Generating 3-Connected Quadrangulations on Surfaces

Momoko Nagashima, Atsuhiro Nakamoto*, Seiya Negami †, Yusuke Suzuki‡

Abstract

Let G be a simple quadrangulation on a closed surface F^2. A *face-contraction* and a 4-*cycle removal* are two reductions for quadrangulations defined in this paper.

- G is *irreducible* if any face-contraction breaks the simplicity of G,
- G is $\mathcal{D}_3$-*irreducible* if G has minimum degree at least 3 and any face-contraction or any 4-cycle removal either breaks the simplicity or reduces the minimum degree to less than 3,
- G is $\mathcal{K}_3$-*irreducible* if G is 3-connected and any face-contraction or any 4-cycle removal breaks the simplicity or the 3-connectedness of the graph,
- G is $\mathcal{S}_4$-*irreducible* if G has no separating 4-cycle and any face-contraction breaks the simplicity or creates a separating 4-cycle.

In [7], it was shown that except the sphere and the projective plane, the irreducibility and the $\mathcal{D}_3$-irreducibility of quadrangulations are equivalent. In this paper, we shall prove that for all surfaces, the $\mathcal{D}_3$-irreducibility and the $\mathcal{K}_3$-irreducibility of quadrangulations are equivalent. We also prove that for the sphere, the projective plane and the torus, the $\mathcal{D}_3$-irreducibility and the $\mathcal{S}_4$-irreducibility of quadrangulations are equivalent, but this does not hold for surfaces of high genus.

Keywords: quadrangulation, 3-connected, face-contraction

*Department of Mathematics, Faculty of Education and Human Sciences, Yokohama National University, Yokohama 240-8501, Japan Email: nakamoto@edhs.ynu.ac.jp

†Department of Mathematics, Faculty of Education and Human Sciences, Yokohama National University, Yokohama 240-8501, Japan Email: negami@edhs.ynu.ac.jp

‡General Sciences, Tsuruoka National College of Technology, Tsuruoka, Yamagata 997-8511, Japan Email: y-suzuki@tsuruoka-nct.ac.jp

1 Intoroduction

A *quadrangulation* G of a closed surface F^2 is a fixed embedding of some simple graph such that each face is bounded by a cycle of length 4. Clearly, the minimum degree of G is at least 2. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. A *k-cycle* means a cycle of length k and denoted by C_k. A k-cycle is said to be *odd* (resp., *even*) if k is odd (resp., even). A *facial cycle* C of G is a cycle bounding a face of G. (We often denote the facial cycle of a face f by ∂f.) We say that a simple closed curve l on F^2 is *separating* if $F^2 - l$ is disconnected. We say that $S \subset V(G)$ is a *cut* of G if $G - S$ is disconnected. In particular, S is called a *k-cut* if S is a cut with $|S| = k$. A cycle C of G is said to be *separating* if $V(C)$ is a cut set.

Let G be a quadrangulation on a closed surface F^2 and let f be a face of G bounded by a cycle $abcd$. The *face-contraction* of f at $\{a, c\}$ in G is to identify a and c, and replace the two pairs of multiple edges $\{ab, cb\}$ and $\{ad, cd\}$ with two single edges respectively. See Figure 1. We can also define the face-contraction of f at $\{b, d\}$. (The inverse operation of the face-contraction is called a *vertex-splitting*.) If the graph obtained from G by a face-contraction is not simple, then we don't apply it. A quadrangulation G is said to be *irreducible* if no face-contraction can be applied to G.

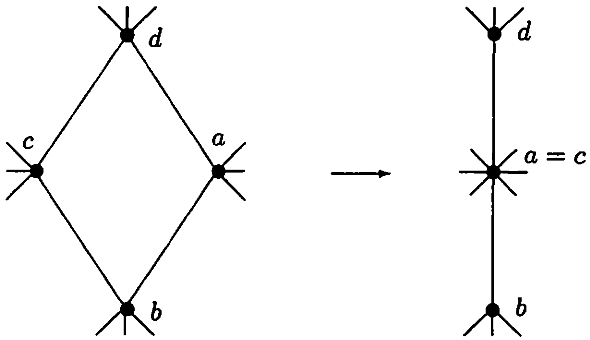
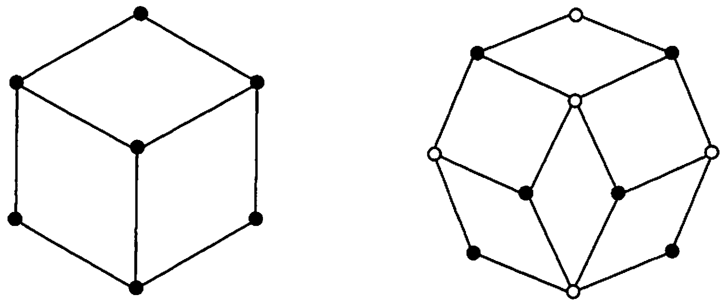


Figure 1: A face-contraction at $\{a, c\}$.

