Decycling Bipartite Tournaments by Deleting Arcs

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Abstract

The *decycling index* of a digraph D is defined to be the minimum number of arcs in a set whose removal from D leaves an acyclic digraph. In this paper, we obtain some results on the decycling index of bipartite tournaments. **keywords :** cycle, digraph, tournament

1 Introduction

One of the nicest theorems in graph theory is the formula for the cycle rank of a graph - the dimension of the cycle space in terms of the numbers of vertices, edges, and components. It is a simple formula for the minimum number of edges that must be removed from a graph to render it cycle-free. In this paper, we investigate the corresponding parameter for directed graphs, for which, unlike the undirected case, there is no simple formula.

Formally, the *decycling index* of a digraph D, denoted $\nabla'(D)$, is the minimum number of arcs in a set whose removal from D leaves a digraph with no directed cycles.

The notation we use is a natural extension of $\nabla(G)$ for the (vertex) decycling number of a graph G [3]. In the study of systems of equations, this parameter has also been called the "arc-feedback number," the name coming from the fact that feedback in sound systems results from the existence of cycles.

To illustrate the concept, we use the two digraphs in Figure 1 (these examples will be considered again later). Clearly, since D_1 is strongly connected, its decycling index is positive, and, since the arc qr is on all cycles, $\nabla'(D_1) = 1$. Similarly, it can be seen that $\nabla'(D_2) = 3$ since D_2 has three arc-disjoint cycles (for example, $wyvw$, $vxzv$, and $wxyzw$) and the deletion of three arcs (such as, yv, zv, and zw) leaves an acyclic digraph.

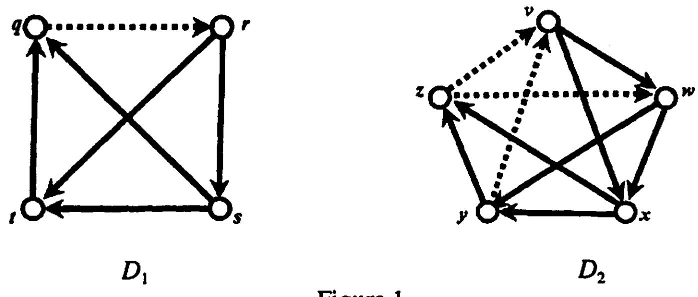


Figure 1

Since every acyclic digraph has at least one vertex of in-degree 0 and at least one of out-degree 0, we have the following elementary result will be useful to us later. (The in-degree and out-degree of a vertex v are denoted id v and od v.)

Lemma 1.1. *The decycling index of a digraph D is*

$$\nabla'(D) = \min_{v \in V}\{\text{id } v + \nabla'(D - v)\} = \min_{v \in V}\{\text{od } v + \nabla'(D - v)\}. \ \square$$

This lemma gives another proof that the decycling index of digraph D_2 in Figure 1 is 3. Note that D_2 is vertex-symmetric with all in-degrees 2, and for each vertex v, D - v is isomorphic to D_1.

In light of Lemma 1.1, it will be useful to have, for a vertex v in a digraph D, notation for id v + $\nabla'(D - v)$. If we denote it $\varphi(v)$, then $\nabla'(D) = \min_{v \in V} \varphi(v)$.

We also observe that the decycling index of a digraph is the sum of the decycling indices of its strong components, so we restrict our attention to strongly connected digraphs.

The decycling index was studied for tournaments in the guise of its complementary parameter, the maximum number of arcs in a cycle-free subdigraph [5]. Stated in terms of decycling, it has been shown that the *maximum arc decycling number* among all tournaments of order n (denoted $\overline{\nabla}'(n)$) is given in Table 1 (the tournaments in Figure 1 illustrate the entries for $n = 4$ and 5).

n	2	3	4	5	6	7	8	9	10	11	12	13
$\overline{\nabla}'(n)$	0	1	1	3	4	7	8	12	15	20	22	28

Table 1

Beyond this, little is known except for bounds.

In this paper we focus on bipartite tournaments, which are to complete bipartite graphs as ordinary tournaments are to complete graphs: An m-by-n *bipartite tournament* is the result of orienting the edges of an m-by-n complete bipartite graph.

We shall use the following notation: The two partite sets of vertices of an m-by-n bipartite tournament T will be denoted by $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$.

