Double hexagonal chains with the extremal third order Randić index *

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Abstract

The third order Randić index of a graph G is defined as $R_3(G) = \sum_{u_1u_2u_3u_4} \frac{1}{\sqrt{d(u_1)d(u_2)d(u_3)d(u_4)}}$, where the summation is taken over all possible paths of length three of G. In this paper, we first give a recursive formula for computing the third order Randić index of a double hexagonal chain. And then we determine the upper and lower bounds of the third order Randić index and characterize the double hexagonal chains with the extremal third order Randić index.

1 Introduction

The connectivity index (or Randić index) of a graph G, denoted by $R(G)$, was introduced by Randić [1] in the study of branching properties of alkanes. It is defined as

$$R(G) = \sum_{uv} \frac{1}{\sqrt{d(u)d(v)}}$$

where $d(u)$ denotes the degree of the vertex u and the summation is taken over all pairs of adjacent vertices of the graph G. Some publications related to the connectivity index can be found in the literature([5,8-14,21-23]).

With the intention of extending the applicability of the connectivity index, Randić, Kier, Hall and co-workers([2] and [3]) considered the higher-order con-

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nectivity index of a graph G as

$$R_k(G) = \sum_{u_1 u_2 \cdots u_{k+1}} \frac{1}{\sqrt{d(u_1)d(u_2)\cdots d(u_{k+1})}}$$

where the summation is taken over all possible paths of length k of G (we do not distinguish between the paths $u_1u_2\cdots u_{k+1}$ and $u_{k+1}u_h\cdots u_1$). This new approach has been applied successfully to an impressive variety of physical, chemical and biological properties (boiling points, solubilities, densities, anesthetic, toxicities etc.) which have appeared in many scientific publications and in two books ([2] and [4]). Results related to the mathematical properties of these indices have been reported in the literature([5] and [6]). Specifically, Rada [15] gave an expression of the second order Randić index of benzenoid systems and found the minimal and maximal value over the set of catacondensed systems. The Randić index of phenylenes has been discussed in [7], the second and the third order Randić indices of phenylenes have been discussed in [16,17]. A recursive formula for computing the third order Randić index of a hexagonal chain is given and the hexagonal chains with the extremal third order Randić index are characterized in [18]. In this paper, we will consider a type of the pericondensed hexagonal system. The double hexagonal chains with the extremal third order Randić index are determined.

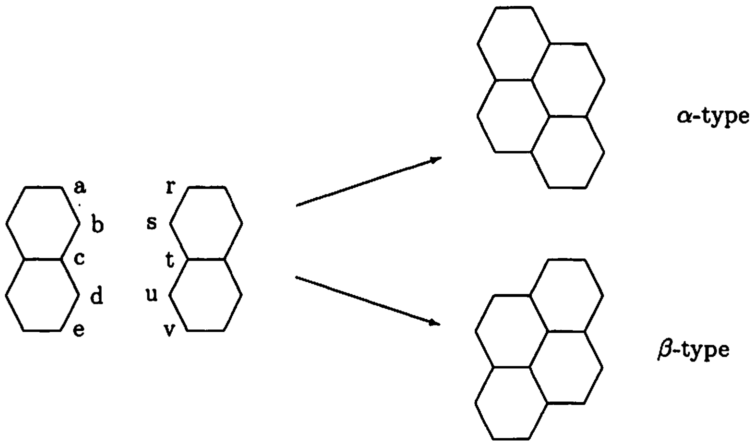


Figure 1. α-type fusing, β-type fusing.

A hexagonal system is a 2-connected plane graph whose every interior face is bounded by a regular hexagon of unit length 1. Hexagonal systems are of considerable importance in theoretical chemistry because they are the natural graph representation of benzenoid hydrocarbons [19]. A vertex of a hexagonal system belongs to, at most, three hexagons. A vertex shared by three hexagons is called an internal vertex of the respective hexagonal system. A hexagonal system H is said to be catacondensed if it does not possess internal vertices, otherwise H is said to be pericondensed. A hexagonal chain is a catacondensed hexagonal system which has no hexagon adjacent to more than two hexagons. An n-tuple hexagonal chain consists of n condensed identical hexagonal chains. When $n = 2$, we call it a double hexagonal chain [19-20].

