

# A 2-parametric generalization of Sierpiński gasket graphs

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## Abstract

Graphs  $S[n, k]$  are introduced as the graphs obtained from the Sierpiński graphs  $S(n, k)$  by contracting edges that lie in no complete subgraph  $K_k$ . The family  $S[n, k]$  is a generalization of a previously studied class of Sierpiński gasket graphs  $S_n$ . Several properties of graphs  $S[n, k]$  are studied in particular, hamiltonicity and chromatic number.

**Key words:** Sierpiński graphs; Sierpiński gasket graphs; Hamiltonicity; Chromatic number

**AMS subject classification (2000):** 05C15, 05C45

## 1 Introduction

Sierpiński-like graphs appear in many different areas of graph theory, topology, probability ([8, 11]), psychology ([16]), etc. The special case  $S(n, 3)$  turns out to be only a step away of the famous Sierpiński gasket graphs  $S_n$ —the graphs obtained after a finite number of iterations that in the limit give the Sierpiński gasket, see [10]. This connection was introduced by Grundy, Scorer and Smith in [23] and later observed in [24].

Graphs  $S(n, 3)$  are also important for the Tower of Hanoi game since they are isomorphic to Hanoi graphs with  $n$  discs and 3 pegs. Metric properties, planarity, vertex and edge coloring were studied by now, see, for instance [1, 5, 6, 7, 14, 22]. Furthermore, in [12] it is proved that graphs  $S(n, 3)$  are uniquely 3-edge-colorable and have unique Hamiltonian cycles.

Graphs  $S(n, 3)$  can be generalized to Sierpiński graphs  $S(n, k)$ ,  $k \geq 3$ , which are also called *Klavžar-Milutinović graphs* and denoted  $KM_{nk}$

[17]. The motivation came from topological studies of the Lipscomb's space [18, 19]. Graphs  $S(n, k)$  independently appeared in [21]. These graphs have many interesting properties, for instance, coding [4] and metric properties [20]. Moreover, in [13] it is shown that graphs  $S(n, k)$  are Hamiltonian and that there are at most two shortest paths between any pair of their vertices. The length between any two vertices can be determined in  $O(n)$  time.

Sierpiński graphs are almost regular. In a natural way, two new families of regular Sierpiński-like graphs  $S^+(n, k)$  and  $S^{++}(n, k)$  were introduced in [15] and their crossing numbers determined (in terms of the crossing number of complete graphs).

Since Sierpiński gasket graphs  $S_n$  are important and are naturally derived from Sierpiński graphs  $S(n, 3)$ , we can apply the same construction for  $S(n, k)$ , where  $k \geq 3$ . We do this in section 2 and denote the new graphs  $S[n, k]$ . In [26] one can find a more algorithmic definition of  $S_n$  based on trilinear coordinates, whereas in [3] one can find the definition of the Sierpiński gasket  $SG_d$  in any Euclidean space of dimension  $d$  with the number of vertices and number of edges. Their graphs  $SG_{d+1}(n)$  are the generalized Sierpiński gasket graphs  $S[n, k]$  introduced in this paper.

It is well known that graphs  $S(n, k)$ ,  $k \geq 3$ , are Hamiltonian [13], as are graphs  $S_n$  [25]. We shall prove the same result for graphs  $S[n, k]$ . The chromatic number, chromatic index and total chromatic number of the Sierpiński graphs  $S(n, k)$  were already determined [20, 9]. In [9], a question was posted for the total chromatic number, where  $k$  is even. The question was answered in [7]. The same chromatic properties were studied for the Sierpiński gasket graphs. The chromatic number was determined in [25], chromatic index in [12] and total chromatic number in [9]. In this paper, we study the chromatic number of graphs  $S[n, k]$ .