To be consistent with other usage (for example, [4]), we denote by $\overline{\nabla}'(G)$ the maximum decycling index among all orientations of a graph G; however, for simplicity we will write $\overline{\nabla}'(m, n)$ instead of $\overline{\nabla}'(K_{m,n})$, is consistent with the notation of Table 1.

As with ordinary tournaments, general results on the decycling index of bipartite tournaments are hard to come by. We have exact values only in the 2-by-n, 3-by-n, 4-by-n, their inverses, and the 5-by-5 case.

The following recursive bound will be useful in our work.

Lemma 1.2. *For all positive integers m, s,and t,*

$$\overline{\nabla}'(m, s+t) \geq \overline{\nabla}'(m, s) + \overline{\nabla}'(m, t).$$

Proof. Let T_1 and T_2 be m-by-s and m-by-t bipartite tournaments that realize the maximum decycling index for these sizes. Then the bipartite tournament obtained by identifying their partite sets of cardinality m cannot be decycled without the removal of at least $\overline{\nabla}'(m, s) + \overline{\nabla}'(m, t)$ arcs. ❑

2 The Cases 2-by-*n* and 3-by-*n*.

In this section, we determine $\nabla'(D)$ when D is a strong m-by-n bipartite tournament for $m = 2, 3$. (Recall our convention that in these cases, one partite set will be $X = \{x_1, x_2\}$ or $X = \{x_1, x_2, x_3\}$ and the other $Y = \{y_1, y_2, \dots, y_n\}$.)

Theorem 2.1. *The decycling index of a strongly connected 2-by-n bipartite tournament T is* $\nabla'(T) = \min\{\text{id}\, x_1, \text{id}\, x_2\}$.

Proof. Let T be a strong 2-by-n bipartite tournament. Since T is strong, every vertex in Y must have both in-degree and out-degree equal to 1. Consequently, od x_1 = id x_2, so there are id x_1 vertices y_i for which $x_2 \to y_i \to x_1$ and id x_2 vertices y_j for which $x_1 \to y_j \to x_2$. Thus, there are $\min\{\text{id}\, x_1, \text{id}\, x_2\}$ arc-disjoint 4-cycles $x_1y_jx_2y_ix_1$. Hence, $\nabla'(T) \geq \min\{\text{id}\, x_1, \text{id}\, x_2\}$. Since the result of deleting all of the arcs either into or out of x_1 is acyclic, we have $\nabla'(T) \leq \min\{\text{id } x_1, \text{id } x_2\}$, and the result follows. ❑

The strong 2-by-n bipartite tournament in which the in-degrees of x_1 and x_2 differ by at most 1 yields the following result.

Corollary 2.2. $\overline{\nabla}'(2,n) = \left\lfloor \frac{n}{2} \right\rfloor$. ❑

In order to treat the 3-by-n case efficiently, we introduce some additional notation and terminology. Note that in a strong 3-by-n bipartite tournament, each vertex y in Y will either have in-degree 1 and out-degree 2 or *vice versa*. We call the arc at y that has the opposite direction to the other two *the exceptional arc at* y. This gives a natural partition of Y into six subsets (some of which may be empty) according to the X-vertices of the exceptional arcs. Formally, for i = 1,2,3, we let Y_i^+ (resp. Y_i^-) be the set of vertices y in Y for which the exceptional arc is directed from (resp., directed to) the vertex x_i. This decomposition is shown in Figure 2, where the heavy lines represent the sets of exceptional arcs.

