On cycle resonant outerplanar graphs *

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Abstract

A connected graph G is said to be k-cycle resonant if, for $0 \leq t \leq k$, for any t disjoint cycles $C_1, C_2, \cdots, C_t$ in G, there is a perfect matching in $G - \cup_{i=1}^{t} V(C_i)$. A connected graph G is said to be cycle resonant if G is k^*-cycle resonant and k^* is the maximum number of disjoint cycles in G. In this paper we prove that for outerplane graphs, 2-cycle resonant is equivalent to cycle resonant and establish a necessary and sufficient condition for an outerplanar graph to be cycle resonant. We also discuss the structure of 2-connected cycle resonant outerplane graphs. Let $\Phi(G)$ denote the number of perfect matchings in G. For any 2-connected cycle resonant outerplane graph G with k chords, we get $k+2 \leq \Phi(G) \leq 2^k + 1$ and give the extremal graphs for the equalities in the inequalities.

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1 Introduction

We consider only finite, simple graphs in this article. In chemical graph theory, a hexagonal system denotes the carbon atom skeleton graph of a benzenoid hydrocarbon, which is a 2-connected plane graph whose every interior face is bounded by a regular hexagon. A hexagonal system H is normal if each of its edges lies in a perfect matching. Fuji Zhang and Rongsi Chen [14] proved that a hexagonal system H is normal if and only if every hexagon of H is resonant, that is, for any hexagon s of H there is a perfect matching M of H such that s is an M-alternating cycle. A hexagonal system H is said to be k-coverable if H contains at least k disjoint hexagons and, for any t disjoint hexagons of H, $1 \leq t \leq k$, there is a perfect matching M such that the t disjoint hexagons are M-alternating cycle.

As a variation and generalization of k-coverable hexagonal systems, Xiaofeng Guo and Fuji Zhang [1] introduced the concept of k-cycle resonant graphs and gave a necessary and sufficient condition for a graph to be k-cycle resonant.

A connected graph G is said to be k-cycle resonant if, for any t disjoint cycles $C_1, C_2, \cdots, C_t$ in G, $0 \leq t \leq k$, there is a perfect matching in $G - \cup_{i=1}^{t} V(C_i)$. A connected graph G is said to be cycle resonant if G is k^*-cycle resonant and k^* is the maximum number of disjoint cycles in G.

Theorem 1.1 *(Guo and Zhang [3]) A connected graph G with at least k disjoint cycles is k-cycle resonant if and only if G is a bipartite graph with perfect matchings and, for $1 \leq t \leq k$ and any t disjoint cycles $C_1, C_2, \cdots, C_t$ in G, $G - \cup_{i=1}^{t} V(C_i)$ contains no odd component.*

The concept of k-cycle resonant is useful in chemistry. It was shown in [1] that in the hexagonal systems with h hexagons obtained from a same parent hexagonal system with $h-1$ hexagons, k^*-cycle resonant systems have greater resonance energies than 1-cycle resonant systems, and 1-cycle resonant systems have greater resonance energies than hexagonal systems not being 1-cycle resonant.

Meanwhile, in the investigation of matching theory, Lovasz et al. [7] - [11] introduced and investigated elementary graphs, 1-extendable graphs and n-extendable graphs etc. A graph G is said to be elementary if there is a connected spanning subgraph G' of G such that any edge of G' belongs to a perfect matching of G. A graph G is said to be n-extendable if any n independent edges of G are contained in some perfect matching of G. We can similarly call k-cycle resonant graphs as k-cycle extendable graphs.

Let $\mathcal{G}_k$ denote the set of all the 2-connected cycle resonant outerplane graphs with k chords and $\Phi(G)$ denote the number of the perfect matchings in a graph G.

In the present paper, we prove that, for outerplane graphs, 2-cycle resonant is equivalent to cycle resonant, and establish a necessary and sufficient condition for an outerplanar graph to be cycle resonant. In addition, we discuss the structure of 2-connected cycle resonant outerplane graphs, prove that for any graph $G \in \mathcal{G}_k$, $k+2 \le \Phi(G) \le 2^k+1$, and give the extremal graphs for the equalities in the inequalities.

2 Some related results on k-cycle resonant graphs

A block of a connected graph G is either a maximal 2-connected subgraph of G or a cut edge of G.

Theorem 2.1 *(Guo and Zhang [3]) Let G be a k-cycle resonant graph. Then*
(i) for a 2-connected block G' of G with the maximum number k^* of disjoint cycles, if $k^* \le k$, G' is k^*-cycle resonant, otherwise G' is k-cycle resonant;
(ii) the forest induced by the vertices of G not in any 2-connected block of G has a unique perfect matching.