## 2 Graphs $S[n, k]$ and their basic properties

First we recall the definition of the Sierpiński graphs  $S(n, k)$ . They are defined for  $n \geq 1$  and  $k \geq 1$  as follows. The vertex set of  $S(n, k)$  consists of all  $n$ -tuples of integers  $1, 2, \dots, k$ , that is,  $V(S(n, k)) = \{1, 2, \dots, k\}^n$ . Two different vertices  $u = (u_1, \dots, u_n)$  and  $v = (v_1, \dots, v_n)$  are adjacent if and only if there exists an  $h \in \{1, \dots, n\}$  such that

- (i)  $u_t = v_t$ , for  $t = 1, \dots, h - 1$ ;
- (ii)  $u_h \neq v_h$ ; and
- (iii)  $u_t = v_h$  and  $v_t = u_h$  for  $t = h + 1, \dots, n$ .

We will shortly write  $u_1 u_2 \dots u_n$  for  $(u_1, u_2, \dots, u_n)$ . See Fig. 1 for  $S(2, 4)$ .

By fixing  $u_1 \in \{1, \dots, k\}$ , we get  $S(n - 1, k)$ . In other words,  $S(n, k)$  is constructed of  $k$  different  $S(n - 1, k)$ . We label each with  $S_i(n, k)$ , for every  $i \in \{1, \dots, k\}$ . Note that  $S_i(n, k)$  and  $S_j(n, k)$ ,  $i \neq j$ , are connected

with a single edge between vertices  $ij\dots j$  and  $ji\dots i$ . As in [9] we call these edges the linking edges of  $S(n, k)$ .

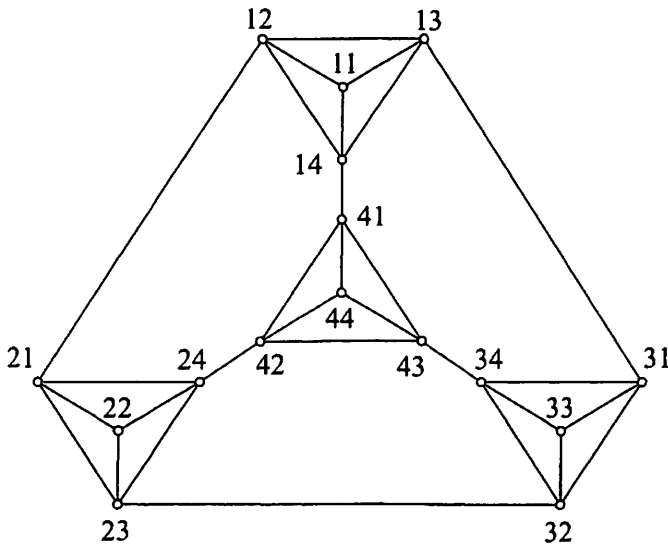


Figure 1: Graph  $S(2,4)$

In [12] in a natural way the Sierpiński gasket  $S_n$  is constructed from the Sierpiński graph  $S(n, 3)$  by contracting all of its edges that lie in no triangle  $K_3$ . We can apply the same construction method for any Sierpiński graph  $S(n, k)$  by contracting all of its edges that lie in no induced subgraph  $K_k$ .

Let  $u_1u_2\dots u_rjl\dots l$  and  $u_1u_2\dots u_rlj\dots j$ ,  $0 \leq r \leq n - 2$ , be two adjacent vertices of graph  $S(n, k)$ . We identify them in one vertex and write  $u_1u_2\dots u_r\{j, l\}$  or shortly  $u_{(r)}\{j, l\}$ ,  $j \neq l$  and  $j, l \in \{1, \dots, k\}$ , where  $u_{(0)}\{j, l\}$  means  $\{j, l\}$ . We get a 2-parametric Sierpiński gasket graph  $S[n, k]$  (shortly  $k$ -Sierpiński gasket graph). We already know the case for  $k = 3$ , namely the Sierpiński gasket  $S_n = S[n, 3]$ . An example of  $S[2, 4]$  is shown in the Fig. 2.

