Adjacent vertex distinguishing edge-coloring of planar graphs with girth at least five*

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Abstract

An adjacent vertex-distinguishing edge coloring, *avd-coloring* for short, of a graph G is a proper edge coloring of G such that no pair of adjacent vertices are incident to the same set of colors. We use $\chi'_{avd}(G)$ to denote the avd-chromatic number of G which is the smallest integer k such that G has an avd-coloring with k-colors, and use $\Delta(G)$ to denote the maximum degree of G. In this paper, we prove that $\chi'_{avd}(G) \leq \Delta(G) + 4$ for every planar graph G without isolated edges whose girth is at least five. This is nearly a sharp bound since $\chi'_{avd}(C_5) = \Delta(C_5) + 3$.

Key words: Edge-coloring, vertex-distinguishing, planar graphs
AMS Subject Classifications (2000): 05C35, 05C75

1 Introduction

In this paper, we only consider simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use $N_G(v)$ to denote the set of neighbors of a vertex v in G, let $d_G(v) = |N_G(v)|$, and let $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$. A vertex of degree k (resp. at least k, at most k) is called a *k-vertex* (resp. k^+-vertex, k^--vertex).

A proper k-edge-coloring of a graph G is a mapping $\phi : E(G) \longrightarrow \{1, 2 \cdots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. Let

*Research partially supported by NSFC projects 10931003 and 11171160 and the Doctoral Fund of Ministry of Education of China

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ϕ be an edge coloring of G. We use $C_\phi(v)$ to denote the set of colors assigned to the edges incident with vertex v, i. e., $C_\phi(v) = \{\phi(uv) \ : \ uv \in E(G)\}$. A proper k-edge-coloring ϕ of G is said to be *an adjacent vertex distinguishing coloring*, or a *k-avd-coloring* for short, if $C_\phi(u) \neq C_\phi(v)$ whenever $uv \in E(G)$. The *avd-chromatic number* of G, denoted by $\chi'_{avd}(G)$, is the smallest integer k such that G has a k-avd-coloring.

The concept of avd-coloring was first introduced and studied by Zhang *et al* [8]. In [8], the authors determined the avd-chromatic number of paths, cycles, trees, complete graphs, and complete bipartite graphs. We say a graph to be *normal* if it has no isolated edges. Zhang *et al* proposed the following conjecture.

Conjecture 1 [8] *If G is a simple connected graph on at least 3 vertices, then $\Delta(G) \leq \chi'_{avd}(G) \leq \Delta(G) + 2$ unless G is the cycle of length 5.*

In [1], Balister *et al* showed that conjecture 1 is true for bipartite graphs and graphs with maximum degree three. In [4], Hatami proved that every normal graph G with $\Delta > 10^{20}$ has $\chi'_{avd}(G) \leq \Delta(G) + 300$. Bu *et al* [2] confirmed conjecture 1 for planar graphs of girth at least 6, where the *girth* $g(G)$ of a graph G is the length of a shortest cycle of G. Chen and Guo [3] determined completely the avd-chromatic number of hypercubes. Wang and Wang [7] studied the avd-colorings of K_4-minor free graphs, they proved that for each connected K_4-minor free graph G with maximum degree $\Delta \geq 4$, $\Delta \leq \chi'_{avd}(G) \leq \Delta + 1$, and $\chi'_{avd}(G) = \Delta + 1$ if and only if G contains no adjacent vertices of maximum degree whenever $\Delta \geq 5$. In [6], Wang and Wang introduced the connection between maximum average degree and avd-colorings, and proved the following results on graphs G with maximum degree $\Delta(G)$ and maximum average degree $mad(G)$: (1) if $mad(G) < 3$ and $\Delta(G) \geq 3$, then $\chi'_{avd}(G) \leq \Delta(G) + 2$; (2) if $mad(G) < \frac{5}{2}$ and $\Delta(G) \geq 4$, or $mad(G) < \frac{7}{3}$ and $\Delta(G) = 3$, then $\chi'_{avd}(G) \leq \Delta(G) + 1$; and (3) if $mad(G) < \frac{5}{2}$ and $\Delta(G) \geq 5$, then $\chi'_{avd}(G) = \Delta(G) + 1$ if and only if G contains adjacent vertices of maximum degree. Since for each planar graph G with girth g, $mad(G) < \frac{2g}{g-2}$, the above conclusion (1) generalizes the result of [2].