The above theorem implies that a non-2-connected k-cycle resonant graph can be constructed from some disjoint 2-connected k^* (or k)-cycle resonant graphs and a forest with a perfect matching by adding some edges between the 2-connected graphs and the forest so that the resultant graph is connected and the added edges are cut edges. Hence we need only consider 2-connected k-cycle resonant graphs.

Let $G=(V,E)$ be a graph, $V_1 \subset V$ and $E_1 \subset E$, denote by $G[V_1]$ and $G[E_1]$ the subgraph induced by V_1 and E_1 respectively. Let G be a connected graph and H a subgraph of G. A vertex in H is said to be an attachment vertex of H if it is incident with an edge in $E(G)\backslash E(H)$. A bridge B of H in G is either an edge in $E(G)\backslash E(H)$ with two end vertices in H, or a subgraph of G induced by all the edges in a connected component B' of $G-V(H)$ together with all the edges with an end vertex in B' and the other in H. The vertices in $V(B)\cap V(H)$ are also attachment vertices of B to H. A bridge with k attachment vertices is called a k-bridge.

The attachment vertices of a k-bridge B of a cycle C in G divide C into k edge-disjoint paths, called the segments of B. Two bridges of C avoid one another if all the attachment vertices of one bridge lie in a single segment of the other bridge, otherwise they overlap.

For a bipartite graph, we always color its vertices black and white so that adjacent vertices have different colors.

Theorem 2.2 *(Guo and Zhang [3]). A 2-connected graph G is planar 1-cycle resonant if and only if G is bipartite and, for any cycle C in G,* *any bridge of C has exactly two attachment vertices which have different colors.*

A plane graph is an embedding of a planar graph. We call the boundary of the unbounded face of a 2-connected plane graph G the outer cycle of G.

Lemma 2.3 *(Xu and Guo [13]) Let G be a 2-connected plane bipartite graph, C is the outer cycle of G. If any bridge of C is an odd-length path, then G is 1-cycle resonant.*

Theorem 2.4 *(Xu and Guo [13]) Let G be a 2-connected plane bipartite graph, then G is 1-cycle resonant if and only if any bridge of the outer cycle of G has exactly two different colored attachment vertices and for any maximal 2-connected subgraph H of any bridge B of C, the following conditions are satisfied:*
(1) H is 1-cycle resonant;
(2) H has exactly two different-colored attachment vertices u and v;
(3) u and v avoid any bridge of the outer cycle of H.

Construction 2.5 *(Xu and Guo [13]) Let $G_0, G_1, \cdots, G_r$ be 2-connected plane 1-cycle resonant graphs, in which all the bridges of the outer cycle C_0 of G_0 are paths. Let $P_0, P_1, \cdots, P_r$ be edge-disjoint paths of odd length on bridges of C_0, u_i and v_i be the end vertices of P_i, $i = 1, 2, \cdots, r$. Let u'_i, v'_i be two different-colored vertices on the outer cycle C_i of G_i which avoid any bridge of C_i. Replace P_i with G_i and identify u_i and u'_i, v_i and v'_i we get a new graph G. Then G is 1-cycle resonant.*

For a 2-connected subgraph B in G with exactly two attachment vertices, we call $G[E(G)\backslash E(B)]$ the complement of B in G, denoted by $\overline{B}$.

A path P of length greater than or equal to one in a graph G is said to be a chain if the degrees of the end vertices of P are not equal to 2 and the degree of any internal vertex of P is equal to 2 in G. The set of internal vertices of a chain P in G is denoted by $V_I(P)$.

A vertex u of a graph G is said to be cycle-related to another vertex v of G if u is contained in a 2-connected block of G and any cycle containing u must also contain v. If v is also cycle-related to u, then u and v are mutually cycle-related.

Theorem 2.6 *(Guo and Zhang [3]). A 2-connected graph G is planar 2-cycle resonant if and only if,*

(i) G is planar 1-cycle resonant,

(ii) for a chain P with even length and end vertices v_1 and v_2, $G - V_I(P)$ has exactly two blocks each of which is 2-connected and v_1 and v_2

are cycle-related to the common vertex of the two blocks,

(iii) for a chain P with odd length and end vertices v_1 and v_2 such that $G - V_I(P)$ is not 2-connected, either (a) $G - V_I(P)$ has exactly three blocks, each of which is 2-connected, and each of v_1 and v_2 is cycle-related to the other attachment vertex of the block containing it, and the attachment vertices of the third block are mutually cycle-related in the third block, or (b) any two 2-connected blocks of $G - V_I(P)$ are disjoint,

(iv) for a 2-connected subgraph B_1 of G with exactly two attachment vertices, if $\overline{B}_1$ is not 2-connected and every block of $\overline{B}_1$ is 2-connected, then $\overline{B}_1$ has exactly three blocks, say B_2, B_3, B_4, and the attachment vertices of each of B_1, B_2, B_3, B_4 are mutually cycle-related in the block.