Since  $S(n, k)$  is built of  $k$  copies of  $S(n - 1, k)$ , the graph  $S[n, k]$  is also built of  $k$  copies of  $S[n - 1, k]$ . We denote each copy with  $S_i[n, k]$ . Note that  $S_i[n, k]$  and  $S_j[n, k]$ ,  $i \neq j$ , share one vertex, that is  $\{i, j\}$ .

It is known, that  $S(1, k)$  is isomorphic to  $K_k$ . Hence,  $S[1, k]$  is also isomorphic to  $K_k$ .

Adjacency of vertices in  $S[n, k]$  is given in the next proposition.

**Proposition 2.1** *Let  $n \geq 2$  and  $u = u_1\dots u_r\{i, j\}$  be a vertex in  $S[n, k]$ ,  $i, j \in \{1, \dots, k\}, i \neq j$ .*

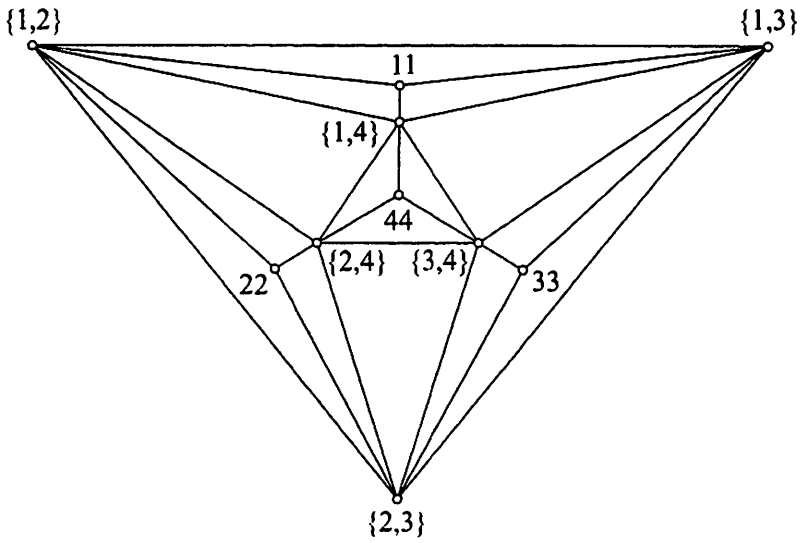


Figure 2: Graph  $S[2, 4]$

(i) If  $0 \leq r \leq n - 3$ , then  $u$  is adjacent to

$$\bar{u}_{(n-2)}\{j, l\}, l \in \{1, \dots, k\} \setminus \{j\},$$

and

$$\bar{\bar{u}}_{(n-2)}\{i, l\}, l \in \{1, \dots, k\} \setminus \{i\}.$$

(ii) If  $r = n - 2$ , then  $u$  is adjacent to

$$\bar{u}_{(n-2)}\{i, l\}, l \in \{1, \dots, k\} \setminus \{i, j\},$$

$$\bar{\bar{u}}_{(n-2)}\{j, l\}, l \in \{1, \dots, k\} \setminus \{i, j\},$$

$$\begin{cases} \bar{u}_{(t-1)}\{i, u_t\}, & t \text{ largest index with } u_t \neq i, 1 \leq t \leq n - 2, \\ i \dots i, & \text{else,} \end{cases}$$

and

$$\begin{cases} \bar{\bar{u}}_{(s-1)}\{j, u_s\}, & s \text{ largest index with } u_s \neq j, 1 \leq s \leq n - 2, \\ j \dots j, & \text{else.} \end{cases}$$