In this paper, we focus on the avd-coloring of normal plane graph with girth $g(G) \geq 5$, and prove the following result.

Theorem 1 *If G is a normal plane graph with girth $g(G) \geq 5$, then $\chi'_{avd}(G) \leq \Delta(G) + 4$.*

Note that C_5 (the cycle of length 5) has avd-chromatic number 5, the bound of Theorem 1 is almost sharp.

Let G be a plane graph. We use $F(G)$ to denote the face set of G. Let f be a face of G. The *boundary* of f, denoted by $\partial(f)$, is the closed walk around f. The *degree* of f, denoted by $d_G(f)$, is the number of edges in its boundary, where each cut edge is counted twice. If $\partial(f) = u_1u_2\cdots u_nu_1$, we simply write $f = [u_1u_2\cdots u_n]$. It is certain that every k-vertex is incident with at most k faces.

In Section 2, we will discuss the structural properties of a counterexample to Theorem 1. The proof of Theorem 1 is completed in Section 3 by applying Euler formula and discharging technique.

2 Some Lemmas

Let G be a counterexample to the theorem such that $|E(G)| + |V(G)|$ is as small as possible. Let H be the graph obtained from G by removing all 1-vertices of G, i. e., $H = G\backslash\{v \in V(G), d_G(v) = 1\}$. Clearly, both G and H are connected.

Lemma 1 *H has the following properties.*

(1) $\delta(H) \geq 2$.

(2) *For every $v \in V(H)$, if $2 \leq d_H(v) \leq 3$, then $d_G(v) = d_H(v)$.*

(3) [5] *Let $uvwx$ be a path in H such that $d_H(v) = d_H(w) = 2$, then $d_G(u) = d_H(u)$ and $d_G(x) = d_H(x)$.*

Proof. The third statement is from [5], we need only to prove (1) and (2).

We prove (1) first. Suppose to the contrary that $\delta(H) \leq 1$. If $\delta(H) = 0$, then it is obvious that H is K_1 and G is a star $K_{1,n-1}$, where $n = |V(G)| \geq 2$. Since G is normal, $\chi'_{avd}(G) = \Delta(G)$, a contradiction. So, we suppose that $\delta(H) = 1$. Let u be a 1-vertex in H and v be the neighbor of u in H. Clearly, $d_G(u) = k \geq 2$. Let $u_i \in N_G(u)\backslash\{v\}$ for $1 \leq i \leq k-1$, and let $G' = G\backslash\{u_{k-1}\}$. By the minimality of G, G' admits a $(\Delta(G)+4)$-avd-coloring ϕ using the color set $C = \{1, 2, 3, \cdots, \Delta(G)+4\}$. Without loss of generality, we may assume that $\phi(uv) = 1$ and $\phi(uu_i) = i+1$ for $1 \leq i \leq k-2$. We color uu_{k-1} with k, either we are done, or $C_\phi(v) = \{1, 2, \cdots, k-1, k\}$. If $C_\phi(v) = \{1, 2, \cdots, k-1, k\}$, then we recolor uu_{k-1} with $k+1$, and get a $(\Delta(G)+4)$-avd-coloring of G, a contradiction.

Now, we prove (2). Suppose to the contrary that $d_G(v) \neq d_H(v)$ for every $v \in V(H)$ with $2 \leq d_H(v) \leq 3$. To complete the proof, we need to consider the following two cases.

Case 1. Let $d_G(v) \neq d_H(v)$ with $d_H(v) = 3$. Then we may assume that $d_G(v) = k \geq 4$. Let v_1, v_2 and v_3 be three neighbors of v in H and v_4, $\cdots$, v_k be the $(k-3)$ 1-vertices adjacent to v in G. Consider

$G' = G - vv_k$. By the minimality of G, G' admits a $(\Delta(G)+4)$-avd-coloring ϕ using the color set $C = \{1,2,3,\cdots,\Delta(G)+4\}$. Without loss of generality, we may assume that $\phi(vv_i) = i$ for $1 \le i \le k-1$. We color vv_k with k. If a neighbor of v, say v_1, verifies $C_\phi(v_1) = \{1,2,\cdots,k\}$, then we recolor vv_k with $k+1$. Otherwise this means that the obtained coloring is still an avd-coloring, we can extend it to G. If we observe that $C_\phi(v_2) = \{1,2,\cdots,k-1,k+1\}$, then we color vv_k with $k+2$, either we are done or verifies $C_\phi(v_3) = \{1,2,\cdots,k-1,k+2\}$. Finally, we color vv_k with $k+3$, the obtained coloring is a $(\Delta(G)+4)$-avd-coloring of G, a contradiction.