Based upon the above theorem, Biao Zhao and Xiaofeng Guo gave a method for constructing any plane 2-cycle resonant graph from smaller plane 2-cycle resonant graphs and simple plane 1-cycle resonant graphs in [15] and later established a linear algorithm for recognizing plane 2-cycle resonant graphs in [16].

3 The resonance of outerplanar graphs

A graph is outerplanar if it has an embedding in the plane such that every vertex lies on the boundary of unbounded face. An outerplane graph is a planar embedding with every vertex on the boundary of the unbounded face. Let G be a 2-connected outerplane graph, then the boundary of the unbounded face of G is a Hamilton cycle C of G and every edge of G not on C is a chord of C. The chain of G is either a chord of C or a path contained in the cycle C. We give C a fixed orientation and for any vertex v, let v^+ and v^- denote the vertex that is successive and predecessor of v respectively. If u and v are two vertices on C, let $C[u, v]$ denote the segment on C between u and v along the given orientation, and let $C(u, v) = C[u^+, v^-]$. We consider $C[u, v]$ and $C(u, v)$ both as paths and as vertex sets. If f is a face of G, denote respectively by $E(f)$ and $V(f)$ the sets of the edges and the vertices on the boundary of f. The degree of a face f denoted by $d(f)$ is the number of edges with which f is incident, cut edges are counted twice.

An edge e in G is said to be subdivided when it is deleted and replaced by a path of length two connecting its ends, the internal vertex of this path is a new vertex. An edge e in G is said to be double-subdivided when it is deleted and replaced by a path of length three connecting its ends, the internal vertices of this path are two new vertices. Conversely, let $P = v_0v_1 \cdots v_t$ ($t \geq 3$) be a chain in G, P is said to be double-contracted if it is deleted and replaced by $P' = v_0v_3v_4 \cdots v_t$. The number of the perfect matchings in a graph G is denoted by $\Phi(G)$. The following properties for double-subdividing and double-contracting are obvious.

Lemma 3.1 *Let G be a graph with a perfect matching, e an edge in G, and $P = v_0 v_1 \cdots v_t$ $(t \geq 3)$ a chain in G. Let e be double-subdivided and the resultant graph be G', and let P be double-contracted and the resultant graph be G''. Then (i) $\Phi(G) = \Phi(G') = \Phi(G'')$; (ii) G is k-cycle resonant if and only if G' (resp. G'') is k-cycle resonant.*

Note that if every bridge of the outer cycle C of a 2-connected plane bipartite graph G is an odd-length path, then by the operations of successive double-contracting of the bridges of C, the final resultant graph is an outerplane (not necessarily simple) graph. Besides, from Construction 2.5 [13] we can see that any plane 1-cycle resonant graph can be constructed from outerplane 1-cycle resonant graphs by the operations of double-subdivision by the construction method in Construction 2.5. Therefore it is important to know the resonance property of the outerplane graphs.

In the following we will give a necessary and sufficient condition for a 2-connected outerplane graph to be 2-cycle resonant and get some properties of 2-cycle resonant 2-connected outerplane graphs.

Let G be a 2-connected outerplane graph and C the outer cycle of G. If G is 1-cycle resonant, then G is bipartite. Conversely, if G is bipartite, then any bridge of C is a chord and therefore an odd-length path, from Lemma 2.3, G is 1-cycle resonant. Thus we have the following theorem.

Theorem 3.2 *Let G be a 2-connected outerplane graph, then G is 1-cycle resonant if and only if G is bipartite.*

The following theorem indicates that for 2-connected outerplane graphs, 2-cycle resonant is equivalent to cycle resonant.

Theorem 3.3 *A 2-connected outerplane graph G is cycle resonant if and only if G is 2-cycle resonant.*

Proof. The necessity is obvious. We need only to prove the sufficiency.

Suppose that G is a 2-connected 2-cycle resonant outerplane graph, but is not cycle resonant, that is, there is a minimum integer number $k > 2$ such that G is $(k-1)$-cycle resonant, but not k-cycle resonant. By Theorem 1.1, there are k disjoint cycles in G, say $C_1, C_2, \cdots, C_k$, such that $G - \bigcup_{i=1,2,\cdots,k} V(C_i)$ has at least two odd components G_1, G_2.

We can assert that every C_i, $i = 1, 2, \cdots, k$, has at least one vertex adjacent to a vertex in G_j for $j = 1, 2$. Otherwise, G would not be $(k-1)$-cycle resonant, a contradiction.