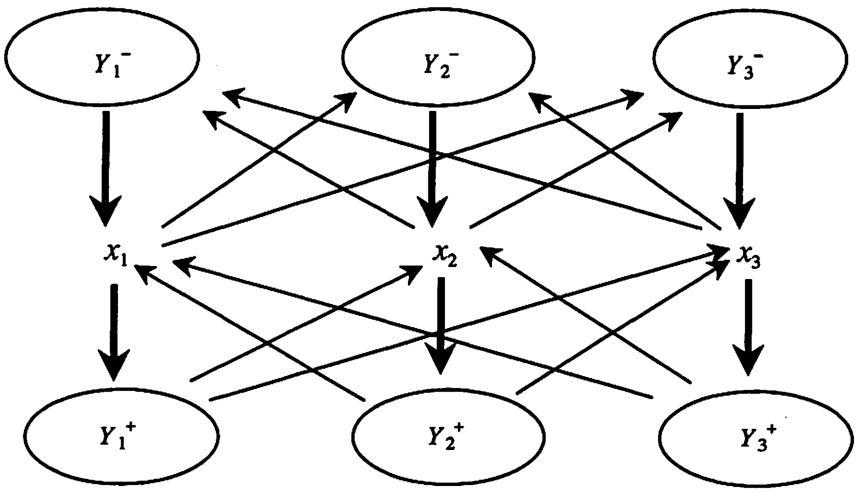


Figure 2

Further, for i = 1,2,3, we let $\alpha_i^{\pm} = |Y_i^{\pm}|$, and, for $i \neq j$, we define the six *exceptionality numbers* $\beta_{ij} = \alpha_i^+ + \alpha_j^-$. For example, in the 3-by-5 bipartite tournament T indicated in Figure 3 (only the X-to-Y arcs are shown), we see that the exceptional arcs are x_1y_1, x_2y_3, x_3y_5, y_2x_2, y_4x_3. Hence, $\alpha_1^- = 0$ while all other α_i^+ and α_j^- are 1, and so $\beta_{2,1} = \beta_{3,1} = 1$ and $\beta_{1,2} = \beta_{1,3} = \beta_{2,3} = \beta_{3,2} = 2$.

We observe that deleting the three arcs x_2y_3, x_3y_5, and y_2x_2 leaves an acyclic digraph. On the other hand, we can find two arc-disjoint cycles, namely, $x_2y_4x_3y_2x_2$, and $x_1y_1x_2y_3x_1$. Notice that no matter which arc we remove from the second 4-cycle, the first 4-cycle and either $x_2y_3x_3y_5x_2$ or $x_1y_1x_3y_5x_1$ remains intact, and so $\nabla'(T) = 3$. Since $n = 5$ and $\max \beta_{ij} = 2$, this is an illustration of the following theorem.

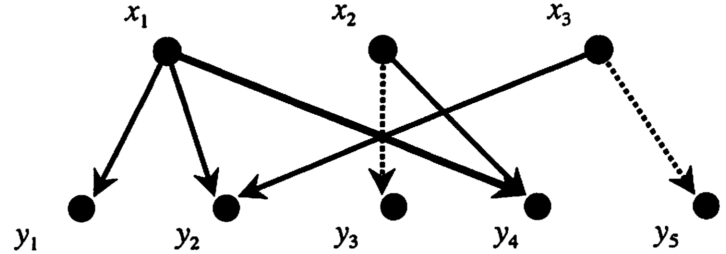


Figure 3

Theorem 2.3. Let T be a strongly connected 3-by-n bipartite tournament having exceptionality numbers β_{ij}. Then the decycling index of T is

$$\nabla'(T) = n - \max_{i \neq j} \beta_{ij}$$

Proof: We first show that $\nabla'(T) \leq n - \max \beta_{ij}$. To this end, we let E be the set of exceptional arcs and let E_{ij} be the subset of those that go neither out of x_i nor into x_j. Without loss of generality, we assume that $\beta_{1,2} = \max \beta_{ij}$. Let $D = T - E_{1,2}$; that is D is the result of removing the exceptional arcs that go into x_1, out of x_2, and those incident with x_3 (see Figure 2). Then in D, Y_1^- and Y_3^- have only in-coming arcs and Y_2^+ and Y_3^+ have only out-going arcs. Hence, every cycle in D must have its Y-vertices in $Y_1^+ \cup Y_2^-$. However, x_1 has arcs going to all of these vertices and x_2 has arcs from all of them, so (since a cycle must have at least two X-vertices) D must be cycle-free. Because there are n exceptional arcs in E and D contains only $\beta_{1,2}$ of them, $\nabla'(T) \leq n - \beta_{1,2} = n - \max \beta_{ij}$.

