Cycles are determined by their domination polynomials

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Abstract

Let G be a simple graph of order n. A dominating set of G is a set S of vertices of G so that every vertex of G is either in S or adjacent to a vertex in S. The domination polynomial of G is the polynomial $D(G,x) = \sum_{i=1}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i. In this paper we show that cycles are determined by their domination polynomials.

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1 Introduction

Throughout this paper we will consider only simple graphs. Let $G = (V, E)$ be a simple graph. The *order* of G denotes the number of vertices of G. For every vertex $v \in V$, the *closed neighborhood* of v is the set $N[v] = \{u \in V | uv \in E\} \cup \{v\}$. For a set $S \subseteq V$, the closed neighborhood of S is $N[S] = \bigcup_{v \in S} N[v]$. A set $S \subseteq V$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V \backslash S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating

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set in G. A dominating set with cardinality $\gamma(G)$ is called a *γ-set*. For a detailed treatment of this parameter, the reader is referred to [4]. Let $\mathcal{D}(G,i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G,i) = |\mathcal{D}(G,i)|$. The *domination polynomial* $D(G,x)$ of G is defined as $D(G,x) = \sum_{i=1}^{|V(G)|} d(G,i)x^i$, (see [1]). Two graphs G and H are said to be *$\mathcal{D}$-equivalent*, written $G \sim H$, if $D(G,x) = D(H,x)$. The *$\mathcal{D}$-equivalence class* of G is defined as $[G] = \{H : H \sim G\}$. A graph G is said to be *$\mathcal{D}$-unique*, if $[G] = \{G\}$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, *join* of G_1 and G_2 denoted by $G_1 \vee G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1)$ and $v \in V(G_2)\}$. We denote the complete graph of order n, the cycle of order n, and the path of order n, by K_n, C_n, and P_n, respectively. Also we denote $K_1 \vee C_{n-1}$ by W_n and call it *wheel* of order n.

Let $n \in \mathbb{Z}$ and p be a prime number. Then if n is not zero, there is a nonnegative integer a such that $p^a \mid n$ but $p^{a+1} \nmid n$, we let $\operatorname{ord}_p n = a$.

In [2], this question was posed: *for every natural number n, C_n is $\mathcal{D}$-unique.*

Also in [2] it was proved that P_n for $n \equiv 0 \pmod 3$ has $\mathcal{D}$-equivalence class of size two and it contains P_n and the graph obtained by adding two new vertices joined to two adjacent vertices of C_{n-2}. In this paper we show that for every positive integer n, C_n is $\mathcal{D}$-unique.

2 $\mathcal{D}$-uniqueness of cycles

In this section we will prove that C_n is $\mathcal{D}$-unique. This answers in affirmative a problem proposed in [2] on $\mathcal{D}$-equivalence class of C_n. As a consequence we obtain that W_n is $\mathcal{D}$-unique. We let $C_1 = K_1$ and $C_2 = K_2$. We begin by the following lemmas.

Lemma 1. [2] *Let H be a k-regular graph with $N[u] \neq N[v]$, for every $u, v \in V(H)$. If $D(G,x) = D(H,x)$, then G is also a k-regular graph.*

Lemma 2. [1, Theorem 2.2.3] *If G has m connected components $G_1, \ldots, G_m$. Then $D(G,x) = \prod_{i=1}^{m} D(G_{n_i}, x)$.*

The next lemma gives a recursive formula for the determination of the domination polynomial of cycles.

Lemma 3. [1, Theorem 4.3.6] *For every $n \geq 4$,*

$$D(C_n, x) = x(D(C_{n-1}, x) + D(C_{n-2}, x) + D(C_{n-3}, x)).$$

Lemma 4. [3, Theorem 1] *For every $n \geq 1$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

Lemma 5. *If n is a positive integer and $\alpha_n := D(C_n, -1)$, then the following holds:*

$$\alpha_n = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 4; \\ -1, & \text{otherwise.} \end{cases}$$

Proof. By Lemma 3, for every $n \geq 4$, $\alpha_n = -(\alpha_{n-1} + \alpha_{n-2} + \alpha_{n-3})$. Now, by induction on n the proof is complete. □