**Proof.** (i) Let  $r \leq n - 3$ . Take two vertices  $\bar{u} = u_1 \dots u_r i j \dots j j$  and  $\bar{\bar{u}} = u_1 \dots u_r j i \dots i i$ , where  $\bar{u}$  ends with at least two  $j$ 's and  $\bar{\bar{u}}$  with at least two  $i$ 's. Then  $\bar{u}$  is in  $S(n, k)$  adjacent to  $u_1 \dots u_r i j \dots j l$ ,  $l \in \{1, \dots, k\} \setminus \{j\}$ . In the construction procedure this vertex contracts to  $\bar{u}_{(n-2)}\{j, l\}$ ,  $l \in \{1, \dots, k\} \setminus \{j\}$  in  $S[n, k]$ . Similarly,  $\bar{\bar{u}}$  is adjacent to  $u_1 \dots u_r j i \dots i l$ ,  $l \in \{1, \dots, k\} \setminus \{i\}$ , which contracts to  $\bar{\bar{u}}_{(n-2)}\{i, l\}$ ,  $l \in \{1, \dots, k\} \setminus \{i\}$  in  $S[n, k]$ . The argument also holds for  $r = 0$ .

(ii) Let  $r = n - 2$ . Then  $\bar{u} = u_1 \dots u_{n-2} i j$  and  $\bar{\bar{u}} = u_1 \dots u_{n-2} j i$ . In  $S(n, k)$ , the vertex  $\bar{u}$  is adjacent to  $u_1 \dots u_{n-2} i l$ ,  $l \in \{1, \dots, k\} \setminus \{i, j\}$ , which contracts to  $\bar{u}_{(n-2)}\{i, l\}$ ,  $l \in \{1, \dots, k\} \setminus \{i, j\}$  in  $S[n, k]$ . It is also adjacent to  $x = u_1 \dots u_{n-2} i i$ . If  $u_1 = \dots = u_{n-2} = i$ , then  $\bar{u}$  is adjacent to the extreme vertex  $i \dots i$  in  $S[n, k]$ . If not all of  $u_1, \dots, u_{n-2}$  are equal to  $i$ , then let  $t$  be the largest index that  $u_t \neq i$ . Then the vertex  $\bar{u}$  is adjacent to  $x = u_1 \dots u_t i \dots i$ ,  $t \in \{1, \dots, n - 2\}$ . In this case,  $\bar{u}$  is adjacent to  $\bar{u}_{(t-1)}\{i, u_t\}$ . Similarly, the vertex  $\bar{\bar{u}}$  is adjacent to  $u_1 \dots u_{n-2} j l$ ,  $l \in \{1, \dots, k\} \setminus \{i, j\}$ , which contracts to  $\bar{\bar{u}}_{(n-2)}\{j, l\}$ ,  $l \in \{1, \dots, k\} \setminus \{i, j\}$  in  $S[n, k]$ . The vertex  $\bar{\bar{u}}$  is also adjacent to  $x = u_1 \dots u_{n-2} j j$ . If  $u_1 = \dots = u_{n-2} = j$ , then  $\bar{\bar{u}}$  is adjacent to the extreme vertex  $j \dots j$  in  $S[n, k]$ . If not all of  $u_1, \dots, u_{n-2}$  are equal to  $j$ , then let  $s$  be the largest index that  $u_s \neq j$ . Then the vertex  $\bar{\bar{u}}$  is adjacent to  $y = u_1 \dots u_s j \dots j$ ,  $s \in \{1, \dots, n - 2\}$ . Therefore,  $\bar{\bar{u}}$  is adjacent to  $\bar{\bar{u}}_{(s-1)}\{j, u_s\}$ .  $\square$

It is easy to see that the graphs  $S[n, k]$  have  $k$  extreme vertices of degree  $k - 1$  and  $|V(S[n, k])| - k$  remaining vertices of degree  $(k - 1) + (k - 1) = 2(k - 1)$ . We immediately get:

**Proposition 2.2** *Graphs  $S[n, k]$  are Eulerian if and only if  $k$  is odd.*

To gain more knowledge of the structure of  $S[n, k]$ , we give the number of vertices and edges of  $S[n, k]$ .