But if both G_1 and G_2 are adjacent to every C_i, that is, $E(V(G_j), V(C_i)) \neq \phi$ for $i = 1, 2, \cdots, k$ and $j = 1, 2$, then G would not be an outerplane graph, again a contradiction.

The proof is thus complete. □

From the above theorem and by the light of Theorem 2.6 we can get some properties of the interior faces which turn out to be a sufficient condition for a connected outerplane bipartite graph to be cycle resonant.

Theorem 3.4 *Let G be a 2-connected cycle resonant outerplane graph that is not a cycle, C the outer cycle of G, and f an interior face of G. Then*

(i) If $E(f) \cap E(C) = \emptyset$, then $d(f) = 4$ and the degrees of the vertices incident with f are all four;

(ii) If $E(f) \cap E(C)$ contains a chain $P = C[x, y]$ of even length in C, then $d(x) = d(y) = 3$ and $E(f) = \{xv, yv\} \cup E(P)$, where v is vertex in $C(y^+, x^-)$;

(iii) If $E(f) \cap E(C)$ contains only one chain $P = C[x, y]$ of odd length, then either (a) the length of P is greater than one, xy is a chord in G and $E(f) = E(P) \cup \{xy\}$; or (b) there are two vertices u and w in $C(y^+, x^-)$ such that $E(f) = E(P) \cup \{yu, uw, wx\}$, $d(x) = d(y) = 3$, u and w are cycle-related in $G[C[u, w]]$;

(iv) If $E(f) \cap E(C)$ contains more than one chain on C, then the lengths of the chains contained in $E(f) \cap E(C)$ are all odd and any two chords contained in $E(f) \backslash E(C)$ have no common end vertex.

Proof. (i) Since G is 2-connected bipartite and the edges on the boundary of f form a cycle in G, $d(f)$ is not odd. Now if the boundary vertices of f are $v_1 v_2 \cdots v_t$ consecutively, then each $B_i = G[C[v_i, v_{i+1}]]$, $i = 1, 2, ...t$, $v_{t+1} = v_1$, is a 2-connected subgraph with two different-colored attachment vertices, by Theorem 2.6(iv), $t = 4$ and the attachment vertices of each of B_1, B_2, B_3, B_4 are mutually cycle-related in the block. Thus we have $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 4$.

(ii) Since P is a chain of even length with end vertices x and y, from Theorem 2.6 (ii), $G - V_I(P)$ has exactly two blocks each of which is 2-connected and x and y are cycle-related to the common vertex of the two blocks, let the common vertex of the two blocks be v, then xv and yv are chords of G and $d(x) = d(y) = 3$, $E(f) = \{xv, yv\} \cup E(P)$.

(iii) Let P be the only one chain of length odd with end vertices x and y contained in $E(f) \cap E(C)$. If $G - V_I(P)$ is 2-connected then xy is a chord in G. If $G \backslash V_I(P)$ is not 2-connected, from Theorem 2.6(iii)(a), $G - V_I(P)$ has exactly three blocks, each of which is 2-connected, and x (resp. y) is cycle-related to the other attachment vertex of the block containing it, say u (resp. w), and the attachment vertices u and w of the third block are mutually cycle-related in the third block. Thus $E(f) \backslash E(C) = \{xu, uw, wy\}$, and $d(x) = d(y) = 3$.

(iv) From Theorem 2.6(ii) we know that $E(f) \cap E(C)$ contains no chain of even length. Let P be a chain of odd length contained in $E(f) \cap E(C)$. Since P is not the only chain contained in $E(f) \cap E(C)$, $G \backslash V_I(P)$ is not 2-connected. By Theorem 2.6(iii)(b), any two 2-connected blocks of $G - V_I(P)$ are disjoint, therefore any two chords contained in $E(f) \backslash E(C)$ have no common end vertex. □

Let f be an interior face of a 2-connected outerplane cycle resonant graph G. We call f a face of type 1, 2, 3, 4 if f satisfies condition (i),(ii),(iii),(iv) in the above lemma respectively. More specifically, let f be of type 3, if $E(f)$ contains only one chord, we call f of type 3(1) and if $E(f)$ contains more than one chord, we call f of type 3(2).

Theorem 3.5 *Let G be a 2-connected outerplane bipartite graph that is not a cycle and C be the outer cycle of G. Then G is cycle resonant if and only if every interior face f of G is of type i, $i \in \{1, 2, 3, 4\}$.*

Proof. The necessity has been proven in Theorem 3.4 and we need only show the sufficiency here.