Now let S be any decycling set of arcs in T, and let S^* be the set of exceptional arcs whose Y-vertices are incident with an arc in S. Then $|S^*| \leq |S|$ (since each Y-vertex has only one exceptional arc). Furthermore, S^* is a decycling set since no non-exceptional arc in S can be on a cycle in $T - S^*$ because its Y-vertex can't be. It is easy to see that if T - S^* contains arcs from two of the sets Y_i^-, or from two of the Y_i^+, or from some Y_i^+ and Y_i^-, then it has a cycle. Hence, S^* must contain some E_{ij}, and so $|S^*| \geq n - \max \beta_{ij}$. This suffices to complete the proof. ❑

Corollary 2.4. $\overline{\nabla}'(3,n) = \left\lfloor \frac{2n}{3} \right\rfloor$

Proof: Let T be a strong 3-by-n bipartite tournament with exceptionality numbers β_{ij}. Then $\sum \beta_{ij} = 2n$ (since each α_i^+ and α_j^- is counted twice), and so the maximum β_{ij} is at least $\left\lceil \frac{n}{3} \right\rceil$. Hence $\overline{\nabla}'(3,n) \leq n - \left\lceil \frac{n}{3} \right\rceil = \left\lfloor \frac{2n}{3} \right\rfloor$.

To demonstrate the reverse inequality, we look to the digraph for which any two of the six $\alpha_i^{\pm}$ values differ by at most 1, and their sum is n. Specifically the following should be true.

$$\alpha_1^- \leq \alpha_2^- \leq \alpha_3^- \leq \alpha_1^+ \leq \alpha_2^+ \leq \alpha_3^+$$

$$\alpha_3^+ \leq \alpha_1^- + 1$$

$$\alpha_1^- + \alpha_2^- + \alpha_3^- + \alpha_1^+ + \alpha_2^+ + \alpha_3^+ = n$$

It is routine to verify (by cases modulo 6) that $\max \beta_{ij} = \left\lceil \frac{n}{3} \right\rceil$, so the 3-by-$n$ bipartite tournament with these exceptional values has decycling index $\left\lfloor \frac{2n}{3} \right\rfloor$.
❑

3 The 4-by-n Case.

In this section, we determine the maximum value of the decycling index of a 4-by-n bipartite tournament. We begin with an upper bound.

Lemma 3.1. The maximum decycling index of a 4-by-n bipartite tournament satisfies this inequality:

$$\overline{\nabla}'(4,n) \leq \begin{cases} \left\lfloor \frac{7n}{6} - 6 \right\rfloor & \text{if } n \equiv 1 \pmod 6 \\ \left\lfloor \frac{7n}{6} \right\rfloor & \text{otherwise} \end{cases}$$

Proof: Let T be a 4-by-n bipartite tournament. By Lemma 1.1, $\overline{\nabla}'(T) = \min_v \{\text{id } v + \overline{\nabla}'(T - v)\}$. If $v \in X$, then id v or od v is at most $\left\lfloor \frac{n}{2} \right\rfloor$ and $\overline{\nabla}'(T - v) \leq \overline{\nabla}'(3, n) = \left\lfloor \frac{2n}{3} \right\rfloor$ (by Corollary 2.4). Hence, $\overline{\nabla}'(4,n) \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{2n}{3} \right\rfloor$. It is straightforward to show, by considering congruence classes modulo 6, that this bound equals those in the statement of the lemma. ❑

We now proceed to show that equality holds except in the $n \equiv 3(mod 6)$ case. This is done by determining $\overline{\nabla}'(4, n)$ for some small values and then using induction.

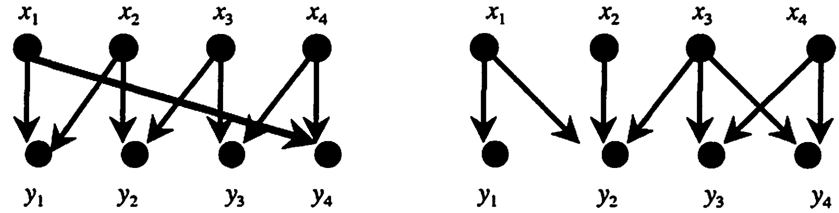
Lemma 3.2. *Let T be a 4-by-4 bipartite tournament. Then $\nabla'(T) \leq 4$, with equality holding if and only if T is regular.*

Proof: Let T be a 4-by-4 bipartite tournament. We observe first that if d is the minimum in-degree in T, then by Lemma 1.1 and Corollary 2.4, $\nabla'(T) \leq d + 2$. Hence, if T is regular, $\nabla'(T) \leq 4$, while if it is not, $\nabla'(T) \leq 3$.