A double hexagonal chain can be constructed inductively. Let us orient the naphthalene so that its interior edges is horizontal. There are two types of fusion of two naphthalenes: (i) $b \equiv r$, $c \equiv s$, $d \equiv t$, $e \equiv u$; (ii) $a \equiv s$, $b \equiv t$, $c \equiv u$, $d \equiv v$ as shown in Figure 1. We call them α-type and β-type fusing, respectively. Any double hexagonal chain can be obtained from a naphthalene B by a stepwise fusion of new naphthalene, and at each step a θ-type fusion is selected, where $\theta \in \{\alpha, \beta\}$.

Let $B(\theta_1, \theta_2, \cdots, \theta_n)$ be the double hexagonal chain with $2(n+1)$ hexagons obtained from a naphthalene B by θ_1-type, θ_2-type, $\cdots$, θ_n-type fusing, successively. $B(\alpha, \alpha, \cdots, \alpha)$ or $B(\beta, \beta, \cdots, \beta)$, i.e., $\theta_i = \theta_{i+1}$ for each i, is called the double linear hexagonal chain and denoted by DL_n; if $\theta_i \neq \theta_{i+1}$ for each i, then $B(\theta_1, \theta_2, \cdots, \theta_n)$ is called the double zig-zag hexagonal chain and denoted by DZ_n (see Figure 2).

Let

$$\overline{\theta} = \begin{cases} \beta, & \text{if } \theta = \alpha; \\ \alpha, & \text{if } \theta = \beta. \end{cases}$$

Then $B(\theta_1, \theta_2, \cdots, \theta_n) \cong B(\overline{\theta_1}, \overline{\theta_2}, \cdots, \overline{\theta_n})$. If $n \geq 1$, the double hexagonal chain $B(\theta_1, \theta_2, \cdots, \theta_n)$ is a pericondensed hexagonal system.

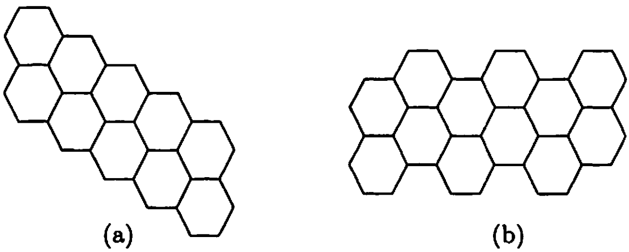
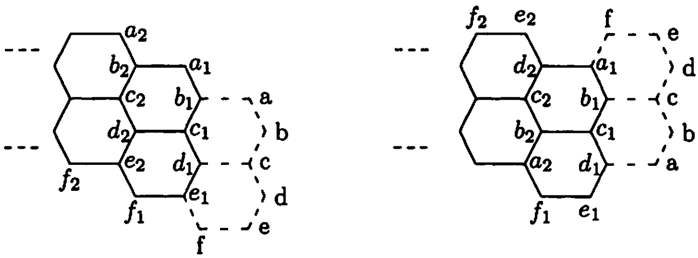


Figure 2. (a) The double linear hexagonal chain $DL_4 = B(\alpha, \alpha, \alpha, \alpha)$;
(b) The double zig-zag hexagonal chain $DZ_5 = B(\beta, \alpha, \beta, \alpha, \beta)$.

2 A recursive formula for computing the third order Randić index of a double hexagonal chain

In this section, we give a recursive formula for computing the third order Randić index of a double hexagonal chain $B(\theta_1, \theta_2, \cdots, \theta_n)$.

Let $B_n = B(\theta_1, \theta_2, \cdots, \theta_n)$ be a double hexagonal chain with $2(n+1)$ hexagons, as in Figure 3.