Let G be a 2-connected outerplane bipartite graph with the property that every interior face f of G is of type 1, or 2, or 3, or 4. It is easy to verify that G satisfies the conditions (i) - (iv) of Theorem 2.6, and so G is 2-cycle resonant. Hence G is also cycle resonant by Theorem 3.3. □

4 The structure of cycle resonant outerplanar graphs

In this section we further discuss the structure of 2-connected cycle resonant outerplane graphs. A characterization for a 2-connected cycle resonant outerplane graphs with maximum degree $\Delta = 3$ is given. In addition, it is showed that, any 2-connected cycle resonant outerplane graph can be constructed by a series of operations from a 2-connected cycle resonant outerplane graphs with maximum degree $\Delta = 3$.

Theorem 4.1 *Let G be a 2-connected bipartite outerplane graph with maximum degree $\Delta = 3$ and C be the outer cycle of G. Let $v_1, v_2, \cdots, v_t$ be the consecutive vertices of degree 3 along C with the given orientation. Then the following statements are equivalent:*

(i) G is cycle resonant;

(ii) v_i and v_{i+1} have different colors for $i = 1, 2, \cdots, t-1$;

(iii) there is no even chain in G;

(iv) every interior face of G is type 3(1) or type 4.

Proof. (i)$\Longrightarrow$(ii). Since G is cycle resonant and $\Delta = 3$, from Theorem 3.6, every face of G is type 3(1) or type 4, therefore any two vertices v_i and v_{i+1} have different colors for $i = 1, 2, \cdots, t-1$.

(ii)$\Longrightarrow$(i). If any two consecutive vertices v_i and v_{i+1} have different colors for $i = 1, 2, \cdots, t-1$, since $\Delta = 3$, every face of G is type 3(1) or type 4, by Theorem 3.6 G is cycle resonant.

It is easy to see that for 2-connected outerplane graph with the maximum degree $\Delta = 3$, (ii), (iii) and (iv) are equivalent. □

Theorem 4.2 *Suppose G is a 2-connected cycle resonant outerplane graph, C is the outer cycle of G, and v is a vertex of G with $d(v) \geq 4$. Let the neighbors of v be $v_1, v_2, \cdots, v_t$ along C with $vv_1, vv_t \in E(C)$ and $vv_2, vv_3, \cdots, vv_{t-1}$ being chords. Denote by f_i the face containing the chords vv_i and vv_{i+1}. Then*

(i) f_1 and f_{t-1} are type 3(1) or type 4;

(ii) if $d(v) = 4$, then f_2 is type 1, 2 or 3(2);

(iii) if $d(v) \geq 6$, then $f_3, f_4, \cdots, f_{t-3}$ are all type 2, $d(v_i) = 3$, $i = 3, 4, \cdots, t-2$;

(iv) if $d(v) \geq 5$, then f_2 and f_{t-2} are type 2 or type 3(2), $d(v_3) = d(v_{t-2}) = 3$;

Proof. (i) We prove the result for f_1 and similarly we will get the result for f_{t-1}. We first show that for any $x \in C(v, v_2^-)$, $v_2x \notin E(G)$. Otherwise let $v_2x \in E(G)$ with $x \in C(v, v_2^-)$. Since G is bipartite, $v_1v_2 \notin E(G)$. Then $G - vv_1$ contains either two 2-connected blocks or three 2-connected blocks and v is cycle related to v_2 by Theorem 2.6. This contradicts that $vv_{t-1} \in E(G)$. Now if $C[v, v_2]$ is a chain, f_1 is type 3(1). Otherwise $C[v, v_2]$ contains at least two chains and is type 4.

(ii) If $d(v) = 4$, f_2 contains two adjacent chords vv_2 and vv_3 and can only be type 1, 2 or 3(2).

(iii) Assume $d(v) \geq 6$ and f_i is a face with $i \in \{3, 4, \cdots, t-2\}$. It is clear that f_i is not type 1, 3(1) or 4. Since v is cycle-related neither to v_i nor to v_{i+1}, f_i is not type 3(2) with $E(f_i) = \{vv_i, vv_{i+1}, v_iu\} \cup C[u, v_{i+1}]$ or $E(f_i) = \{vv_i, vv_{i+1}, v_{i+1}u\} \cup C[v_i, u]$, in which $u \in C(v_i, v_{i+1})$. Now f_i can only be a face of type 2 and $d(v_i) = 3$, $i = 3, 4, \cdots, t-2$.

(iv) Since $d(v) \geq 5$, f_2 and f_{t-2} can only be type 2 or type 3(2).

If $d(v) \geq 6$, from (iii) we have $d(v_3) = d(v_{t-2}) = 3$.