Case 2. Let $d_G(v) \neq d_H(v)$ with $d_H(v) = 2$. By repeating above procedure, we can show that if $d_H(v) = 2$, then $d_G(v) = 2$. This proves (2). ∎

A 2-vertex is called *bad* if it is adjacent to a 2-vertex, otherwise we call it *good*. The following Lemma 2 from [6] says that in graph H every 2-vertex is adjacent to at least one 3^+-vertex.

Lemma 2 [6] *There is no path uvw in H such that $d_H(u) = d_H(v) = d_H(w) = 2$.*

Lemma 3 *Let uv be an edge with $d_G(u) = d_G(v) = 2$. Then, for any $(\Delta(G)+4)$-avd-coloring ϕ of $G - uv$, the edge incident with u receives the same color as the edge incident with v.*

Proof. Let x be the neighbor of u different from v, and let y be the neighbor of v different from u. If $\phi(ux) \neq \phi(vy)$, then we color uv with $\alpha \in \{1,2,\cdots,\Delta(G)+4\} \backslash \{\phi(ux), \phi(vy)\}$, and thus extend ϕ to G, contradicting the choice of G. ∎

Next lemma shows that H contains no adjacent 3-vertex and 2-vertex.

Lemma 4 *There is no 3-vertex in H adjacent to 2-vertices.*

Proof. Suppose to the contrary that H contains a 3-vertex u adjacent to a 2-vertex u_1. By Lemma 1(2), $d_H(u_1) = d_G(u_1) = 2$, and $d_G(u) = d_H(u) = 3$. Let u_2 and u_3 be the remaining neighbor of u. Let v_1 be the neighbor of u_1 different from u. We consider two possible cases as follows.

Case 1. $d_H(v_1) = 2$. Let w_1 be the neighbor of v_1 different from u_1. Clearly, by Lemma 1(2) and (3), $d_H(v_1) = d_G(v_1) = 2$. Let $G' = G - u_1v_1$. Then, by the minimality of G, G' admits a $(\Delta(G)+4)$-avd-coloring ϕ using the color set $C = \{1,2,3,\cdots,\Delta(G)+4\}$. By Lemma 3, we may suppose $\phi(uu_1) = \phi(v_1w_1) = 1$ and $\phi(uu_i) = i$ for $2 \le i \le 3$. We first erase the color of uu_1. Since $\Delta(G)+4 \geq 7$ ($\Delta(G) \geq 3$), we can always find a color different from 1, say α, to recolor uu_1 such that $C_\phi(u) \neq C_\phi(u_2)$ and $C_\phi(u) \neq C_\phi(u_3)$. Now, we can color u_1v_1 with a color different from 1 and α to get a $(\Delta(G)+4)$-avd-coloring of G, a contradiction.

Case 2. $d_H(v_1) \geq 3$. Let $G' = G - uu_1$. By the minimality of G, G' admits a $(\Delta(G)+4)$-avd-coloring ϕ. Without loss of generality, we may assume that $\phi(uu_i) = i$ for $2 \leq i \leq 3$ and $\phi(v_1u_1) = 1$. Since $\Delta(G)+4 \geq 7$ ($\Delta(G) \geq 3$), we can always find a color different from 1, 2, and 3, say α, to color uu_1 such that $C_\phi(u) \neq C_\phi(u_2)$ and $C_\phi(u) \neq C_\phi(u_3)$. This result in a $(\Delta(G)+4)$-avd-coloring of G, a contradiction. ∎

Below Lemma 5 from [1] shows that we can suppose that $\Delta(G) \geq 4$.

Lemma 5 [1] *If G is a normal graph with $\Delta(G) = 3$, then $\chi'_{avd}(G) \leq 5$.*

Now we show for a 3-vertex v of H, $N_H(v)$ contains at most one 3-vertex.