(1) $B(\cdots, \alpha, \alpha)$ (2) $B(\cdots, \beta, \alpha)$

Figure 3.

Case I. $\theta_n = \theta_{n-1} = \alpha$. Then B_n is obtained from B_{n-1} by a α-type fusing (see Figure 3 (1)). The following paths of length 3 are added:

$abcd$, $abcd_1$, $ab_1a_1b_2$, $ab_1c_1d_2$, $ab_1c_1d_1$, $bcde$, bcd_1c_1, bcd_1e_1, bab_1a_1, bab_1c_1, $cdef$, cd_1e_1f, $cd_1e_1f_1$, $cd_1c_1d_2$, $cd_1c_1b_1$, $cbab_1$, $defe_1$, dcd_1c_1, dcd_1e_1, efe_1f_1, efe_1d_1, $edcd_1$, $fe_1f_1e_2$, $fe_1d_1c_1$.

Table 1. The weights of some paths of length 3 in B_n.

$abcd$	$abcd_1$	$ab_1a_1b_2$	$ab_1c_1d_2$	$ab_1c_1d_1$	$bcde$
$\frac{1}{2\sqrt{6}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$
bcd_1c_1	bcd_1e_1	bab_1a_1	bab_1c_1	$cdef$	cd_1e_1f
$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$
$cd_1e_1f_1$	$cd_1c_1d_2$	$cd_1c_1b_1$	$cbab_1$	$defe_1$	dcd_1c_1
$\frac{1}{3\sqrt{6}}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$
dcd_1e_1	efe_1f_1	efe_1d_1	$edcd_1$	$fe_1f_1e_2$	$fe_1d_1c_1$
$\frac{1}{3\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3\sqrt{6}}$

Table 2. The weights of some paths of length 3 in B_n and B_{n-1}.

	$a_2b_2a_1b_1$	$b_2a_1b_1c_1$	$c_2b_2a_1b_1$	$c_2d_2c_1b_1$	$c_2d_2c_1d_1$
W_1	$\frac{1}{6}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{9}$
W_2	$\frac{1}{2\sqrt{6}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{3\sqrt{6}}$
	$d_2c_1b_1a_1$	$d_2c_1d_1e_1$	$d_2e_2f_1e_1$	$e_2d_2c_1b_1$	$e_2d_2c_1d_1$
W_1	$\frac{1}{3\sqrt{6}}$	$\frac{1}{9}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{9}$	$\frac{1}{9}$
W_2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$
	$e_2f_1e_1d_1$	$f_2e_2f_1e_1$	$a_1b_1c_1d_1$	$b_1c_1d_1e_1$	$c_1d_1e_1f_1$
W_1	$\frac{1}{3\sqrt{6}}$	$\begin{cases} \frac{1}{6}, & \text{if } d(f_2)=2; \\ \frac{1}{3\sqrt{6}}, & \text{if } d(f_2)=3 \end{cases}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{9}$	$\frac{1}{3\sqrt{6}}$
W_2	$\frac{1}{2\sqrt{6}}$	$\begin{cases} \frac{1}{2\sqrt{6}}, & \text{if } d(f_2)=2; \\ \frac{1}{6}, & \text{if } d(f_2)=3 \end{cases}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$

Except for the paths:
$a_2b_2a_1b_1$, $b_2a_1b_1c_1$, $c_2b_2a_1b_1$, $c_2d_2c_1b_1$, $c_2d_2c_1d_1$, $d_2c_1b_1a_1$, $d_2c_1d_1e_1$, $d_2e_2f_1e_1$, $e_2d_2c_1b_1$, $e_2d_2c_1d_1$, $e_2f_1e_1d_1$, $f_2e_2f_1e_1$, $a_1b_1c_1d_1$, $b_1c_1d_1e_1$, $c_1d_1e_1f_1$,
the other paths of length 3 in B_n are the same in B_{n-1}.