Assume $d(v) = 5$. Since v is not cycle-related to v_3 in $G[C[v_3, v]]$ or $G[C[v, v_3]]$, f_2 is not type 3(2) with $E(f_2) = \{vv_2, vv_3, v_3u\} \cup C[v_2, u]$ in which $u \in C(v_2, v_3)$ and f_3 is not type 3(2) with $E(f_3) = \{vv_3, vv_4, v_3u\} \cup C[u, v_4]$ in which $u \in C(v_3, v_4)$, thus we have $d(v_3) = 3$, and $d(v_{t-2}) = 3$ similarly. □

From Theorem 4.2(iii) we can get the following corollary.

Corollary 4.3 *Suppose G is a 2-connected cycle resonant outerplane graph without face of type 2, then the maximum degree of G is at most 5.*

Furthermore, we can reduce a 2-connected cycle resonant outerplane graph G into a a 2-connected cycle resonant outerplane graph with maximum degree $\Delta = 3$ as follows.

If each interior face of G is type 3(1) or type 4, by Theorem 4.1 G is a 2-connected cycle resonant outerplane graphs with the maximum degree $\Delta = 3$. If G contains faces of type 1, 2 or 3(2), we can use the following operations to get a 2-connected cycle resonant outerplane graphs with $\Delta = 3$. (a) If f is type 1 with $E(f) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$, then any face adjacent to f is type 3(1) or 4, delete the edges v_1v_2 and v_3v_4, the face containing v_2v_3, v_4v_1 is type 4. (b) If f is type 2 with $E(f) = \{vu', vu''\} \cup C[u', u'']$, we can delete f from G by deleting the vertices in $C(u', u'')$, identifying the vertices u' and u'' to a new vertex u, and changing the chords vu' and vu'' to a new chord vu. (c) If f is type 3(2) in G with $E(f) = \{v_1v_2, v_2v_3, v_3v_4\} \cup C[v_4, v_1]$, since v_2 and v_3 are cycle-related in $G[C[v_2, v_3]]$, the face containing v_2v_3 other than f is type 3(1) or type 4, after deleting the edge v_2v_3 the face containing v_1v_2, v_3v_4 is type 4.

Reversely, any 2-connected cycle resonant outerplane graph G can be constructed from a 2-connected cycle resonant outerplane graph G' with maximum degree $\Delta = 3$ by the converse operations of the above operations.

5 The extremal problem of the number of perfect matchings in cycle resonant outerplane graphs

A 2-connected cycle resonant outerplane graph G is called standard if the length of its any even chain is 2, the length of its any odd chain is 1 or 3, and the length of a chain $C[x, y]$ is 3 if and only if xy is a chord of G. Let G_1 and G_2 be two 2-connected cycle resonant outerplane graphs. We denote $G_1 \approx G_2$ if the final resultant graphs obtained respectively from G_1 and G_2 by successively double-contracting are isomorphic. Note that if G is standard, then G can not be double-contracted. From Lemma 3.1 we have $\Phi(G_1) = \Phi(G_2)$.

Suppose G is a plane graph. The dual graph [12] G^* of G is a plane graph having a vertex for each region in G. The edges of G^* correspond to the edges of G as follows: if e is an edge of G that has region X on one side and region Y on the other side, then the corresponding dual edge $e^* \in E(G^*)$ is an edge joining the vertices x, y of G^* that correspond to the faces X, Y of G. The weak dual of a plane graph G is the graph obtained from the dual G^* by deleting the vertex corresponding to unbounded face of G. The weak dual of an outerplane graph is a forest and the weak dual of an 2-connected outerplane graph is a tree.

Let G be a 2-connected outerplane graph, T the weak dual of G, and f a face of G which corresponds a leaf of T. Then f is type 3(1) and the only chord $uv \in E(f)$ is called a leaf chord of G. We denote by $G-u-v$ the graph obtained from G by deleting vertices u and v and $G-uv$ the graph obtained from G by deleting the edge uv from G.

Lemma 5.1 *Let G be a 2-connected outerplane graph with a perfect matching, uv be a chord of the outer cycle of G. Then $\Phi(G) = \Phi(G-u-v) + \Phi(G-uv)$.*

Proof. To each perfect matching M of $G-u-v$ there corresponds a perfect matching $M \cup uv$ of G containing the chord uv and each perfect matching M' of $G-uv$ is also a perfect matching in G not containing the chord uv. It follows that $\Phi(G) = \Phi(G-u-v) + \Phi(G-uv)$. □

Lemma 5.2 *Let G be a 2-connected cycle resonant outerplane graph and uv a leaf chord of G. Then $\Phi(G-u-v) < \Phi(G-uv)$ and $2\Phi(G-uv) > \Phi(G)$.*