Lemma 6. *For every positive integer n,*

$$\operatorname{ord}_3 D(C_n, -3) = \begin{cases} \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 0 \pmod 3; \\ \lceil \frac{n}{3} \rceil \text{ or } \lceil \frac{n}{3} \rceil + 1, & \text{if } n \equiv 1 \pmod 3; \\ \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $a_n := D(C_n, -3)$. By Lemma 3, for any $n \geq 4$, $a_n = -3(a_{n-1} + a_{n-2} + a_{n-3})$. Since $D(C_1, x) = x$, $D(C_2, x) = x^2 + 2x$, and $D(C_3, x) = x^3 + 3x^2 + 3x$, one has $a_1 = -3$, $a_2 = 3$, $a_3 = -9$. Now, by induction on n one can easily see that $\operatorname{ord}_3 a_n \geq \lceil \frac{n}{3} \rceil$. Suppose that $a_n = (-1)^n 3^{\lceil \frac{n}{3} \rceil} b_n$. By Lemma 3 it follows that for every n, $n \geq 4$ the following hold,

$$b_n = \begin{cases} 3b_{n-1} - 3b_{n-2} + b_{n-3}, & \text{if } n \equiv 0 \pmod 3; \\ b_{n-1} - b_{n-2} + b_{n-3}, & \text{if } n \equiv 1 \pmod 3; \\ 3b_{n-1} - b_{n-2} + b_{n-3}, & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{1}$$

It turns out that b_i, for every i, $1 \leq i \leq 30$ modulo 9, are as follows:
1, 1, 3, 3, 7, 6, 2, 7, 3, 7, 7, 3, 3, 4, 6, 5, 4, 3, 4, 4, 3, 3, 1, 6, 8, 1, 3, 1, 1, 3.

So for every n, $1 \leq n \leq 30$, $9 \nmid b_n$. By (1) and induction on t, it is easily seen that for every t, $t \geq 1$, $b_{t+27} \equiv b_t \pmod 9$. Hence for any positive integer n, $9 \nmid b_n$ or equivalently $\operatorname{ord}_3 b_n \leq 1$. By induction on n, and using (1), we find that if $n \equiv 0 \pmod 3$, $\operatorname{ord}_3 b_n = 1$, and $\operatorname{ord}_3 b_n \in \{0, 1\}$ for $n \equiv 1 \pmod 3$, and $\operatorname{ord}_3 b_n = 0$ for $n \equiv 2 \pmod 3$. This completes the proof. □

Remark 1. Since for every t, $t \geq 1$, $b_{t+27} \equiv b_t \pmod 9$, by considering b_n for $1 \leq n \leq 30$, we conclude that in the case $n \equiv 1 \pmod 3$,

$$\operatorname{ord}_3 a_n = \begin{cases} \lceil \frac{n}{3} \rceil + 1, & \text{if } n \in \{4, 13, 22\} \, (mod\, 27); \\ \lceil \frac{n}{3} \rceil, & \text{otherwise.} \end{cases}$$

Now, we prove our main result.

Theorem 1. *For every positive integer n, cycle C_n is $\mathcal{D}$-unique.*

Proof. The assertion is trivial for $n = 1, 2, 3$. Now, let $n \geq 4$. Let G be a simple graph with $D(G, x) = D(C_n, x)$. By Lemma 1, G is 2-regular and so it is a disjoint union of cycles $C_{n_1}, \ldots, C_{n_k}$. Hence, by Lemma 2, $D(G, x) = \prod_{i=1}^{k} D(C_{n_i}, x)$. Thus $n = n_1 + \cdots + n_k$ and by Lemma 4, $\lceil \frac{n}{3} \rceil = \lceil \frac{n_1}{3} \rceil + \cdots + \lceil \frac{n_k}{3} \rceil$. Therefore at least $k - 2$ numbers of $n_1, \ldots, n_k$, are divisible by 3. On the other hand,

$$\operatorname{ord}_3 D(C_n, -3) = \sum_{i=1}^{k} \operatorname{ord}_3 D(C_{n_i}, -3).$$

Now, by Lemma 6 it is easily seen that $k \leq 3$.

Now, let $\alpha_n := D(C_n, -1)$, for every positive integer n. Since $D(C_n, x) = \prod_{i=1}^{k} D(C_{n_i}, x)$, therefore $\alpha_n = \prod_{i=1}^{k} \alpha_{n_i}$. By Lemma 5, $\alpha_n \in \{-1, 3\}$. If $\alpha_n = 3$, then only one of the numbers $n_1, \ldots, n_k$ is divisible by 4, and therefore k is an odd number. If $\alpha_n = -1$, then for every i, $1 \leq i \leq k$, $\alpha_{n_i} = -1$, and thus k is an odd number. Since $k \leq 3$, then $k \in \{1, 3\}$.