In [8], it was proved that a 4-cycle is the unique irreducible quadrangulation on the sphere, and that there exist precisely two irreducible quadrangulations of the projective plane shown in Figure 2, where Q_P^1 and Q_P^2 are the unique quadrangular embeddings of K_4 and $K_{3,4}$ on the projective plane, respectively. The irreducible quadrangulations on the torus and the

Klein bottle have also been determined in [5, 6]. There exist precisely eight and ten irreducible quadrangulations of the torus and the Klein bottle, respectively. In general, by bounding the number of vertices by a linear function of the Euler characteristic of F^2 [4], it was proved that for any closed surface F^2, there exist only finitely many irreducible quadrangulations on F^2, up to homeomorphism.



Q_P^1 $\qquad$ Q_P^2

Figure 2: Irreducible quadrangulations on the projective plane.

Clearly, every quadrangulation G on any closed surface F^2 is 2-connected and hence the minimum degree of G is at least 2. If G has a vertex of degree 2, then $G - v$ is also a quadrangulation on F^2.

Let G be a quadrangulation on a closed surface F^2 with minimum degree at least 3, and let f be a face of G bounded by $v_1v_2v_3v_4$. The *4-cycle addition* to f is to put a 4-cycle $u_1u_2u_3u_4$ inside f in G and join v_i and u_i for $i = 1, 2, 3, 4$, as shown in Figure 3. (The inverse operation of a 4-cycle addition is called the *4-cycle removal.*) Consider the quadrangulation, denoted by $\tilde{G}$, on F^2 obtained from G by applying a 4-cycle addition to all faces of G. Then each face-contraction applied to $\tilde{G}$ yields a vertex of degree 2. Therefore, in order to reduce quadrangulations preserving the minimum degree at least 3, we need more operations other than a face-contraction.

Embed a $2n$-cycle $v_1u_1v_2u_2\dots v_nu_n$ $(n \geq 3)$ into the sphere, put vertex x on one side and vertex y on the other side and add edges xv_i and yu_i for $i = 1, \dots, n$. The resulting quadrangulation on the sphere with $2n + 2$ vertices is said to be the *pseudo double wheel* and denoted by W_{2n}. The smallest pseudo double wheel is W_6, which is the cube. See, for example, the left side of Figure 4.

Embed a $(2n-1)$-cycle $C = v_1v_2\dots v_{2n-1}$ $(n \geq 2)$ into the projective plane so that the tubular neighborhood of C forms a Möbius band. Next, put a vertex x on the center of the unique face of the embedding and join x with v_i for all i so that the resulting graph is a quadrangulation. The resulting quadrangulation on the projective plane with $2n$ vertices is said

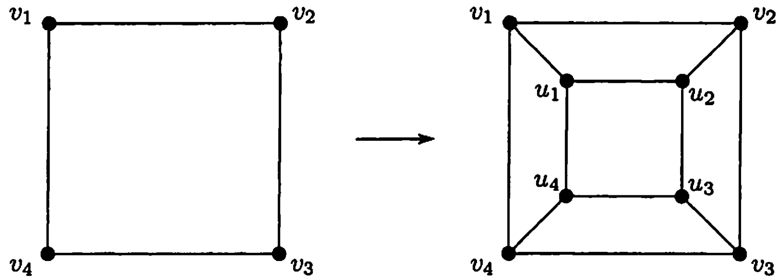


Figure 3: 4-cycle addition.

to be the *Möbius wheel* and denoted by $\tilde{W}_{2n-1}$. See, for example, the right side of Figure 4, where the outer decagon represents the projective plane.

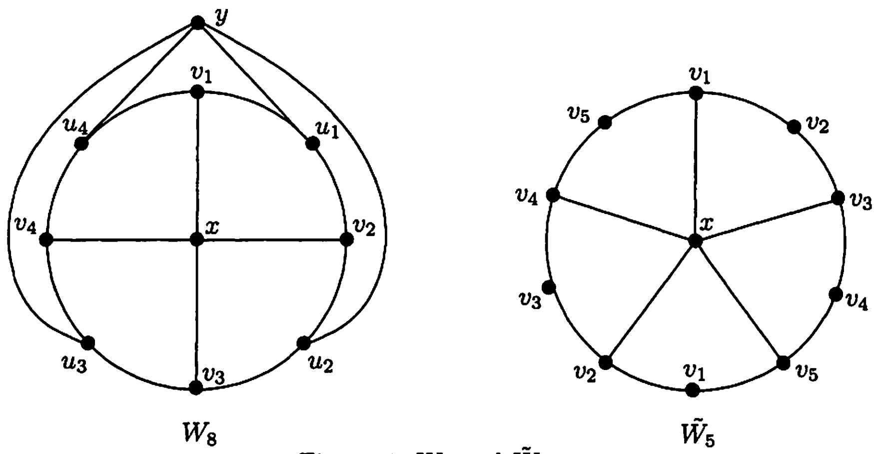


Figure 4: W_8 and $\tilde{W}_5$.