Lemma 6 *There is no 3-vertex in H adjacent to two 3-vertices.*

Proof. By contradiction, suppose that H contains a 3-vertex x adjacent to two 3-vertices x_1 and x_2. By Lemma 1(2), $d_G(x) = d_H(x) = 3$ and $d_G(x_i) = d_H(x_i) = 3$ for $1 \leq i \leq 2$. Let x_3 be the remaining neighbor of x. If $d_H(x_3) \geq 4$, then any changes of the color on the edges incident with x does not destroy the fact that $C_\phi(x) \neq C_\phi(x_3)$. So we suppose $d_H(x_3) = 3$. Let y_i and z_i be the remaining neighbors of x_i for $1 \leq i \leq 3$. By Lemma 5, we have nothing to do if $\Delta(H) = 3$. Thus we may assume that $\Delta(H) \geq 4$. Let $G' = G - xx_1$, By the minimality of G, G' has a $(\Delta(G)+4)$-avd-coloring ϕ using the color set $C = \{1, 2, 3, \cdots, \Delta(G)+4\}$. Without loss of generality, we may assume that $\phi(xx_i) = i$ for $2 \leq i \leq 3$. Now, we consider several possible cases as follows.

Case 1. $\{\phi(x_1y_1), \phi(x_1z_1)\} \bigcap \{\phi(xx_2), \phi(xx_3)\} = \emptyset$. Without loss of generality, we may assume that $\phi(x_1y_1) = 4$ and $\phi(x_1z_1) = 5$. Since $\Delta(G)+4 \geq 8$, we can always choose a color, say 8, to color xx_1 to extend ϕ to G such that $C_\phi(x) \neq C_\phi(x_1)$, $C_\phi(x) \neq C_\phi(x_2)$, $C_\phi(x_1) \neq C_\phi(y_1)$ and $C_\phi(x_1) \neq C_\phi(z_1)$. If $C_\phi(x) \neq C_\phi(x_3)$, we are done. Otherwise, we have $C_\phi(x) = C_\phi(x_3) = \{2, 3, 8\}$. In this case, we erase the color of xx_2 and recolor it with 1. If a neighbor of x_2, say y_2, verifies $C_\phi(y_2) = \{1, 3, 7\}$, then we color xx_2 with 4. If the other neighbor z_2 of x_2, verifies $C_\phi(z_2) = \{3, 4, 7\}$, then we color xx_2 with 5. Now, we observe that $C_\phi(x_1) = \{8, 4, 5\}$, $C_\phi(x_3) = \{8, 2, 3\}$, $C_\phi(x_2) = \{3, 5, 7\}$ and $C_\phi(x) = \{8, 3, 5\}$. So the resulted coloring is an $(\Delta(G)+4)$-avd-coloring of G.

Case 2. $|\{\phi(x_1y_1), \phi(x_1z_1)\} \bigcap \{\phi(xx_2), \phi(xx_3)\}| = 1$. Without loss of generality, we may assume that $\phi(x_1y_1) = 2$ and $\phi(x_1z_1) = 4$. Since $\Delta(G) + 4 \geq 8$ ($\Delta(G) \geq \Delta(H) \geq 4$), we can always choose a color, say 8, to color xx_1 to extend ϕ to G such that $C_\phi(x) \neq C_\phi(x_1)$, $C_\phi(x) \neq C_\phi(x_2)$, $C_\phi(x) \neq C_\phi(x_3)$, $C_\phi(x_1) \neq C_\phi(y_1)$ and $C_\phi(x_1) \neq C_\phi(z_1)$. The extended coloring is an $(\Delta(G) + 4)$-avd-coloring of G, a contradiction.