**Proposition 2.3** *Graphs  $S[n, k]$  have  $\frac{k}{2}(k^{n-1} + 1)$  vertices and  $\frac{k-1}{2} \cdot k^n$  edges.*

**Proof.** A vertex in  $S[n, k]$  is of the form  $u_1 u_2 \dots u_r \{j, l\}$ . We have  $k$  possibilities for every  $u_i$ ,  $i \in \{1, \dots, r\}$ , and  $\binom{k}{2}$  possibilities for the unordered pair  $\{j, l\}$ . There are also  $k$  extreme vertices left. We get:

$$|V(S[n, k])| = k + \sum_{r=0}^{n-2} k^r \cdot \binom{k}{2} = \frac{k}{2}(k^{n-1} + 1).$$

We determine the number of edges by using the formula:

$$\begin{aligned}
 |E(S[n, k])| &= \frac{1}{2} \sum_{u \in S[n, k]} \text{deg}(u) \\
 &= \frac{1}{2} \left( k(k-1) + \left( \frac{k}{2} (k^{n-1} + 1) - k \right) \cdot 2(k-1) \right) \\
 &= \binom{k}{2} k^{n-1} = \frac{k-1}{2} \cdot k^n.
 \end{aligned}$$

□

**Remark 2.4** *Alternatively, for all  $k \geq 1$ , a proof can be based on the recurrence relations  $|V(S[n+1, k])| = k \cdot |V(S[n, k])| - \binom{k}{2}$ ,  $n \geq 1$ ,  $|V(S[1, k])| = k$ , since  $\binom{k}{2}$  vertices are merged and  $S[1, k]$  is isomorphic to  $K_k$ . Further,  $|E(S[n+1, k])| = k \cdot |E(S[n, k])|$ ,  $n \geq 1$ ,  $|E(S[1, k])| = \binom{k}{2}$ , since the graph  $S[n+1, k]$  consists of  $k$  copies of  $S[n, k]$ .*

**Corollary 2.5** *For all  $n, k \in \mathbb{N}$ :  $|E(S[n, k])| = (k-1) \cdot |V(S[n, k])| - \binom{k}{2}$ .*

Applying Proposition 2.3 and setting  $k = 3$  we get:

**Corollary 2.6** *Graphs  $S_n$  have  $\frac{3}{2}(3^{n-1} + 1)$  vertices and  $3^n$  edges.*

### 3 Hamiltonicity

In [25], Tequia and Godbole proved that graphs  $S_n$  are Hamiltonian. In this section we generalize their statement to  $S[n, k]$ . First, we need a lemma.

**Lemma 3.1** *Graphs  $S[n, k]$ ,  $k \geq 2$ , have a Hamiltonian path connecting two arbitrary extreme vertices.*

**Proof.** The statement holds for  $k = 2$  because  $S[n, 2]$  is isomorphic to a path on  $2^{n-1} + 1$  vertices.

Let  $k \geq 3$ . This statement is true for  $n = 1$ , since  $S[1, k]$  is isomorphic to  $K_k$ . Let  $n \geq 2$ . Without loss of generality we start in vertex  $1 \dots 1$ . Since  $S_1[n, k]$  is isomorphic to  $S[n-1, k]$  we can find a Hamiltonian path from  $1 \dots 1$  to the vertex  $\{1, 2\}$ . With the same argument, we can find a Hamiltonian path in  $S_2[n, k]$  from the vertex  $\{1, 2\}$  to the vertex  $\{2, 3\}$ . Next we find a Hamiltonian path in  $S_3[n, k]$  from the vertex  $\{2, 3\}$  to the vertex  $\{3, 4\}$  by avoiding  $\{1, 3\}$  (since we locally have an induced complete graph, avoiding is possible). We continue the procedure in  $S_4[n, k]$  by

finding a Hamiltonian path between vertices  $\{3, 4\}$  and  $\{4, 5\}$  by avoiding vertices  $\{1, 4\}$  and  $\{2, 4\}$ . In general we find a Hamiltonian path in  $S_i[n, k]$ ,  $i \in \{3, \dots, k-1\}$ , from the vertex  $\{i-1, i\}$  to the vertex  $\{i, i+1\}$  by avoiding vertices  $\{1, i\}, \{2, i\}, \dots, \{i-2, i\}$ . Finally, we find a Hamiltonian path in  $S_k[n, k]$  from vertex  $\{k-1, k\}$  to vertex  $k \dots k$  by avoiding vertices  $\{1, k\}, \{2, k\}, \dots, \{k-2, k\}$ . All together we have constructed a Hamiltonian path between vertices  $1 \dots 1$  and  $k \dots k$ .