Let $W_1(P)$ and $W_2(P)$ be the weight $\frac{1}{\sqrt{d(v_1)d(v_2)d(v_3)d(v_4)}}$ of the path $P = v_1v_2v_3v_4$ in B_n and B_{n-1}, respectively.

Note that $d(f_2) = 2$ if $\theta_{n-2} = \alpha$ and $d(f_2) = 3$ if $\theta_{n-2} = \beta$. By the definition of the third order Randić, we have

$$R_3(B_n) = R_3(B_{n-1}) + \begin{cases} \frac{4}{3} + \frac{7}{9}\sqrt{6}, & \text{if } \theta_{n-2} = \theta_{n-1} = \theta_n = \alpha; \\ 1 + \frac{11}{12}\sqrt{6}, & \text{if } \theta_{n-2} = \beta \text{ and } \theta_{n-1} = \theta_n = \alpha. \end{cases}$$

And

$$R_3(B_n) = R_3(B_{n-1}) + \begin{cases} \frac{4}{3} + \frac{7}{9}\sqrt{6}, & \text{if } \theta_{n-2} = \theta_{n-1} = \theta_n = \beta; \\ 1 + \frac{11}{12}\sqrt{6}, & \text{if } \theta_{n-2} = \alpha \text{ and } \theta_{n-1} = \theta_n = \beta \end{cases}$$

since $B(\theta_1, \theta_2, \cdots, \theta_n) \cong B(\overline{\theta_1}, \overline{\theta_2}, \cdots, \overline{\theta_n})$.

Case II. $\theta_{n-1} = \alpha$ and $\theta_n = \beta$. Then B_n is obtained from B_{n-1} by a β-type fusing (see Figure 3 (2)). The following paths of length 3 are added:
$abcd$, $abcb_1$, $ad_1e_1f_1$, $ad_1c_1b_2$, $ad_1c_1b_1$, bad_1e_1, bad_1c_1, bcd_1c_1, bcb_1a_1, $bcde$, $cbad_1$, $cb_1c_1d_1$, $cb_1c_1b_2$, $cb_1a_1d_2$, cb_1a_1f, $cdef$, dcb_1c_1, dcb_1a_1, $defa_1$, $edcb_1$, efa_1b_1, efa_1d_2, $fa_1b_1c_1$, $fa_1d_2c_2$, $fa_1d_2e_2$.

Table 3. The weights of some paths of length 3 in B_n.

$abcd$	$abcb_1$	$ad_1e_1f_1$	$ad_1c_1b_2$	$ad_1c_1b_1$	bad_1e_1	bad_1c_1
$\frac{1}{2\sqrt{6}}$	$\frac{1}{6}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{6}$
bcd_1c_1	bcb_1a_1	$bcde$	$cbad_1$	$cb_1c_1d_1$	$cb_1c_1b_2$	$cb_1a_1d_2$
$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
cb_1a_1f	$cdef$	dcb_1c_1	dcb_1a_1	$defa_1$	$edcb_1$	efa_1b_1
$\frac{1}{3\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{6}$	$\frac{1}{6}$
efa_1d_2	$fa_1b_1c_1$	$fa_1d_2c_2$	$fa_1d_2e_2$			
$\frac{1}{6}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{6}$			

Table 4. The weights of some paths of length 3 in B_n and B_{n-1}.

	$f_2e_2d_2a_1$	$e_2d_2a_1b_1$	$d_2a_1b_1c_1$	$c_2d_2a_1b_1$	$c_2b_2c_1a_1$
W_1	$\begin{cases} \frac{1}{6}, & \text{if } d(f_2)=2; \\ \frac{1}{3\sqrt{6}}, & \text{if } d(f_2)=3 \end{cases}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
W_2	$\begin{cases} \frac{1}{2\sqrt{6}}, & \text{if } d(f_2)=2; \\ \frac{1}{6}, & \text{if } d(f_2)=3 \end{cases}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3\sqrt{6}}$
	$c_2b_2c_1d_1$	$b_2c_1b_1a_1$	$b_2c_2d_2a_1$	$b_2c_1d_1e_1$	$a_2b_2c_1b_1$
W_1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{9}$
W_2	$\frac{1}{3\sqrt{6}}$	$\frac{1}{6}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{6}$	$\frac{1}{3\sqrt{6}}$
	$a_2b_2c_1d_1$	$a_2f_1e_1d_1$	$a_1b_1c_1d_1$	$b_1c_1d_1e_1$	$c_1d_1e_1f_1$
W_1	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{3\sqrt{6}}$	$\frac{1}{6}$
W_2	$\frac{1}{3\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{2\sqrt{6}}$

