$, whenever integer $k \neq 0, j-i$;*

(ii) *For each vertex pair $\{v_i, v_j\}$ of T, we have always $v_i v_j \notin E(T)$ whenever $i+j \leq d$ or $i+j > d+\varepsilon$.*

7 CRITERION FOR SEQUENTIAL GRAPH

Definition 4. Given an n-by-n matrix $A = (a_{ij})$, the *kth sub-diagonal* of A is the vector $D_k(A) = (a_{ij} | i+j = k)$, $(k = 1, 2, \ldots, 2n-1)$, namely

$$D_k(A) = (a_{1,k}, a_{2,k-1}, \ldots, a_{k,1}).$$

A 1-*sub-diagonal* of the adjacency matrix A is a sub-diagonal $D_k(A)$ in which there is exactly one pair of entries (a_{ij}, a_{ji}) with $a_{ij} = a_{ji} = 1$ and the other entries are all zero. An adjacency matrix A is called *n-continuous* if there are n 1-sub-diagonals $D_l(A), D_{l+1}(A), D_{l+2}(A), \ldots, D_{l+n-1}(A)$ of A, for some l. Obviously, an adjacency matrix of a graph with ε edges can be at most ε-continuous.

The following results depict a sequential graph by the concept of n-continuous.

Theorem 17. *Given a non-tree simple graph G, then G is sequential if and only if $n(G) \leq \varepsilon(G)$ and the vertex closure G° has an ε-continuous adjacency matrix.*

Proof. If a non-tree simple graph G is sequential, then G must satisfy $n(G) \leq \varepsilon(G)$ by Theorem 9. And the condition that there is an ε-continuous adjacency matrix $A(G^\circ)$ is equivalent to that the vertex closure G° has a sequential adjacency matrix; Applying Theorem 11 we obtain the desired conclusion. □

Similarly we have the following result.

Theorem 18. *A tree T is sequential if and only if there is an ε-continuous adjacency matrix $A(T)$.*

8 CONCLUDING REMARK

The sequential graph is defined in two cases, non-tree graphs or trees, respectively, by Grace. In fact, the characterizations of non-tree sequential graphs have been presented by Theorem 11, 15 and 17, because a non-tree graph G with $n(G) > \varepsilon(G)$ is not a sequential graph (Theorem 9). And special non-tree sequential graphs are depicted by Theorem 13 and 14.

The characterizations of sequential trees are described by Theorem 8, 16 and 18. Both non-tree sequential graphs and sequential trees are strongly connected with ε-continuous adjacency matrices. These characterizations can be used to structure sequential graphs and to verify any sequential graph.

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