Let G be a quadrangulation on a closed surface F^2 with minimum degree at least 3. We say that G is $\mathcal{D}_3$*-irreducible* if each face-contraction or each 4-cycle removal applied to G breaks the simplicity or the minimum degree at least 3 of the graph.

THEOREM 1 (Nakamoto[7]) • *If G is a $\mathcal{D}_3$-irreducible quadrangulation on the sphere, then G is isomorphic to a pseudo double wheel.*

- *If G is a $\mathcal{D}_3$-irreducible quadrangulation on the projective plane, then G is isomorphic to either Q_P^1 in Figure 2 or a Möbius wheel.*

- *Let G be a quadrangulation on a closed surface other than the sphere and the projective plane. Then G is $\mathcal{D}_3$-irreducible if and only if G is irreducible.*

In this paper, we consider 3-connected quadrangulations on closed surfaces. A 3-connected quadrangulation G on a closed surface F^2 is said to be *$\mathcal{K}_3$-irreducible* if each face-contraction or each 4-cycle removal applied to G breaks the simplicity or the 3-connectedness.

The following theorem is our first result.

THEOREM 2 *For any closed surface F^2, a quadrangulation G is $\mathcal{D}_3$-irreducible if and only if G is $\mathcal{K}_3$-irreducible.*

The spherical case has already been proved in [1]. Moreover, they restricted face-contractions to faces incident to a vertex of degree at most 3, but we cannot do so in general, since a quadrangulation on a closed surface with high genus might have no vertex of degree at most 3.

In the final section, we also consider the class of all quadrangulations without separating 4-cycles. A quadrangulation G is said to be *$\mathcal{S}_4$-irreducible* if G has no separating 4-cycle and if each face-contraction breaks the simplicity or makes a separating 4-cycle. (Note that we don't need the 4-cycle removal since a quadrangulation for which a 4-cycle removal can be applied has a separating 4-cycle.) In [1], it was shown that for the sphere, the $\mathcal{S}_4$-irreducible quadrangulations coincide with the $\mathcal{D}_3$-irreducible quadrangulations. The following theorem claims that the same facts hold for the projective plane and the torus, too.

THEOREM 3 *For the sphere, the projective plane and the torus, a quadrangulation G is $\mathcal{S}_4$-irreducible if and only if G is $\mathcal{K}_3$-irreducible.*

However, a theorem such as Theorem 3 does not hold for closed surfaces with high genera, as shown in the following theorem.

THEOREM 4 *There exists a closed surface F^2, both orientable and nonorientable, which admits a $\mathcal{S}_4$-irreducible quadrangulation but is not $\mathcal{K}_3$-irreducible.*

Let $\mathcal{S}$ be an infinite set of graphs. Assume that there exists a subset $\mathcal{S}_0 \subset \mathcal{S}$ such that for any graph $G = G_0$ in $\mathcal{S}$, there is a sequence of graphs $G_0, G_1, \ldots, G_l$ in $\mathcal{S}$ where $G_l \in \mathcal{S}_0$ and each G_{i+1} is obtained from G_i by one of specific operations of graphs, for $0 \leq i \leq l+1$. This implies that all the members of $\mathcal{S}$ can be constructed from the graphs in $\mathcal{S}_0$ by applying the inverse of the corresponding operations. Therefore, our theorems are generating theorems for quadrangulations on surfaces.

2 3-Connected quadrangulations

In this section, we shall prove Theorem 2. Before doing it, we prepare several lemmas.

LEMMA 5 *A quadrangulation on a closed surface F^2 has no separating odd cycle corresponding a separating simple closed curve on F^2.*

Proof. Suppose that a quadrangulation G on F^2 has such a separating odd cycle C. Cutting F^2 along C, we obtain two surfaces, each of whose boundary component is C. Pasting a disk to the boundary of one of the surfaces, we get an embedding on the closed surface with all quadrilateral faces, except one bounded by C. Since C has odd length, the dual of the graph has exactly one vertex of odd degree, which is impossible. ∎

Let G be a quadrangulation on a closed surface F^2 and let f be a face of G bounded by a 4-cycle $abcd$. We say that f is *contractible* at $\{a, c\}$ in G if the graph obtained from the face-contraction of f at $\{a, c\}$ is simple. We say that f is $\mathcal{D}_3$*-contractible* at $\{a, c\}$ in G if f is contractible at $\{a, c\}$, and if the graph obtained from the face-contraction at $\{a, c\}$ has minimum degree at least three. Similarly, we can define $\mathcal{K}_3$*-contractible* and $\mathcal{S}_4$*-contractible* faces.