Similarly, we can find a Hamiltonian path between any two different extreme vertices in  $S[n, k]$ .  $\square$

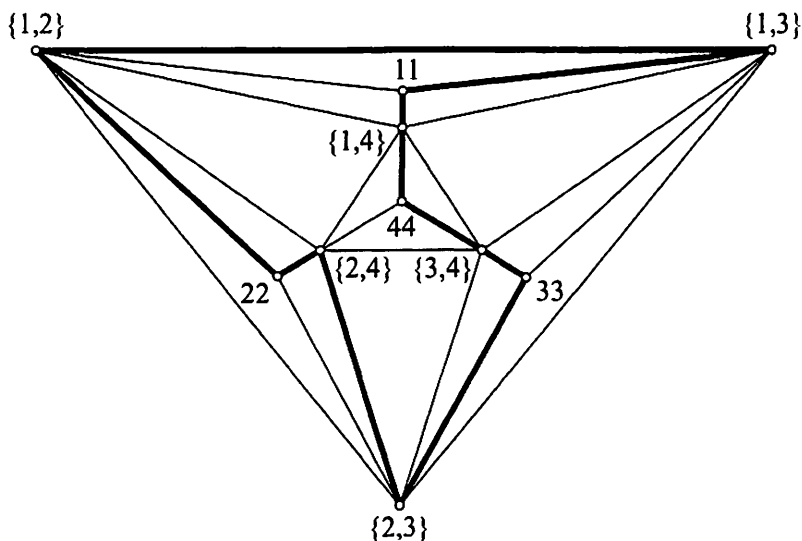


Figure 3: A Hamiltonian cycle in  $S[2, 4]$

**Theorem 3.2** *Graphs  $S[n, k]$  are Hamiltonian, for any  $n \geq 1$  and  $k \geq 3$ .*

**Proof.** The statement is true for  $n = 1$ , since  $S[1, k]$  is a complete graph.

Let  $n \geq 2$ . By Lemma 3.1, we can find a Hamiltonian path from the vertex  $\{k, 1\}$  to the vertex  $\{1, 2\}$  in  $S_1[n, k]$ . Similarly, we can find a Hamiltonian path between vertices  $\{1, 2\}$  and  $\{2, 3\}$  in  $S_2[n, k]$ . Now we find a Hamiltonian path in  $S_3[n, k]$  from between vertices  $\{2, 3\}$  and  $\{3, 4\}$  by avoiding vertex  $\{1, 3\}$ . In general, we find a Hamiltonian path in  $S_i[n, k]$  between vertices  $\{i-1, i\}$  and  $\{i, i+1\}$ ,  $i \in \{3, \dots, k-1\}$ , by avoiding vertices  $\{1, i\}, \{2, i\}, \dots, \{i-2, i\}$ . Finally, we find a Hamiltonian path in  $S_k[n, k]$  from the vertex  $\{k-1, k\}$  to the vertex  $\{k, 1\}$  by avoiding vertices  $\{2, k\}, \dots, \{k-2, k\}$ . Again, all the avoiding is possible because

locally we have an induced complete graph. All the paths together form a Hamiltonian cycle in  $S[n, k]$ .  $\square$

Fig. 3 shows a Hamiltonian cycle in  $S[2, 4]$  obtained with the above construction. Once again, by setting  $k = 3$  we get:

**Corollary 3.3** [25] *Graphs  $S_n$  are Hamiltonian.*

## 4 The chromatic number

Since  $S[n, k]$  is built of complete graphs  $K_k$ , it is obvious that  $\chi(S[n, k]) \geq k$ . We prove the following.