Case 3. $|\{\phi(x_1y_1), \phi(x_1z_1)\} \bigcap \{\phi(xx_2), \phi(xx_3)\}| = 2$. Without loss of generality, we may assume that $\phi(x_1y_1) = 2$ and $\phi(x_1z_1) = 3$. Then we first erase the color of xx_2. If $\{\phi(x_2y_2), \phi(x_2z_2)\} \bigcap \{3\} = \emptyset$, then we may assume that $\phi(x_2y_2) = 4$ and $\phi(x_2z_2) = 5$. We recolor xx_2 with 1. However, it is possible that $C_\phi(y_2) = \{1, 4, 5\}$. Then we color xx_2 with 6. Similarly, it is also possible that $C_\phi(z_2) = \{4, 5, 6\}$. Thus we color xx_2 with 7, and hence $|\{\phi(xx_2), \phi(xx_3)\} \bigcap \{\phi(x_1y_1), \phi(x_1z_1)\}| = 1$, we get back to case 2. If $|\{\phi(x_2y_2), \phi(x_2z_2)\} \bigcap \{3\}| = 1$, then we may assume that $\phi(x_2y_2) = 3$ and $\phi(x_2z_2) = 4$, the following discussion is similar to the above procedure. Finally, we can conclude that ϕ can be extended to a $(\Delta(G) + 4)$-avd-coloring of G, a contradiction. This proves Lemma 6. ∎

With almost the same arguments as used in the proof of Lemma 4, one can verify the following conclusion. We omit its proof.

Lemma 7 *Each 4-vertex is not adjacent to any 2-vertex, and each 5-vertex is adjacent to at most one 2-vertex. Moreover, if a 5-vertex u is adjacent to a 2-vertex, then u is not adjacent to any 3-vertex.*

Intuitively from the definition of avd-coloring, adjacent vertices of distinct degree make little influence on the avd-chromatic number. We show that if a vertex is adjacent to two 2-vertices, then most neighbors of it would have the same degree as itself.

Lemma 8 *Let d, k, l be positive integers, and let D be a graph of maximum degree d such that D is not $(d + l)$-avd-colorable with minimum $|V(D)| + |E(D)|$. Let H is the graph obtained from D by removing all vertices of degree 1 in D. If a k-vertex u of H is adjacent to two 2-vertices, then $N_H(u)$ contains at most $(k - l - 1)$ vertices of degree not equal to k.*

Proof. By contradiction, suppose that $N_H(u)$ contains at least $k-l$ vertices of degree not equal to k, and hence $N_H(u)$ contains at most l vertices of the same degree as u. Let $N_H(u) = \{u_1, u_2, \cdots, u_k\}$ with $d_H(u_1) = d_H(u_2) = 2$ and $d_H(u_i) = k$ for $k - l + 1 \leq i \leq k$, and let v_1 be the neighbor of u_1 different from u. We consider two cases.

Case 1. $d_H(v_1) = 2$.

Let w_1 be the neighbor of v_1 different from u_1. Clearly, by Lemma 1(2) and (3), $d_G(u) = d_H(u)$ and $d_H(v_1) = d_G(v_1) = 2$. Let $G' = G - u_1v_1$. By the minimality of G, G' admits a $(d+l)$-avd-coloring ϕ with the color set $C = \{1, 2, 3, \cdots, d+l\}$. By Lemma 3, we may suppose $\phi(uu_1) = \phi(v_1w_1) = 1$ and $\phi(uu_i) = i$ for $2 \leq i \leq k$. Without loss of generality, we may assume that $C_\phi(u_{k-l+i}) = \{2, 3, \cdots, k, k+i\}$ for $1 \leq i \leq l$. Let v_2 be the neighbor of u_2 different from u. In this case, we first erase the colors of uu_1 and uu_2. If u_2 is a bad 2-vertex, then we recolor uu_2 with $\alpha \in C \backslash \{2, 3, \cdots, k\} \bigcup C_\phi(v_2)$ and recolor uu_1 with $\beta \in C \backslash \{1, 2, 3, \cdots, k, \alpha\}$. Now $C_\phi(u) \neq C_\phi(u_i)$ for $k-l+1 \leq i \leq k$ and $\phi(uu_1) = \beta \neq \phi(v_1w_1)$. If u_2 is a good 2-vertex, then we color uu_2 with $\alpha \in C \backslash \{2, 3, \cdots, k\} \bigcup \{\phi(u_2v_2)\}$ and recolor uu_1 with $\beta \in C \backslash \{1, 2, 3, \cdots, k, \alpha\}$. Now $C_\phi(u) \neq C_\phi(u_i)$ for $k-l+1 \leq i \leq k$ and $\phi(uu_1) = \beta \neq \phi(v_1w_1)$. So we can color u_1v_1 with $c \in C \backslash \{k+l, 1\}$, and get a $(d+l)$-avd-coloring of G, a contradiction.