Let G be a quadrangulation on a closed surface F^2 and let f be a face of G bounded by a 4-cycle $v_0v_1v_2v_3$. Then a pair $\{v_i, v_{i+2}\}$ is called a *diagonal pair* of f in G, where the subscripts are taken modulo 4. A closed curve l on F^2 is said to be a *diagonal k-curve* for G if l passes only through distinct k faces $f_0, \dots, f_{k-1}$ and distinct k vertices $x_0, \dots, x_{k-1}$ of G such that for each i, f_i and f_{i+1} share x_i, and that for each i, $\{x_{i-1}, x_i\}$ forms a diagonal pair of f_i of G, where the subscripts are taken modulo k.

LEMMA 6 *Let G be a quadrangulation on a closed surface F^2 with a 2-cut $\{x, y\}$. Then there exists a separating diagonal 2-curve for G only through x and y.*

Proof. Observe that every quadrangulation on any closed surface F^2 is 2-connected and admits no closed curve on F^2 crossing G at most once. Thus there exists a simple separating closed curve l on F^2 crossing only x and y, since $\{x, y\}$ is a cut set of G.

We shall show that l is a diagonal 2-curve. Suppose that l passes through two faces f_1 and f_2 meeting at two vertices x and y. If l is not a diagonal 2-curve, then x and y are adjacent on ∂f_1 or ∂f_2. Since G has no multiple edges between x and y, and since $\{x, y\}$ is a 2-cut of G, we may suppose that x and y are adjacent in ∂f_1, but not in ∂f_2. Here we can take a separating 3-cycle along l. This contradicts Lemma 5. ∎

Lemma 7 *Let G be a 3-connected quadrangulation on a closed surface F^2, and let f be a face of G bounded by abcd. If the face-contraction of f at $\{a, c\}$ breaks the 3-connectedness of the graph but preserves the simplicity, then G has a separating diagonal 3-curve passing through a, c and x for some $x \in V(G) - \{a, b, c, d\}$.*

Proof. Let G' be the quadrangulation on F^2 obtained from G by the face-contraction of f at $\{a, c\}$. Since G' has connectivity 2, G' has a 2-cut. By Lemma 6, G' has a separating diagonal 2-curve l' passing through two vertices of the 2-cut. Clearly, one of the two vertices must be $[ac]$ of G', which is the image of a and c by the contraction of f. (If not, G would not be 3-connected, a contradiction.) Let x be another vertex of G' on l' other than $[ac]$. Note that x is not a neighbor of $[ac]$ in G'.

Now apply the inverse operation of the face-contraction of f at $\{a, c\}$ to G' to obtain G. Then a diagonal 3-curve for G passing through only a, c and x arises from l' for G'. ■

Lemma 8 *Let G be a $\mathcal{K}_3$-irreducible quadrangulation on a closed surface F^2. If G has a separating 4-cycle $C = x_0x_1x_2x_3$, then there is no face f of G such that*

(i) one of the diagonal pairs of f is $\{x_i, x_{i+2}\}$ for some i, and

(ii) f has a separating diagonal 3-curve l intersecting C only at x_i and x_{i+2} transversely.

Proof. For getting contradictions, we suppose that G has a separating 4-cycle $C = x_0x_1x_2x_3$ and a face f bounded by ax_1cx_3. Since C is separating, G has two subgraphs G_R and G_L such that $G_R \cup G_L = G$ and $G_R \cap G_L = C$. Suppose that f is contained in G_R. Furthermore, we assume that G_R contains as few vertices of G as possible.

Since C is separating, we have $\partial f \neq C$. By (ii), f has a separating diagonal 3-curve l through x_1, x_3 and some vertex x. Note that $x \in V(G_L) - V(C)$ by (ii). Since G is $\mathcal{K}_3$-irreducible, f is not $\mathcal{K}_3$-contractible at $\{a, c\}$. Observe that l (or the 3-cut $\{x_1, x, x_3\}$) separates a from c. Further, G does not have both of edges ax and cx since $\partial f \neq C$. Therefore, there is no path of G of length at most 2 joining a and c other than ax_1c and ax_3c. Moreover, if $\{a, c\} \cap \{x_0, x_2\} = \emptyset$, then f has no separating diagonal 3-curve joining a and c, either. This contradicts the $\mathcal{K}_3$-irreducibility of G. Therefore we may suppose that $a = x_0$ and $c \neq x_2$, and f has a separating diagonal 3-curve, say γ, through a $(= x_0)$ and c.