What remains to be shown is that if T is regular, $\nabla'(T) \geq 4$. To this end, we note that the underlying graph of the eight X-to-Y arcs must be 2-regular and so must be either an 8-cycle or two 4-cycles. Consequently, up to isomorphism, there are only two possibilities for T ; these are indicated as T_1 and T_2 in Figure 4 (as in Figure 3, only the X-to-Y arcs are shown). It is not difficult to find sets of four arc-disjoint cycles in each :

T_1	T_2
$x_1\, y_1\, x_3\, y_3\, x_1$	$x_1\, y_1\, x_3\, y_3\, x_1$
$x_1\, y_4\, x_2\, y_2\, x_1$	$x_1\, y_2\, x_4\, y_4\, x_1$
$x_2\, y_1\, x_4\, y_3\, x_2$	$x_2\, y_1\, x_4\, y_3\, x_2$
$x_3\, y_2\, x_4\, y_4\, x_3$	$x_2\, y_2\, x_3\, y_4\, x_2$

Hence, if T is regular, $\nabla'(T) \geq 4$, which completes the proof. ❑



T_1 T_2

Figure 4

Lemma 3.3. $\overline{\nabla}'(4,5) \geq 5$.

Proof: The orientation T described by Directed Adjacency Matrix 1 below ("+" indicates the orientation $x_i y_j$ and "–" indicates the orientation $y_j x_i$) attains the upper bound, as we now show.

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \left[\begin{array}{ccccc} + & - & + & - & + \\ + & - & - & + & - \\ - & + & + & - & - \\ - & + & - & + & + \end{array}\right] \end{array}$$

Directed Adjacency Matrix 1

Each $X \cup \{ y_i, y_j : 1 \leq i < j \leq 5 \}$ contains at least one 4-cycle, in fact both $X \cup \{ y_1, y_2 \}$ and $X \cup \{ y_3, y_4 \}$ contain two arc-disjoint 4-cycles. Therefore to decycle this graph using exactly 4 arcs, we would have to remove one arc incident with each y_j for $j = 1, 2, 3, 4$ or a cycle containing y_5 would remain. Once an arc incident with y_1 is chosen we are forced to remove a specific arc incident with y_2, since otherwise a 4-cycle containing y_2 and either y_1 or y_5 remains. Now the remaining arcs incident with y_1 and y_2 along with either the arcs incident with y_3 or the arcs incident with y_4 contain two arc-disjoint 4-cycles. A third arc-disjoint cycle can be found in the arcs incident with y_5 and the arcs incident with either y_4 or y_3 (whichever was not used before). Hence we must remove at least 5 arcs, and $\overline{\nabla}'(T) \geq 5$. ❑

Lemma 3.4. $7 \leq \overline{\nabla}'(4,6) \leq \overline{\nabla}'(4,7)$.

Proof: We show that the bipartite tournament T described in Directed Adjacency Matrix 2 has decycling index 7 (the arcs in parentheses are an example of a decycling set). In order to make certain properties of the digraph T easier to see, we will, for this example, use double subscripts to label the vertices in Y (y_{ij} $1 \leq i \leq j \leq 4$). We create T by orienting the edges of a $K_{4,6}$ from x_k to y_{ij} if and only if $k = i$ or j. Using Theorem 2.3, it can be shown that $\overline{\nabla}'(T - x_4) = 4$, so $\varphi(x_4) = 7$. On the other hand, $T - y_{2,3}$ is isomorphic to the bipartite tournament in Table 2, whose decycling index is 5, so $\varphi(y_{2,3}) = 7$ as well. It is easy to see that the orbits of the vertices under the automorphism group of T are X and Y, hence the lemma follows. ❑

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{array}{c}\begin{array}{cccccc} y_{1,2} & y_{3,4} & y_{1,3} & y_{2,4} & y_{1,4} & y_{2,3} \end{array}\\ \left[\begin{array}{cccccc} (+) & - & + & - & (+) & - \\ + & (-) & (-) & + & - & + \\ - & + & + & - & - & + \\ - & (+) & - & (+) & (+) & - \end{array}\right]\end{array}$$