Case 2. $d_H(v_1) \geq 3$.

Suppose that $d_G(v) = d_H(v)$. We may assume by symmetry that u_2 has no neighbor of degree 2. Let $G' = G - uu_1$. By the minimality of G, G' admits a $(d+l)$-avd-coloring ϕ. Without loss of generality, assume that $\phi(uu_i) = i$ for $2 \leq i \leq k$ and $\phi(u_1v_1) = 1$. If $C_\phi(u_{k-l+i}) = \{2, 3, \cdots, k, k+i\}$ for $1 \leq i \leq l$, then we erase the color of uu_2. Since u_2 has no neighbor of degree 2, we can recolor uu_2 with $\alpha \in C \backslash \{2, 3, \cdots, k, k+l\} \bigcup \{\phi(u_2v_2)\}$. Now, we can color uu_1 with $\beta \in C \backslash \{1, 2, 3, \cdots, k, \alpha\}$ such that $C_\phi(u) \neq C_\phi(u_i)$ for $k-l+1 \leq i \leq k$ and $\phi(uu_1) = \beta \neq \phi(u_1v_1)$, and get a $(d+l)$-avd-coloring of G, a contradiction.

Now we suppose $d_G(u) = m > k$. Let $v_{k+1}, \cdots, v_l$ be the $(m-k)$ 1-vertices adjacent to u in G. Let $G' = G - vv_m$. Then, G' admits a $(d+l)$-avd-coloring ϕ. Without loss of generality, we assume that $\phi(uu_i) = i$ for $1 \leq i \leq m-1$. We color uu_m with $m+i-1$, either we are done or verifies $C_\phi(u_{k-l+i}) = \{1, 2, \cdots, m-1, m+i-1\}$, where $1 \leq i \leq l$. In this case, we recolor uu_m with $m+l$, and get a $(d+l)$-avd-coloring of G, a contradiction. ∎

Applying Lemma 8 with $l = 4$ to G, we have the following two conclusions.

Corollary 1 *Let v be a k-vertex of H. If v of H is adjacent to two 2-vertices, then $N_H(v)$ contains at most $k-5$ vertices of degree not equal to k.*

Corollary 2 *There is no k-vertex in H adjacent to $(k-4)$ 2-vertices for $k \geq 6$.*

Let f be a 5-face of G. By Lemma 2, a 2-vertex incident with f is adjacent to at most one 2-vertex. We will show that $\partial(f)$ has at most two 2-vertices.

Lemma 9 *H does not contains a face $f = [x_1x_2 \cdots x_5]$ such that $d_H(x_i) = 2$ for all x_i except x_1 and x_3.*

Proof. Suppose to the contrary that H contains a face $f = [x_1x_2 \cdots x_5]$ such that $d_H(x_i) = 2$ for all x_i except x_1 and x_3. By Lemma 1(2) and (3), we know that $d_G(x_i) = d_H(x_i)$ for $1 \leq i \leq 5$. Let $G' = G - x_4x_5$. By the minimality of G, G' admits a $(\Delta(G) + 4)$-avd-coloring ϕ using the color set $C = \{1, 2, 3, \cdots, \Delta(G) + 4\}$. If $\phi(x_1x_5) \neq \phi(x_3x_4)$, then we can color x_4x_5 with $c \in C \backslash \{\phi(x_1x_5), \phi(x_3x_4)\}$, and get an *avd* – *coloring* of G. If $\phi(x_1x_5) = \phi(x_3x_4)$, without loss of generality, we assume that $\phi(x_1x_5) = \phi(x_3x_4) = 1$, $\phi(x_1x_2) = 2$ and $\phi(x_2x_3) = 3$. Now, we exchange the color between x_3x_4 and x_2x_3 such that $\phi(x_4x_3) = 3$ and $\phi(x_2x_3) = 1$, and get a new $(\Delta(G)+4)$-avd-coloring of G'. Then we can color x_4x_5 with a color $\alpha \in C \backslash \{1, 3\}$, and get a $(\Delta(G)+4)$-avd-coloring of G, a contradiction. ∎

We still need a lemma from [2]. It describes the distribution of 2-vertices around a 6-face.