Since γ separates x_1 and x_3 and since x_2 is a common neighbors of x_1 and x_3, γ must pass through x_2, and hence we can find a face f' of G_R one of whose diagonal pair is $\{c, x_2\}$. Let C' be the 4-cycle $x_1x_2x_3c$ of G.

Since $\deg(c) \geq 3$, we have $\partial f' \neq C'$, and hence C' is a separating 4-cycle in G_R such that $C' \neq C$. Moreover, γ and C' cross transversely at x_2 and c. Therefore, C' and f' are a 4-cycle and a face which satisfy the assumption of the lemma, and moreover, C' can cut a strictly smaller graph than G_R from G. Therefore, this contradicts the choice of C. ■

Now we shall prove Theorem 2.

Proof of Theorem 2. Let G be a quadrangulation on F^2. We first claim that if G is $\mathcal{D}_3$-irreducible, then G is 3-connected. Suppose that this does not hold. Since G must have connectivity 2, G has a 2-cut $\{x, y\}$ of G. By Lemma 6, there is a separating diagonal 2-curve l passing through x and y. Let f_1 and f_2 be the two faces through which l passes, where $\partial f_i = xa_iyb_i$, for $i = 1, 2$. Since f_1 and f_2 intersect only at x and y (otherwise, G has a vertex of degree 2), we have $\deg(x), \deg(y) \geq 4$. Hence, f_1 is $\mathcal{D}_3$-contractible at $\{a_1, b_1\}$. (It is easy to see that there is no path of length at most 2 joining a_1 and b_1, but intersecting neither x and y.) This contradicts that G is $\mathcal{D}_3$-irreducible. Hence G is 3-connected.

We shall prove the necessity. Since G is $\mathcal{D}_3$-irreducible, each face-contraction and each 4-cycle removal transforms G into a quadrangulation with a vertex of degree 2, loop or multiple edges. Since G is 3-connected as shown above, this implies that G is $\mathcal{K}_3$-irreducible.

Next we shall prove the sufficiency. For getting a contradiction, assume that the sufficiency does not hold. Let G be a $\mathcal{K}_3$-irreducible quadrangulation on F^2. Suppose that a 4-cycle removal in G yields a new face F such that the resulting graph, denoted by G', has the minimum degree ≥ 3, but is not 3-connected. Then, by Lemma 6, there is a separating diagonal 2-curve through F and f in G', where f is a face of G'. However by Lemma 8, it is impossible since we can easily find a separating diagonal 3-curve in G passing through f and two faces of G placed in the region corresponding to F.

We consider the reduction of G by face-contractions. Let f_1 be a face of G which is bounded by $x_1a_1x_2b_1$ and $\mathcal{D}_3$-contractible at $\{x_1, x_2\}$, but is not $\mathcal{K}_3$-contractible. Let G' be the quadrangulation obtained from G by a face-contraction of f_1 at $\{x_1, x_2\}$. Then G' is a simple quadrangulation with minimum degree at least 3, but is not 3-connected. By Lemma 7, G has a sequence of three faces, say f_1, f_2 and f_3 bounded by $x_1a_1x_2b_1$, $x_2a_2x_3b_2$ and $x_3a_3x_1b_3$, respectively, such that a separating diagonal 3-curve l passing through x_1, x_2 and x_3, where we suppose that a_1, a_2, a_3 and b_1, b_2, b_3 are separated by l. We denote the 3-cut $\{x_1, x_2, x_3\}$ of G by X. Further, let A and B be two components of $G - X$, which contain a_1, a_2, a_3 and b_1, b_2, b_3, respectively. (See Figure 6. To obtain the configuration of annular part of F^2, identify two horizontal broken lines.)

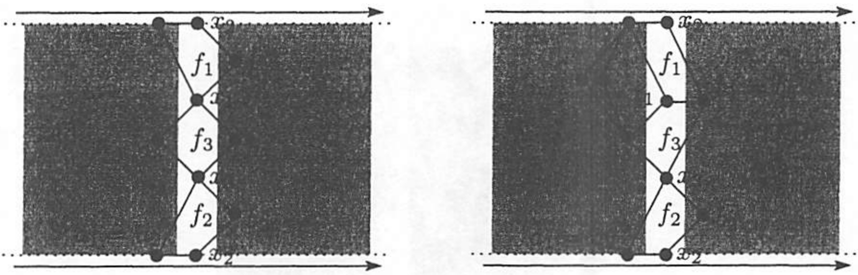


Figure 5: Structures of $f_1 \cup f_2 \cup f_3$ in Case 1.