**Theorem 4.1** *For any  $n \geq 1$  and any  $k \geq 1$ ,  $\chi(S[n, k]) = k$ .*

**Proof.** For  $k = 1$  and  $k = 2$  we get a vertex and a path on  $2^{n-1} + 1$  vertices respectively for which the statement is true. Let  $k \geq 3$ . For  $n = 1$  we have a complete graph on  $k$  vertices. It is well known that  $\chi(K_k) = k$ . Let  $n \geq 2$ .

Graph  $S[n, k]$  consists of  $k$  copies of  $S[n - 1, k]$ , each denoted with  $S_i[n, k]$ ,  $i \in \{1, \dots, k\}$ . Two copies, say  $S_i[n, k]$  and  $S_j[n, k]$ ,  $i \neq j$ , share a common vertex  $\{i, j\}$ . For every  $i, j \in \{1, \dots, k\}$ ,  $i \neq j$ , expand this vertex into two vertices  $ij \dots j$  and  $ji \dots i$  connected with a linking edge.

Note that for  $n \geq 3$  the expansion process described above is not the inverse process of getting  $S[n, k]$  from the  $S(n, k)$ , since we do not expand all the vertices that were contracted in  $S(n, k)$ .

By induction assumption graph  $S_1[n, k]$  can be colored with  $k$  colors. Denote the color of the vertex  $1j \dots j$ ,  $j \in \{1, \dots, k\}$ , with  $c_{1+j-2 \pmod k}$ , where  $c_{1+j-2 \pmod k} \in \{1, \dots, k\}$ . Then graphs  $S_i[n, k]$ ,  $i \in \{2, \dots, k\}$ , can also be colored in such a way that a vertex  $ij \dots j$  receives color  $c_{i+j-2 \pmod k} \in \{1, \dots, k\}$ . This coloring is the same as the coloring used to color graph  $S_1[n, k]$ , only to be rotated clockwise. In other words we color graph  $S[n - 1, k]$  like  $S_1[n, k]$  and rotate it clockwise  $(i - 1)$ -times to get the coloring of graph  $S_i[n, k]$ . See Fig. 4 for the visualization of the rotated colorings.

A quick observation is that we obviously do not get a proper vertex coloring of the expanded graph  $S[n, k]$  since vertices  $ij \dots j$  and  $ji \dots i$ ,  $i \neq j$ , receive the same color, that is  $c_{i+j-2 \pmod k}$ . By contracting the previously expanded linking edges, and getting  $S[n, k]$ , the merged vertex  $\{i, j\}$  receives color  $c_{i+j-2 \pmod k}$ . Therefore, we get a proper vertex coloring of  $S[n, k]$ .  $\square$

By setting  $k = 3$  we immediately get the next result:



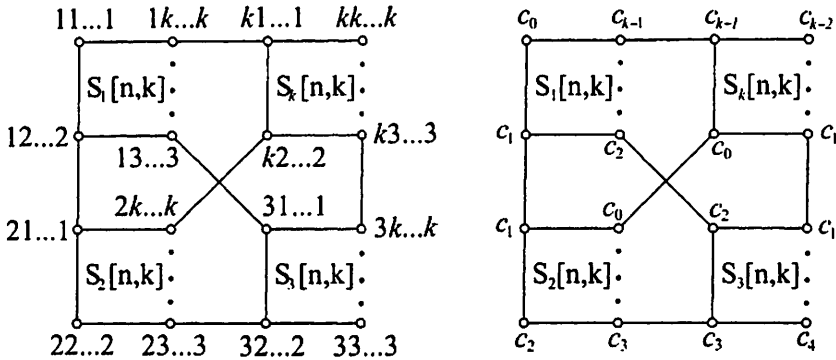


Figure 4: Vertex coloring of graph  $S[n, k]$

**Corollary 4.2** [25] *For any  $n \geq 1$ ,  $\chi(S_n) = 3$ .*

We conclude the paper by asking what is the chromatic index and the total chromatic number of  $S[n, k]$ ?

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