Lemma 10 [2] *H does not contain a face $f = [x_1x_2 \cdots x_6]$ such that $d_H(x_i) = 2$ for all x_i except x_1 and x_4.*

3 Proof of Theorem 1

Now, we are ready to prove Theorem 1.

First, we define a weight function w as follow. For each $v \in V(H)$, let $w(v) = \frac{6}{5}d_H(v) - 4$. For each $f \in F(H)$, let $w(f) = \frac{4}{5}d_H(f) - 4$. Then, $\sum_{v \in V(H)} (\frac{6}{5}d_H(v) - 4) + \sum_{f \in F(H)} (\frac{4}{5}d_H(f) - 4) = -8$, which follows from the Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$ and the Degree Sum Formula $\sum_{v \in V(H)} d_H(v) = \sum_{f \in F(H)} d_H(f) = 2|E(H)|$. Next, we will design a discharging rule and redistribute weights between elements of $V(H) \cup F(H)$. Once the discharging procedure is completed, we get a new weight function w'. However, the sum of all weights is kept fixed during the discharging process. We will show that $w'(x) \geq 0$ for all $x \in V(H) \cup F(H)$. This leads to the contradiction $0 \leq -8$, and hence proves the nonexistence of the counterexample.

A face $f = [u_1u_2u_3\cdots]$ of G is called a weak face belonging to u_2 if $d_H(u_2) \geq 5$ and either $d_H(u_1) = 2$ or $d_H(u_3) = 2$. The discharging rule are defined as follows.

(R_1) If v is a 2-vertex incident with a face f, then f gives $\frac{4}{5}$ to v for each occurrence of v in $\partial(f)$.

(R_2) If v is a 4^+-vertex, then v gives $\frac{1}{5}$ to each adjacent 3-vertex.

(R_3) If v is a 5^+-vertex, then v gives $\frac{4}{5}$ to each weak face belonging to v.

Now, we calculate ω'. Let v be a d-vertex of H. Then $d \geq 2$.

If $d = 2$, then $w'(v) = \frac{6}{5} \times 2 - 4 + \frac{4}{5} \times 2 = 0$ by (R_1) and Lemma 2.

If $d = 3$, then $w'(v) \geq \frac{6}{5} \times 3 - 4 + \frac{1}{5} \times 2 = 0$ by (R_2) and Lemmas 4 and 6.

If $d = 4$, then $w'(v) \geq \frac{6}{5} \times 4 - 4 - \frac{1}{5} \times 4 = 0$ by (R_2) and Lemma 7.

If $d = 5$, then by Lemma 7, v is adjacent to at most one 2-vertex, and is not adjacent to any 3-vertex if it is adjacent to a 2-vertex. So, there are at most two weak faces belonging to v, and thus $w'(v) > \frac{6}{5} \times 5 - 4 - \frac{4}{5} \times 2 > 0$ by (R_3).

Suppose that $d \geq 6$. For $i \in \{2, 3\}$, let n_i be the number of i-vertices adjacent to v. By Corollaries 1 and 2, $n_2 + n_3 \leq d - 5$.

If $d = 6$, then $n_2 + n_3 \leq 1$, and hence there are at most two weak faces belonging to v. So, $w'(v) \geq \frac{6}{5} \times 6 - 4 - \frac{4}{5} \times 2 - \frac{1}{5} \times 5 > 0$ by (R_2) and (R_3).

If $d = 7$, then $n_2 + n_3 \leq 2$, and there are at most four weak faces belonging to v. Therefore, $w'(v) \geq \frac{6}{5} \times 7 - 4 - \frac{4}{5} \times 4 > 0$ by (R_3).

If $d = 8$, then $n_2 + n_3 \leq 3$, and there are at most six weak faces belonging to v. Hence, $w'(v) \geq \frac{6}{5} \times 8 - 4 - \frac{4}{5} \times 6 > 0$ by (R_3).

If $d = 9$, then $n_2 + n_3 \leq 4$, there are at most eight weak faces belonging to v. Hence, $w'(v) \geq \frac{6}{5} \times 9 - 4 - \frac{4}{5} \times 8 > 0$ by (R_3).