We consider the coincidence of a_1, a_2, a_3, b_1, b_2 and b_3. Since l is separating, we clearly have $\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\} = \emptyset$. Since G' is simple and has minimum degree at least 3, we do not have the two cases when $a_1 = a_2 = a_3$ and when $a_1 \neq a_2 = a_3$, and similarly, we don't have the two cases when $b_1 = b_2 = b_3$ and when $b_1 \neq b_2 = b_3$.

Case 1. $a_1 = a_2 \neq a_3$ (or $b_1 = b_2 \neq b_3$).

See the left-hand side of Figure 5. Consider the face-contraction of f_3 at $\{a_3, b_3\}$ in G. Since G is $\mathcal{K}_3$-irreducible, G has either a path of length at most two joining a_3 and b_3, intersecting neither x_1 nor x_3, or a separating diagonal 3-curve for f_3 through a_3 and b_3 (by Lemma 7). First, we assume the former. If the face-contraction at $\{a_3, b_3\}$ yields a loop, then G has an edge joining a_3 and b_3. However, it is impossible since X separates a_3 and b_3 in G. If the contraction yields multiple edges, then G should have two edges a_3x_2 and b_3x_2. However, this is also impossible since $a_1 = a_2 \neq a_3$.

Hence we assume the latter case, that is, there is a separating diagonal 3-curve γ for f_3 through a_3 and b_3 and the third vertex v in G. If $v \in X$, we have $v = x_1$ or x_3 and there is a separating 3-cycle of G, contrary to Lemma 5. Next, we suppose the case of $v \in B$. In the case, there should be a face bounded by a 4-cycle $a_3x_ivx_j$ for distinct $i, j \in \{1, 2, 3\}$, through which γ passes. However, it requires the edge a_3x_2 since $\{i, j\} \neq \{1, 3\}$, a contradiction. (Observe that x_2 has the unique neighbor a_1 in A.)

Thus, we may assume that $v \in A$ and have $v = a_1$, clearly. Moreover, we have $b_1 = b_3$ since $b_2 \neq b_3$ (see the right-hand of Figure 6). Now, observe that a 4-cycle $C' = a_1x_3a_3x_1$ does not bound a face of G since $\deg(a_3) \geq 3$. Therefore, C' is a separating 4-cycle, in which some quadrilateral face is incident to a_1 and a_3, and a separating diagonal 3-curve γ crossing C' transversely at a_1 and a_3. This contradicts Lemma 8.

Case 2. a_1, a_2, a_3, b_1, b_2 and b_3 are distinct (See Figure 6 again).

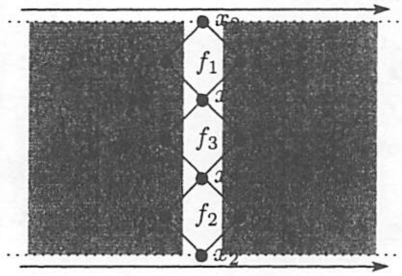


Figure 6: Structures of $f_1 \cup f_2 \cup f_3$ in Case 2.

We first consider the case when the face-contraction of f_i at $\{a_i, b_i\}$ breaks the 3-connectedness of the graph but preserves the simplicity, for some i, say $i = 3$. Then, by Lemma 7 again, G has a separating diagonal 3-curve γ through a_3 and b_3. Moreover, since a_1, b_1, a_2, b_2, a_3 and b_3 are all distinct, the third vertex v on γ must be in X. However, by the similar argument in Case 1, we can find a separating odd cycle, contrary to Lemma 5.

Secondly we may suppose that the face-contraction of f_2 at $\{a_2, b_2\}$ breaks the simplicity of the graph. Since a_1, b_1, a_2, b_2, a_3 and b_3 are distinct, both a_2 and b_2 are adjacent to x_1. Now we have the contradiction for the choice of f_1, since the face-contraction of f_1 at $\{x_1, x_2\}$ makes two edges x_2a_2 and x_1a_2 be multiple edges. Therefore, the sufficiency also holds. ■

3 Without separating 4-cycles

In this section, we consider quadrangulations without separating 4-cycles. Clearly, a quadrangulation G with no separating 4-cycle admits no separating diagonal 2-curve, and hence G is 3-connected, by Lemma 6. Now we shall prove Theorem 3.

Proof of Theorem 3. We first prove that if G is $\mathcal{K}_3$-irreducible, then G has no separating 4-cycle. (Note that the statement holds only on the sphere, the projective plane or the torus.) For getting a contradiction, we suppose that G has a separating 4-cycle $C = abcd$. Since C bounds a quadrilateral 2-cell region, denoted by R, on the surface, we may suppose that there is no separating 4-cycle in R, except C. Let $\bar{C}$ be the subgraph of G consisting of the vertices and the edge of C and in the interior of C. By Lemma 6 in [1], the plane quadrangulation $\bar{C}$ must be a pseudo double wheel.