It remains to show that $k \neq 3$. Let $\beta_n := D'(C_n, -1)$, for every $n \geq 1$, where $D'(C_n, x)$ is the derivative of $D(C_n, x)$ with respect to x. Then by the recursive formula given in Lemma 3 we conclude that for every n, $n \geq 4$, $\beta_n = -(\alpha_n + \beta_{n-1} + \beta_{n-2} + \beta_{n-3})$. Now, by induction on n and using Lemma 5, we have:

$$\beta_n = \begin{cases} -n, & \text{if } n \equiv 0 \pmod 4; \\ n, & \text{if } n \equiv 1 \pmod 4; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Let $\theta_n := D''(C_n, -1)$, for every n, $n \geq 1$. By Lemma 3, we conclude that for every $n \geq 4$, $\theta_n = -2\alpha_n - 2\beta_n - (\theta_{n-1} + \theta_{n-2} + \theta_{n-3})$. Now, by induction on n, using Lemma 5 and relation (2), we obtain the following:

$$\theta_n = \begin{cases} n(n-4)/2, & \text{if } n \equiv 0 \pmod 4; \\ -n(n-1)/2, & \text{if } n \equiv 1 \pmod 4; \\ n(n+2)/4, & \text{if } n \equiv 2 \pmod 4; \\ 0, & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{3}$$

Now, let $k = 3$. Thus

$$D(C_n, x) = \prod_{i=1}^{3} D(C_{n_i}, x). \tag{4}$$

By putting $x = -1$ in relation (4), we find that $\alpha_n = \alpha_{n_1}\alpha_{n_2}\alpha_{n_3}$. Since $n = n_1 + n_2 + n_3$, by Lemma 5, ten cases can be considered:

1) $n \equiv 0 \pmod 4$, $n_1 \equiv 0 \pmod 4$, $n_2 \equiv 1 \pmod 4$, $n_3 \equiv 3 \pmod 4$;

2) $n \equiv 0 \pmod 4$, $n_1 \equiv 0 \pmod 4$, $n_2 \equiv 2 \pmod 4$, $n_3 \equiv 2 \pmod 4$;

3) $n \equiv 1 \pmod 4$, $n_1 \equiv 1 \pmod 4$, $n_2 \equiv 1 \pmod 4$, $n_3 \equiv 3 \pmod 4$;

4) $n \equiv 1 \pmod 4$, $n_1 \equiv 1 \pmod 4$, $n_2 \equiv 2 \pmod 4$, $n_3 \equiv 2 \pmod 4$;

5) $n \equiv 1 \pmod 4$, $n_1 \equiv 3 \pmod 4$, $n_2 \equiv 3 \pmod 4$, $n_3 \equiv 3 \pmod 4$;

6) $n \equiv 2 \pmod 4$, $n_1 \equiv 1 \pmod 4$, $n_2 \equiv 2 \pmod 4$, $n_3 \equiv 3 \pmod 4$;

7) $n \equiv 2 \pmod 4$, $n_1 \equiv 2 \pmod 4$, $n_2 \equiv 2 \pmod 4$, $n_3 \equiv 2 \pmod 4$;

8) $n \equiv 3 \pmod 4$, $n_1 \equiv 1 \pmod 4$, $n_2 \equiv 1 \pmod 4$, $n_3 \equiv 1 \pmod 4$;

9) $n \equiv 3 \pmod 4$, $n_1 \equiv 1 \pmod 4$, $n_2 \equiv 3 \pmod 4$, $n_3 \equiv 3 \pmod 4$;

10) $n \equiv 3 \pmod 4$, $n_1 \equiv 2 \pmod 4$, $n_2 \equiv 2 \pmod 4$, $n_3 \equiv 3 \pmod 4$.

For instance, if Case 1 occurs, by derivative of two sides of the equality (4) and putting $x = -1$, we find that $\beta_n = \beta_{n_1}\alpha_{n_2}\alpha_{n_3} + \alpha_{n_1}\beta_{n_2}\alpha_{n_3} + \alpha_{n_1}\alpha_{n_2}\beta_{n_3}$. Now, by Lemma 5 and relation (2) we obtain that $n_3 = 2n_2$ which is impossible. Similarly, in cases 2, 3, 4, 5, 6, 8, and 9 we obtain a contradiction.

If the Case 7 occurs then, by the second derivative of two sides of equality (4) and putting $x = -1$, we conclude that:

$$\theta_n = \theta_{n_1}\alpha_{n_2}\alpha_{n_3} + \alpha_{n_1}\theta_{n_2}\alpha_{n_3} + \alpha_{n_1}\alpha_{n_2}\theta_{n_3} + 2\beta_{n_1}\beta_{n_2}\alpha_{n_3} + 2\beta_{n_1}\beta_{n_3}\alpha_{n_2} + 2\beta_{n_2}\beta_{n_3}\alpha_{n}$$

Now, by Lemma 5, and using relations (2) and (3) we find that, $n_1n_2 + n_1n_3 + n_2n_3 = 0$ which is impossible. Similarly, for case 10 we get a contradiction. Thus $k = 1$ and the proof is complete. □

By the following lemma and Theorem 1, the next corollary follows immediately.

Lemma 7. [2, Corollary 2] *If G is $\mathcal{D}$-unique, then for every m, $m \geq 1$, $G \vee K_m$ is $\mathcal{D}$-unique.*

Corollary 1. *For every two positive integers m and n, $K_m \vee C_n$ is $\mathcal{D}$-unique. In particular W_n is $\mathcal{D}$-unique.*

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