Directed Adjacency Matrix 2

Note that for each pair of vertices y and y^* in Y, $X \cup \{y, y^*\}$ contains at least one 4-cycle and in fact $X \cup \{y_{1,2}, y_{3,4}\}$, $X \cup \{y_{1,3}, y_{2,4}\}$, and $X \cup \{y_{1,4}, y_{2,3}\}$ each contain 2 arc-disjoint 4-cycles. We will call the pairs $\{y_{1,2}, y_{3,4}\}$, $\{y_{1,3}, y_{2,4}\}$, and $\{y_{1,4}, y_{2,3}\}$ *complementary pairs.*

Lemma 3.5. For $n \geq 2$,

$$\overline{\nabla}'(4,n) = \begin{cases} \left\lfloor \frac{7n}{6} \right\rfloor - 1 & \text{if } n \equiv 1 \pmod 6 \\ \left\lfloor \frac{7n}{6} \right\rfloor & \text{if } n \equiv 0, 2, 4, 5 \pmod 6 \end{cases}$$

Proof: The values for $n = 2, 4, 5, 6$, and 7 follow from Corollaries 2.2 and 2.4, and Lemmas 3.1, 3.2, 3.3, and 3.4.
For the remaining values of n, the upper bound is given in Lemma 3.1. The lower bound follows from applying Lemma 1.2 to the 4-*by*-n bipartite tournament created by taking the appropriate number of copies of the digraph T described by Directed Adjacency Matrix 2, the tournament which gave the lower bound in from Corollary 2.2 or 2.4, or Lemma 3.2, 3.3, or 3.4, and then identifying their X-sets. ❑

We now consider the case $n \equiv 3 \pmod 6$ which was not covered in Lemma 3.5.

Lemma 3.6: For $k \geq 0$, $\overline{\nabla}'(4, 6k + 3) = 7k + 2$.

Proof: The lower bound follows from Lemma 3.5 since $\overline{\nabla}'(4, 6k + 2) = 7k + 2$. In order to demonstrate the upper bound, we will need to consider 3 cases, depending upon the minimum in- or out-degree d of the vertices in Y. Let T be a 4-by-$(6k + 3)$ bipartite tournament. Without loss of generality, we may assume $id(y_1) = d < od(y_1)$.

<u>*Case 1*</u>. $d = 0$: Here $\nabla'(T) = \nabla'(T - y_1) \leq \overline{\nabla}'(4, 6k + 2) = \left\lfloor \frac{7(6k + 2)}{6} \right\rfloor = 7k + 2$.

Case 2. $d = 1$: Without loss of generality we hay assume that x_1 is the vertex with an arc into y_1. Either $id(x_1)$ or $od(x_1)$ is at most $3k+1$, so by Lemma 1.1, $\nabla'(T) \leq \nabla'(T - x_1) + 3k + 1 = \nabla'(T - x_1 - y_1) + 3k + 1 \leq \overline{\overline{\nabla}}'(3, 6k+2) + 3k + 1 = 4k + 1 + 3k + 1 = 7k + 2$.

Case 3. $d = 2$: In this case each y_i must have the adjacency pattern of some y_{ij} from Directed Adjacency Matrix 2. For $1 \leq i \leq j \leq 4$ let $Y_{ij} = \{ y \in Y \mid y$ has the same adjacency pattern as $y_{ij}\}$and let a be the minimum value of $|Y_{ij}|$.

If $a < k$ then without loss of generality assume $|Y_{3,4}| = a$ and $|Y_{1,4}| \leq |Y_{2,3}|$. We can choose a decycling set of two arcs incident with each vertex in either $Y_{3,4}$ or $Y_{1,4}$, none from $Y_{2,3}$, and one from each of the other y vertices. The order of the set will be:

$$|Y_{1,2}| + 2|Y_{3,4}| + |Y_{1,3}| + |Y_{2,4}| + 2|Y_{1,4}| \leq$$
$$|Y_{1,2}| + 2|Y_{3,4}| + |Y_{1,3}| + |Y_{2,4}| + |Y_{1,4}| + |Y_{2,3}| =$$
$$6k + 3 + a < 7k + 3$$

If $a = k$ then some pair of Y_{ij} associated with a complementary pair must have an odd number of vertices. Without loss of generality, assume that $|Y_{1,4}| + |Y_{2,3}|$ is odd, $|Y_{1,4}| < |Y_{2,3}|$, and $|Y_{3,4}| = a$.

$$|Y_{1,2}| + 2|Y_{3,4}| + |Y_{1,3}| + |Y_{2,4}| + 2|Y_{1,4}| <$$
$$|Y_{1,2}| + 2|Y_{3,4}| + |Y_{1,3}| + |Y_{2,4}| + |Y_{1,4}| + |Y_{2,3}| =$$
$$6k + 3 + a = 7k + 3$$

In all three cases we can see that $\overline{\overline{\nabla}}'(4, 6k+3) = 7k + 2$. ❑

Theorem 3.9. $\overline{\overline{\nabla}}'(5,5) = 6$.