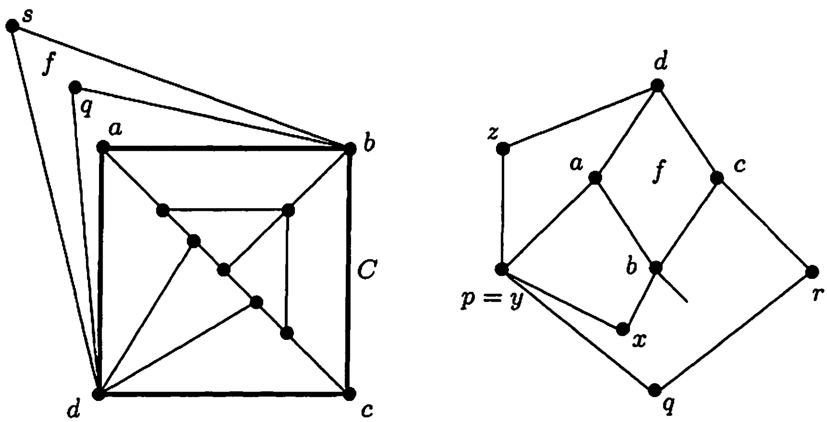


Figure 7: Configurations in the proof of Theorem 3.

If $\bar{C}$ is a cube then it is $\mathcal{K}_3$-removal while if $\bar{C}$ is not a cube then we can find a $\mathcal{K}_3$-contractible face, since there is no face outside R whose diagonal pair is $\{a, c\}$ or $\{b, d\}$, by Lemma 8. (See the left-hand of Figure 7. There does not exist a face like f in the figure.) Therefore, G has no separating 4-cycle.

We shall show the sufficiency. For getting a contradiction, we suppose that G is $\mathcal{K}_3$-irreducible, but has a $\mathcal{S}_4$-contractible face f. (Note that the 4-cycle removal cannot be applied to G since G has no separating 4-cycle.) Since G is $\mathcal{K}_3$-irreducible, the graph obtained from G by contracting f is not 3-connected, then it has a separating diagonal 2-curve by Lemma 6, and hence we can find a separating 4-cycle along it. Therefore, we can conclude that G is $\mathcal{S}_4$-irreducible.

Next we shall prove the necessity. If the necessity does not hold, then there exists a quadrangulation G which is $\mathcal{S}_4$-irreducible but not $\mathcal{K}_3$-irreducible. That is, G has a face f with $\partial f = abcd$ such that the graph, say G', obtained from G by contracting f at $\{a, c\}$ has a separating 4-cycle C', but it is still 3-connected. Therefore, we have $\deg_G(b), \deg_G(d) \geq 4$. Let $[ac]$ be the vertex of G' obtained by identifying a and c. Clearly, $[ac]$ is a vertex of C'; for otherwise, G would have a separating 4-cycle, a contradiction. Suppose that b is contained in the 2-cell region bounded by C'. Let $C' = [ac]pqr$ and C be the pre-image of C' which is the path $apqrc$ of length 4 in G.

Consider the face-contraction of f at $\{b, d\}$ in G, and let G'' be the resulting graph. Since G is $\mathcal{S}_4$-irreducible, either G'' is non-simple or it has a separating 4-cycle. In the former case, G has a path of length at most 2 joining b and d, which must cross C. Although we might have $d = q$, we have $d \neq r$ and $d \neq p$ since G has no separating 4-cycle. However in any case, q and b must be joined by an edge (since the hexagon $abcrqp$ is a 2-cell region and by Lemma 5). Since $\deg_G(b) \geq 4$, G must have a separating 4-cycle, a contradiction.

In the latter case, G has a path, say $P = bxyzd$, of length 4 joining b and d, which must cross C. If P passes through q (in this case, either $x = q$ or $z = q$, by Lemma 5), then we can find a separating 4-cycle ($abqp, bcrq, dapq$ or $dcrq$), a contradiction. Hence we suppose that P passes through either p or r, say p, then $p = y$. (See the right-hand of Figure 7.) Now consider the face-contraction of the face $abxp$ at $\{a, x\}$. (Note that the 4-cycle $abxp$ bound a face since G has no separating 4-cycle.) We have $x \neq q$, since P does not pass through q, by the assumption. Since G is $\mathcal{S}_4$-irreducible, the resulting graph either is non-simple or has a separating 4-cycle. The former case can be easily excluded, and consider the latter case. In this case, d and x must be joined by a path of length 3 intersecting neither b nor p. Since $\deg(b) \geq 4$ (and by our assumptions), the latter case does not happen unless G has a separating 4-cycle, a contradiction. Therefore, G is also $\mathcal{K}_3$-irreducible. ■

The sufficiency of Theorem 3 does not hold even on the Klein bottle. Consider the quadrangulation on the Klein bottle obtained from two copies of Q^1_{P} by pasting them to identify a facial cycle of one copy and a facial cycle of the other. Then it is $\mathcal{K}_3$-irreducible since each face-contraction breaks the simplicity of Q^1_{P}. However, it clearly has a separating 4-cycle. In the same way, we can construct such a quadrangulation on each closed surface other than the sphere, the projective plane and the torus.