Except for the paths:
$f_2e_2d_2a_1$, $e_2d_2a_1b_1$, $d_2a_1b_1c_1$, $c_2d_2a_1b_1$, $c_2b_2c_1a_1$, $c_2b_2c_1d_1$, $b_2c_1b_1a_1$, $b_2c_2d_2a_1$, $b_2c_1d_1e_1$, $a_2b_2c_1b_1$, $a_2b_2c_1d_1$, $a_2f_1e_1d_1$, $a_1b_1c_1d_1$, $b_1c_1d_1e_1$, $c_1d_1e_1f_1$,
the other paths of length 3 in B_n are the same in B_{n-1}.

The weight of these paths are given in Tables 3 and 4, where $W_1(P)$ and $W_2(P)$ are the weight of the path P in B_n and B_{n-1}, respectively.

Note that $d(f_2)=2$ if $\theta_{n-2}=\beta$ and $d(f_2)=3$ if $\theta_{n-2}=\alpha$. By the definition of the third order Randić, we have

$$R_3(B_n) = R_3(B_{n-1}) + \begin{cases} \frac{7}{3}+\frac{7}{18}\sqrt{6}, & \text{if } \theta_{n-2}=\beta,\ \theta_{n-1}=\alpha,\ \theta_n=\beta; \\ 2+\frac{19}{36}\sqrt{6}, & \text{if } \theta_{n-2}=\theta_{n-1}=\alpha,\ \theta_n=\beta. \end{cases}$$

And

$$R_3(B_n) = R_3(B_{n-1}) + \begin{cases} \frac{7}{3}+\frac{7}{18}\sqrt{6}, & \text{if } \theta_{n-2}=\alpha,\ \theta_{n-1}=\beta,\ \theta_n=\alpha; \\ 2+\frac{19}{36}\sqrt{6}, & \text{if } \theta_{n-2}=\theta_{n-1}=\beta,\ \theta_n=\alpha. \end{cases}$$

since $B(\theta_1, \theta_2, \cdots, \theta_n) \cong B(\overline{\theta_1}, \overline{\theta_2}, \cdots, \overline{\theta_n})$.

So, we have the following recursive formula

Theorem 1. Let $B_n = B(\theta_1, \theta_2, \cdots, \theta_n)$ and $B_{n-1} = B(\theta_1, \theta_2, \cdots, \theta_{n-1})$ be double hexagonal chains, $n \geq 3$. Then

$$R_3(B_n) = R_3(B_{n-1}) + \begin{cases} \frac{4}{3} + \frac{7}{9}\sqrt{6}, & \theta_{n-2} = \theta_{n-1} = \theta_n = \alpha \\ & or \quad \theta_{n-2} = \theta_{n-1} = \theta_n = \beta; \\ 1 + \frac{11}{12}\sqrt{6}, & \theta_{n-2} = \beta, \theta_{n-1} = \theta_n = \alpha \\ & or \quad \theta_{n-2} = \alpha, \theta_{n-1} = \theta_n = \beta; \\ \frac{7}{3} + \frac{7}{18}\sqrt{6}, & \theta_{n-2} = \beta,\ \theta_{n-1} = \alpha,\ \theta_n = \beta \\ & or \quad \theta_{n-2} = \alpha,\ \theta_{n-1} = \beta,\ \theta_n = \alpha; \\ 2 + \frac{19}{36}\sqrt{6}, & \theta_{n-2} = \theta_{n-1} = \alpha,\ \theta_n = \beta \\ & or \quad \theta_{n-2} = \theta_{n-1} = \beta,\ \theta_n = \alpha. \end{cases}$$

3 The extremal double hexagonal chains with respect to the third order Randić index

In this section, we will give the upper and lower bounds on the third order Randić indices of double hexagonal chains and characterize the extremal graphs.