If $d = 10$, then the number of faces incident with v is at most 10. If v has to give $\frac{4}{5}$ to each face incident with it, then $n_2 = 5$. Since $n_3 = 0$ whenever $n_2 = 5$, $w'(v) \geq \frac{6}{5} \times 10 - 4 - \frac{4}{5} \times 10 = 0$.

Let v be an 11-vertex. Similarly, if the number of weak faces incident with v is 11, then $n_2 = 6$, and $n_2 = 6$ implies that $n_3 = 0$. So, $w'(v) \geq \frac{6}{5} \times 11 - 4 - \frac{4}{5} \times 11 > 0$.

If $d = 12$, then v is incident with at most 12 faces. If v has to give $\frac{4}{5}$ to each face incident with it, then $n_2 = 6$. Note that $n_2 + n_3 \leq 6$. We have $w'(v) \geq \frac{6}{5} \times 12 - 4 - \frac{4}{5} \times 12 - \frac{1}{5} > 0$ by (R_2) and (R_3).

If $d \geq 13$, and v has to give $\frac{4}{5}$ to each face incident with it, then $n_2 \geq \lceil \frac{k}{2} \rceil$. Now, $n_3 \leq \lfloor \frac{d}{2} \rfloor$, and thus $w'(v) \geq \frac{6}{5} \times d - 4 - \frac{4}{5} \times d - \frac{1}{5} \times \lfloor \frac{d}{2} \rfloor \geq 0$ for $d \geq 13$ by (R_2) and (R_3).

To complete the proof of Theorem 2, it is enough to show that the value $w'(f)$ of each face f is nonnegative. Let f be a k-face, and let $\xi(f)$ denote the number of occurrences of 2-vertices in $\partial(f)$. Clearly, $k \geq 5$ since G has girth at least 5. If $k \geq 7$, then by Lemma 2, we have

$$\xi(f) \leq \begin{cases} 2m, & \text{for } k = 3m & (3.1) \\ 2m, & \text{for } k = 3m+1 & (3.2) \\ 2m+1, & \text{for } k = 3m+2 & (3.3) \end{cases}$$

For every vertex $v \in \partial(f)$, we use $\beta(f)$ to denote the number of vertices in $\partial(f)$ which have to give $\frac{4}{5}$ to f. By lemmas 2, 4 and 7, $\beta(f) \geq \lceil \frac{k}{3} \rceil$. Therefore, according to the discharging rules we know that $w'(f) \geq \frac{4}{5} \times k - 4 - (\lfloor \frac{k}{3} \rfloor \times 2 + 1) \times \frac{4}{5} + \lceil \frac{k}{3} \rceil \times \frac{4}{5} \geq 0$ for $k \geq 9$. So, we suppose that $k \leq 8$

If $k = 5$, then there are at most two 2-vertices in $\partial(f)$ by Lemma 9. Moreover if $\xi(f) = 2$, then $\beta(f) \geq 2$. Hence $w'(f) \geq \frac{4}{5} \times 5 - 4 - \frac{4}{5} \times 2 + \frac{4}{5} \times 2 = 0$.

If $k = 6$, then there are at most three 2-vertices in $\partial(f)$ by Lemma 10. Moreover if $\xi(f) = 3$, then $\beta(f) = 3$. Hence $w'(f) \geq \frac{4}{5} \times 6 - 4 - \frac{4}{5} \times 3 + \frac{4}{5} \times 3 > 0$.

If $k = 7$, then there are at most four 2-vertices in $\partial(f)$ by (3.2). Moreover if $\xi(f) = 4$, then $\beta(f) = 3$. Hence $w'(f) \geq \frac{4}{5} \times 7 - 4 - \frac{4}{5} \times 4 + \frac{4}{5} \times 3 = \frac{4}{5} > 0$.

If $k = 8$, then we know that there are at most five 2-vertices in $\partial(f)$ by (3.3). Moreover if $\xi(f) = 5$, then $\beta(f) = 3$. Hence $w'(f) \geq \frac{4}{5} \times 8 - 4 - \frac{4}{5} \times 5 + \frac{4}{5} \times 3 = \frac{4}{5} > 0$.

This completes the proof. ∎

4 Acknowledgements

The authors would like to thank the referee for his/her thorough reviewing with useful suggestions and comments to the paper.

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