Proof. Without loss of generality we assume that G is standard and let f be the face which corresponds a leaf in the weak dual of G with $E(f) = \{ux, xy, yv, uv\}$. Let $G-u-v = G_1 \cup \{xy\}$, then $\Phi(G-u-v) = \Phi(G_1)$. For each perfect matching M of G_1, $M \cup \{ux, yv\}$ is a perfect matching in $G-uv$. Besides, $G-uv$ has at least one perfect matching containing the edge xy, it follows that $\Phi(G-u-v) < \Phi(G-uv)$, from Lemma 5.1 we have $2\Phi(G-uv) > \Phi(G)$. □

Let $H_k \in \mathcal{G}_k$ be a graph with the maximum degree $\Delta = 3$ whose the consecutive vertices of degree three are $v_1, v_2, \cdots, v_{2k}$ along C and whose chords are $v_i v_{2k+1-i}, i = 1, 2, \cdots, k$. Let $Q_k \in \mathcal{G}_k$ be a graph with the maximum degree $\Delta = 3$ whose consecutive vertices of degree three are $v_1, v_2, \cdots, v_{2k}$ and whose chords are $v_i v_{i+1}, i = 1, 3, 5, \cdots, 2k-1$. Let $J_k \in \mathcal{G}_k$ be a graph with exactly one vertex of degree $k+2$, k vertices of degree 3, and all the other vertices being of degree 2. We will show that for any $G \in \mathcal{G}_k$ with the maximum degree $\Delta(G) = 3$, $\Phi(H_k) \le \Phi(G) \le \Phi(Q_k)$, and that for any $G \in \mathcal{G}_k$, $\Phi(L_k) \le \Phi(G) \le \Phi(Q_k)$.

The Fibonacci numbers are the sequence of numbers $\{F_n, n = 1, 2, \cdots\}$ defined by the linear recurrence equation $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = F_2 = 1$.

Lemma 5.3 $\Phi(H_k) = F_{k+3}$ *and* $\Phi(Q_k) = 2^k + 1$, $k = 0, 1, 2, \cdots$.

Proof. We prove the two equalities by induction on k. For $k = 0$, $\Phi(H_0) = \Phi(Q_0) = 2$ and $F_3 = 2$, the equalities are true.

For the chord $v_1 v_{2k}$ of H_k, by the induction hypothesis, $\Phi(H_k - v_1 - v_{2k}) = \Phi(H_{k-2})$ and $\Phi(H_k - v_1 v_{2k}) = \Phi(H_{k-1})$, therefore

$$\Phi(H_k) = \Phi(H_k - v_1 - v_{2k}) + \Phi(H_k - v_1 v_{2k})$$

$$= \Phi(H_{k-2}) + \Phi(H_{k-1}) = F_{k+1} + F_{k+2} = F_{k+3}.$$

For the chord v_1v_2 of Q_k, since $\Phi(Q_k - v_1 - v_2) = 2^{k-1}$ and by the induction hypothesis $\Phi(Q_k - v_1v_2) = \Phi(Q_{k-1})$,

$$\Phi(Q_k) = \Phi(Q_k - v_1 - v_2) + \Phi(Q_k - v_1v_2)$$

$$= 2^{k-1} + \Phi(Q_{k-1}) = 2^{k-1} + 2^{k-1} + 1 = 2^k + 1. \quad \square$$

Lemma 5.4 $F_{r+s+4} < F_{r+3}\ F_{s+3}$.

Proof. Honsberger [5] (1985, p. 107) gives the following relation

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}.$$

Thus we have

$$F_{r+s+4} = F_{(r+2)+(s+2)} = F_{r+1}F_{s+2} + F_{r+2}F_{s+3}$$

$$< F_{r+1}F_{s+3} + F_{r+2}F_{s+3} = F_{r+3}F_{s+3}. \quad \square$$

Theorem 5.5 *For any graph $G \in \mathcal{G}_k$ with the maximum degree $\Delta(G) = 3$, $F_{k+3} \leq \Phi(G) \leq 2^k + 1$. Furthermore, the left hand equality holds only when $G \approx H_k$ and the right hand equality holds only when $G \approx Q_k$.*

Proof. We prove by induction on k. For $k = 0$, $\Phi(H_0) = \Phi(Q_0) = \Phi(G) = 2$ and the result is true. Let uv be a leaf chord of G, from Lemma 5.1 $\Phi(G) = \Phi(G - u - v) + \Phi(G - uv)$. By the induction hypothesis, $F_{k+2} \leq \Phi(G - uv) \leq 2^{k-1} + 1$ and the left hand equality holds only if $G - uv \approx H_{k-1}$ (that is, $G \approx H_k$) and the right hand equality holds only if $G - uv \approx Q_{k-1}$ (that is, $G \approx Q_k$).