Theorem 2. Let $B_n = B(\theta_1, \theta_2, \cdots, \theta_n)$ be a double hexagonal chain with $2(n+1)$ hexagons. Then

(i) $R_3(B_n) \geq \frac{12n+7}{9} + \frac{7n+10}{9}\sqrt{6}$
with equality if and only if $(\theta_1, \theta_2, \cdots, \theta_n) = (\alpha, \alpha, \cdots, \alpha)$, or $(\beta, \beta, \cdots, \beta)$, i.e., B_n is the double linear hexagonal chain DL_n;

(ii) $R_3(B_n) \leq \frac{21n-2}{9} + \frac{7n+27}{18}\sqrt{6}$
with equality if and only if $(\theta_1, \theta_2, \cdots, \theta_n) = (\beta, \alpha, \beta, \alpha, \cdots)$ or $(\alpha, \beta, \alpha, \beta, \cdots)$, i.e., B_n is the double zig-zag hexagonal chain DZ_n.

Proof. It can be computed out directly that

$R_3(B) = \frac{11}{6} + \frac{2}{3}\sqrt{6}$;

$R_3(B_1) = R_3(B(\alpha)) = R_3(B(\beta)) = \frac{19}{9} + \frac{17}{9}\sqrt{6}$;

$$R_3(B_2) = \begin{cases} R_3(B(\alpha, \alpha)) = R_3(B(\beta, \beta)) = \frac{31}{9} + \frac{8}{3}\sqrt{6} \\ R_3(B(\alpha, \beta)) = R_3(B(\beta, \alpha)) = \frac{40}{9} + \frac{41}{18}\sqrt{6} \end{cases}$$

and

$$\frac{31}{9} + \frac{8}{3}\sqrt{6} < \frac{40}{9} + \frac{41}{18}\sqrt{6};$$

$$R_3(B_3) = \begin{cases} R_3(B(\alpha,\alpha,\alpha)) = R_3(B(\beta,\beta,\beta)) = \frac{43}{9} + \frac{31}{9}\sqrt{6} \\ R_3(B(\alpha,\alpha,\beta)) = R_3(B(\beta,\beta,\alpha)) = \frac{49}{9} + \frac{115}{36}\sqrt{6} \\ R_3(B(\alpha,\beta,\alpha)) = R_3(B(\beta,\alpha,\beta)) = \frac{61}{9} + \frac{8}{3}\sqrt{6} \\ R_3(B(\alpha,\beta,\beta)) = R_3(B(\beta,\alpha,\alpha)) = \frac{49}{9} + \frac{115}{36}\sqrt{6} \end{cases}$$

and

$$\frac{43}{9} + \frac{31}{9}\sqrt{6} < \frac{49}{9} + \frac{115}{36}\sqrt{6} < \frac{61}{9} + \frac{8}{3}\sqrt{6}.$$

From Theorem 1 and $R_3(B_1) = \frac{19}{9} + \frac{17}{9}\sqrt{6}$, it can be computed out easily that

$R_3(DL_n) = \frac{12n+7}{9} + \frac{7n+10}{9}\sqrt{6}$

$R_3(DZ_n) = \frac{21n-2}{9} + \frac{7n+27}{18}\sqrt{6}$.

In the following, using the inductive method on n, we prove that $R_3(DL_n) \leq R_3(B_n) \leq R_3(DZ_n)$ with the left (or right) equality if and only if B_n is DL_n (or DZ_n).

The result is true for $n = 1, 2$ from above.