Proof: Let T be an orientation of $K_{5,5}$. If any vertex has indegree 0, 1, 4, or 5, then Lemma 1.1 and Theorem 3.6 give the upper bound.

We assume that T is *near regular* (for any two vertices u and v the difference in their in-degrees is at most 1). We can also assume WOLOG that the sum of the in-degrees of the y_i is less than the sum of the out-degrees. In this case, the y_i in-degree sequence σ_y for the y_i (and hence the out-degree sequence for the x_j) must be either (2, 2, 2, 3, 3); (2, 2, 2, 2, 3); or (2, 2, 2, 2, 2). We consider these cases individually.

In the case where the sequence is 2, 2, 2, 3, 3 either some y_i with in-degree 3 has an arc to an x_j of out degree 2 or none of them do. If none do, there is a set of 4 arcs whose removal decycles T. If x_j does, then one of the out arcs must be incident with a y_k of in-degree 2. If we remove the 3 X–to-Y arcs incident with these two vertices, then all we must do is decycle $T - x_j - y_k$, which is not regular, and hence by Lemma 3.2 can be decycled by removing at most 3 arcs.

In the case where the sequence is 2, 2, 2, 2, 3 the y_i with in-degree 3 has an arc to an x_j of out degree 2. For x_j, one of the out arcs must be incident with a y_k of in-degree 2. If we remove the 3 X-to-Y arcs incident with these two vertices, then all we must do is decycle $T - x_j - y_k$, which is not regular, and hence by Lemma 3.2 can be decycled by removing at most 3 arcs.
In the case where the sequence is 2, 2, 2, 2, 2 each y_i has an arc to an x_j of out degree 2. If we remove the 3 X-to-Y arcs incident with these two vertices, then all we must do is decycle $T - x_j - y_i$, which is not regular, and hence by Lemma 3.2 can be decycled by removing at most 3 arcs.
In any of the 3 cases $\nabla'(T) \le 6$, which gives the upper bound.
The bipartite tournament T described in Directed Adjacency Matrix 2 proves the lower bound. Clearly removing any vertex from T (by removing its 2 "+" arcs) leaves the same bipartite tournament T^* (up to isomorphism). In T^* there are 4 arc independent 4-cycles. Therefore,

	y_1	y_2	y_3	y_4	y_5
x_1	+	+	-	-	-
x_2	-	+	+	-	-
x_3	-	-	+	+	-
x_4	-	-	-	+	+
x_5	+	-	-	-	+

Directed Adjacency Matrix 3

by Lemma 1.1 $\nabla'(T) \ge 6$, and the theorem is proven. ❑
As this is being written, there are not enough examples to make a guess at a formula for $\overline{\nabla}'(5,n)$.

Conjecture 3.10. $\overline{\nabla}'(6,6) = 10$.

4 References

[1] Barefoot, C. A., Entringer, R., and Swart, H. C.: Vulnerability in Graphs – A Comparative Survey, Journal of Combinatorial Mathematics and Combinatorial Computing 1, 12 - 22 (1987)
[2] Beineke, L. W., Vandell, R. C.: Decycling Graphs, Journal of Graph Theory 25, 59 - 77 (1997)
[3] Chartrand, G., Lesniak, L.: Graphs and Digraphs 3rd edition, Wadsworth & Brooks/Cole, Pacific Grove, California 1990

[4] Vandell, R. C.:The Integrity of a Directed Graph and the Directed Integrity of a Graph, Congressus Numerantium 155, 81 - 88 (2002)
[5] Reid, K. B.: On Sets of Arcs Containing No Cycles in a Tournament, Canadian Mathematics Bulletin 12, 261 - 264 (1969)