The necessity of Theorem 3 does not hold in general, either.

Proof of Theorem 4. It was proved in [2, 3] that the complete graph K_n with n vertices quadrangulates a closed surface F_χ with Euler characteristic χ if and only if $n(n-5)/4 = -\chi$, except the Klein bottle and the double torus. Let G be a quadrangular embedding of K_8 on F_{-6} with vertex set $\{1, \ldots, 8\}$. Note that F_{-6} can be the orientable closed surface of genus 4, and the nonorientable closed surface of crosscap number 8. Suppose that the vertices $2, 3, 4, 5, 6, 7, 8$ lie in the cyclic order around the vertex 1 in the embedding G. Let G' be the embedding on the same surface obtained from G by removing the vertex "1" and add a vertex a to join $2, 3, 4, 5$ and a vertex b to join $5, 6, 7, 8$. See Figure 3. Since $G' - \{a, b\} = G - \{1\}$ is complete, each diagonal pair $\{x, y\}$ of a quadrilateral face of G' is adjacent if $\{x, y\} \subset \{2, 3, 4, 5, 6, 7, 8\}$. For other pairs, since G' has no quadrilateral face whose diagonal pair is $\{a, b\}$, the diagonal pairs are joined by a path of length at most 2.

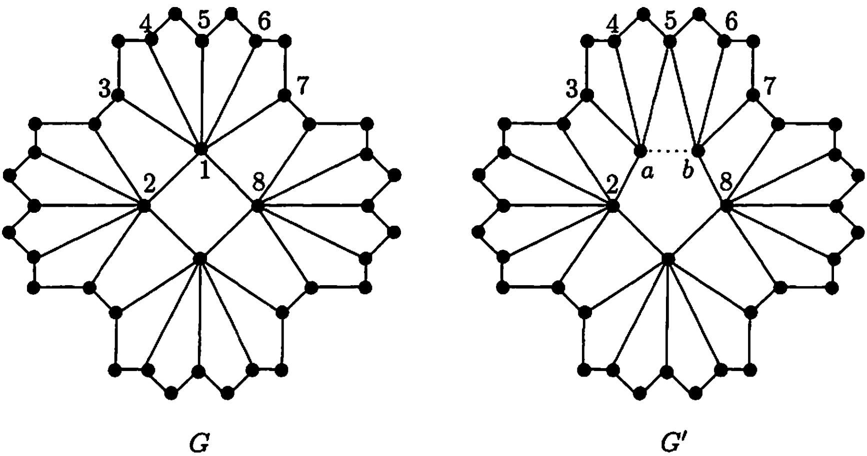


Figure 8: G and G'.

Now prepare another copy of G', denoted by $\tilde{G}'$, whose vertices are $\tilde{a}, \tilde{b}, \tilde{2}, \ldots, \tilde{8}$, corresponding to $a, b, 2, \ldots, 8$ in G', respectively. Let Q be the quadrangulation of F_{-12} obtained, as follows:

1. Add an edge ab and $\tilde{a}\tilde{b}$ to G' and $\tilde{G}'$ respectively, and then each of G' and $\tilde{G}'$ has a unique pentagonal face.

2. Paste G' and $\tilde{G}'$ along the pentagonal faces so that the corresponding vertices of G' and $\tilde{G}'$ are identified.

3. Delete the edge joining the two vertices $a(=\tilde{a})$ and $b(=\tilde{b})$.

Clearly, Q has a face, say g, bounded by $a5b\tilde{5}$, which is $\mathcal{K}_3$-contractible at $\{a, b\}$, but it is not $\mathcal{S}_4$-contractible. Therefore, Q is not $\mathcal{K}_3$-irreducible. However, the face g is non-contractible at $\{5, \tilde{5}\}$, since G' has an edge 25 and since $\tilde{G}'$ has an edge $\tilde{2}\tilde{5}$. Moreover, any other face of Q belongs to either G' or $\tilde{G}'$, and hence it is non-contractible at either diagonal pair. Therefore, Q is $\mathcal{S}_4$-irreducible, but it is not $\mathcal{K}_3$-irreducible. ■

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