We suppose that the result is true for n and $n-1$ ($n \geq 2$), i.e.,

$R_3(DL_{n-1}) \leq R_3(B_{n-1}) \leq R_3(DZ_{n-1})$ with the left (or right) equality if and only if B_{n-1} is DL_{n-1} (or DZ_{n-1}) and $R_3(DL_n) \leq R_3(B_n) \leq R_3(DZ_n)$ with the left (or right) equality if and only if B_n is DL_n (or DZ_n).

Let $B_{n+1} = B(\theta_1, \theta_2, \cdots, \theta_n, \theta_{n+1})$ be a double hexagonal chain with $2(n+2)$ hexagons. Note that

$$\frac{4}{3} + \frac{7}{9}\sqrt{6} < 1 + \frac{11}{12}\sqrt{6} < \frac{7}{3} + \frac{7}{18}\sqrt{6} < 2 + \frac{19}{36}\sqrt{6}.$$

By Theorem 1, we have

(i) $R_3(B_{n+1}) \geq R_3(B_n) + \frac{4}{3} + \frac{7}{9}\sqrt{6}$ with equality if and only if $\theta_{n-1} = \theta_n = \theta_{n+1} = \alpha$ or $\theta_{n-1} = \theta_n = \theta_{n+1} = \beta$. By the inductive hypothesis, $R_3(B_n) \geq R_3(DL_n)$ with equality if and only if B_n is DL_n. So, $R_3(B_{n+1}) \geq R_3(DL_{n+1})$ with equality if and only if B_{n+1} is DL_{n+1}.

(ii)

$$R_3(B_{n+1}) \leq \begin{cases} R_3(B_n) + (2 + \frac{19}{36}\sqrt{6}), & \text{if } \theta_{n-1} = \theta_n = \alpha,\ \theta_{n+1} = \beta \\ & \text{or } \theta_{n-1} = \theta_n = \beta,\ \theta_{n+1} = \alpha; \\ R_3(B_n) + (\frac{7}{3} + \frac{7}{18}\sqrt{6}), & \text{if } \theta_{n-1} = \alpha, \theta_n = \beta, \theta_{n+1} = \alpha \\ & \text{or } \theta_{n-1} = \beta, \theta_n = \alpha, \theta_{n+1} = \beta. \end{cases}$$

$$\leq \begin{cases} R_3(B_{n-1}) + (1 + \frac{11}{12}\sqrt{6}) + (2 + \frac{19}{36}\sqrt{6}), & \text{if } \theta_{n-1} = \theta_n = \alpha,\ \theta_{n+1} = \beta \\ & \text{or } \theta_{n-1} = \theta_n = \beta,\ \theta_{n+1} = \alpha; \\ R_3(B_{n-1}) + 2(\frac{7}{3} + \frac{7}{18}\sqrt{6}), & \text{if } \theta_{n-1} = \alpha, \theta_n = \beta, \theta_{n+1} = \alpha \\ & \text{or } \theta_{n-1} = \beta, \theta_n = \alpha, \theta_{n+1} = \beta. \end{cases}$$

Since $(1 + \frac{11}{12}\sqrt{6}) + (2 + \frac{19}{36}\sqrt{6}) < 2(\frac{7}{3} + \frac{7}{18}\sqrt{6})$, we have

$$R_3(B_{n+1}) \leq R_3(B_{n-1}) + 2(\frac{7}{3} + \frac{7}{18}\sqrt{6})$$

with the equality if and only if $(\theta_{n-2}, \theta_{n-1}, \theta_n, \theta_{n+1}) = (\beta, \alpha, \beta, \alpha)$ or $(\alpha, \beta, \alpha, \beta)$.

By the inductive hypothesis, $R_3(B_{n-1}) \leq R_3(DZ_{n-1})$ with equality if and only if B_{n-1} is DZ_{n-1}. So, $R_3(B_{n+1}) \leq R_3(DZ_{n+1})$ with equality if and only if B_{n+1} is DZ_{n+1}.

Therefore, the result is true for $n \geq 1$ by the inductive method and the proof of Theorem 2 is completed.

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