Now, it suffices to prove that $\Phi(G - u - v)$ is maximum (resp. minimum) only if $G \approx Q_k$ (resp. $G \approx H_k$).

Notice that there are two components in $G - u - v$ one of which is a path with one perfect matching. Denote by G' the other component of $G - u - v$ and let the 2-connected blocks of G' be $G_1, G_2, \cdots, G_t$. Clearly $G - V(G')$ is a cycle containing u and v, say C_{uv}. It is easy to check that $\Phi(G - u - v) = \Phi(G') = \Phi(G_1)\ \Phi(G_2)\ \cdots\ \Phi(G_t)$.

We claim that $\Phi(G - u - v)$ is maximum only if G' has maximum number of 2-connected blocks, that is $t = k - 1$, each G_i is a cycle and $\Phi(G - u - v) = 2^{k-1}$. To the contrary, assume that $\Phi(G - u - v)$ is maximum but $t < k - 1$. Then at least one block of G' has at least one chord. Without loss of generality, suppose that G_1 has a chord. Let xy be a leaf chord of G_1. Let G'' be the graph obtained from G' by removing xy and constructing a new cycle block G_{t+1} disjoint from 2-connected blocks in G' such that $V(G'')$ together with $V(C_{uv})$ still induce a graph in $\mathcal{G}_k$

with the maximum degree $\Delta = 3$, say G^*, and $G^* - u - v$ has $t+1$ 2-connected blocks. From Lemma 5.2, we have $2\Phi(G_1 - xy) > \Phi(G_1)$ and $\Phi(G^* - u - v) = \Phi(G'') = \Phi(G_1 - xy)\,\Phi(G_2)\,\cdots\,\Phi(G_t)\,\Phi(G_{t+1}) = 2\Phi(G_1 - xy)\,\Phi(G_2)\,\cdots\,\Phi(G_t) > \Phi(G_1)\,\Phi(G_2)\,\cdots\,\Phi(G_t) = \Phi(G') = \Phi(G - u - v)$, a contradiction. Hence $\Phi(G - u - v)$ is maximum only if G' has $k-1$ 2-connected blocks, that is, $G \approx Q_k$.

We claim that $\Phi(G - u - v)$ is minimum only if $t = 1$, $G_1 \approx H_{k-2}$ and $\Phi(G - u - v) = F_{k+1}$. Otherwise, let $\Phi(G - u - v)$ be minimum but $t > 1$, without loss of generality, let G_1 and G_2 be two 2-connected blocks of $G - u - v$ which have r chords and s chords respectively. Replace G_1 and G_2 in G' by a new 2-connected block $G_1' \approx H_{r+s+1}$ to obtain the resultant graph G''' such that $V(G'''$ together with $V(C_{uv})$ still induce a graph in $\mathcal{G}_k$ with the maximum degree $\Delta = 3$. From Lemma 5.4 we have $\Phi(H_{r+s+1}) = F_{r+s+4} < F_{r+3}\,F_{s+3} = \Phi(G_1)\,\Phi(G_2)$ and $\Phi(G''') = \Phi(G_1')\,\Phi(G_3)\,\Phi(G_4)\,\cdots\,\Phi(G_t) < \Phi(G_1)\,\Phi(G_2)\,\cdots\,\Phi(G_t) = \Phi(G')$, a contradiction. Hence $\Phi(G - u - v)$ is minimum only if $t = 1$, $G_1 \approx H_{k-2}$, that is, $G \approx H_k$. □

Theorem 5.6 *Let $G \in \mathcal{G}_k$ be a 2-connected cycle resonant outerplane graph with k chords. Then $k + 2 = \Phi(L_k) \leq \Phi(G) \leq \Phi(Q_k) = 2^k + 1$.*

Proof. Since G is a 2-connected bipartite outerplane graph, G has two perfect matchings containing no chord and for each chord uv of G, there is one perfect matching containing a unique chord uv, so $\Phi(G) \geq k + 2 = \Phi(J_k)$.

On the other hand, for a leaf chord uv of G, by a similar argument as in the proof of Theorem 5.5, $\Phi(G - uv) \leq 2^{k-1} + 1 = \Phi(Q_{k-1})$ and the equality holds only if $G \approx Q_k$, and $\Phi(G - u - v)$ is maximum only if $G - u - v$ has maximum number of 2-connected blocks. By Theorem 3.4, it is not difficult to verify that any two 2-connected blocks of $G - u - v$ are disjoint. Now it follows that $\Phi(G - u - v)$ is maximum only if $\Phi(G - u - v)$ has $k-1$ 2-connected blocks and $\Phi(G - u - v) = 2^{k-1}$. Hence $\Phi(G) \leq \Phi(Q_k) = 2^k + 1$.
□

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