On the harmonic index and the matching number of a tree*

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Abstract

The harmonic index of a graph G is defined as the sum of weights $\frac{2}{d(u)+d(v)}$ of all edges uv of G, where $d(u)$ and $d(v)$ are the degrees of the vertices u and v in G, respectively. In this paper, we give a sharp lower bound on the harmonic index of trees with a perfect matching in terms of the number of vertices. A sharp lower bound on the harmonic index of trees with a given size of matching is also obtained.
Key words: Harmonic index; matching number; bound.
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1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Its *order* is $|V(G)|$, denoted by n. For $v \in V(G)$, let $N_G(v)$ (or $N(v)$ for short) be the set of vertices which are adjacent to v in G and let $d_G(v)$ (or $d(v)$ for short) be the degree of v. Clearly, $d(v) = |N(v)|$. We will use $G-v$ to denote the graph that arises from G by deleting the vertex $v \in V(G)$.

The Randić index is one of the most successful molecular descriptors in structure-property and structure-activity relationships studies. The Randić

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index of a graph G is defined in [10] as the sum of the weights $(d(u)d(v))^{-\frac{1}{2}}$ over all edges uv of G. The mathematical properties of this graph invariant have been studied extensively (see recent book [6] and survey [8]). Motivated by the success of Randić index, various generalizations and modifications were introduced, such as the sum-connectivity index [11, 12] and the general sum-connectivity index [2, 3].

Another variant of the Randić index, named the harmonic index $H(G)$, which is defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)},$$

where the summation goes over all edges uv of G. This index was first appeared in [4]. Estimating bounds for $H(G)$ is of great interest, and many results have been obtained. For example, Favaron *et al.* [5] considered the relationship between the harmonic index and the eigenvalues of graphs; Zhong [13, 14] found the minimum and maximum values of the harmonic index for simple connected graphs, trees and unicyclic graphs, and characterized the corresponding extremal graphs, respectively. Li and Shiu [9] studied how the harmonic index behaves when the graph is under perturbations and provided a simpler method for determining the unicyclic graphs with maximum and minimum harmonic index among all unicyclic graphs, respectively. Moreover, the lower bound for harmonic index is also obtained in [9]. Deng *et al.* [1] studied the relationship between the harmonic index and the chromatic number of a graph G, and obtained the lower bound for $H(G)$ in terms of its chromatic number.

In this paper, we consider the relationship between the harmonic index and the matching number of a tree. Lower bounds on the harmonic index of trees with a perfect matching and trees with a given size of matching are obtained, respectively.

2 Preliminaries

Two distinct edges in a graph G are called to be independent if they are not adjacent in G. A matching of G is a set of mutually independent edges in G. The largest matching is called a maximum matching. The cardinality of a maximum matching of G is commonly known as its matching number. Let M be a matching of G. M is called the m-matching of G if M contains exactly m edges of G. A vertex v of G is said to be M-saturated if it is incident with an edge of M, otherwise v is called an M-unsaturated vertex. The matching M of G is called a perfect matching if all vertices of G are M-saturated.

We begin with the following two important results due to Hou and Li [7] for trees with an m-matching.

Lemma 2.1 ([7]) *Let T be a tree of order $n(n \geq 3)$ with a perfect matching. Then T has at least two pendant vertices such that each of them is adjacent to a vertex of degree 2.*

Lemma 2.2 ([7]) *Let T be a tree of order n with an m-matching, where $n > 2m$. Then there is an m-matching M and a pendant vertex v such that v is an M-unsaturated vertex.*

Let $e = uv$ be an edge of a graph G. Let G' be the graph obtained from G by contracting the edge e into a new vertex u_e and adding a new pendent edge $u_e v_e$, where v_e is a new pendent vertex. We say that G' is obtained from G by separating an edge uv (see Fig. 1).

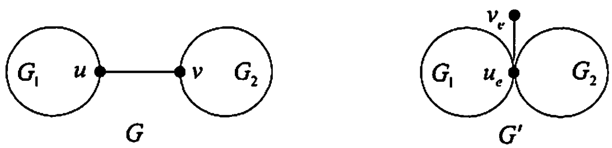

Figure 1: Separating an edge uv.

Lemma 2.3 ([9]) *Let $e = uv$ be a cut edge of a connected graph G and suppose that $G - uv = G_1 \cup G_2$ $(|V(G_1)|, |V(G_2)| \geq 2)$, where G_1 and G_2 are two components of $G - uv, u \in V(G_1)$ and $v \in V(G_2)$. Let G' be the graph obtained from G by separating the edge uv. Then $H(G) > H(G')$.*

Lemma 2.4 *(1) For $x \geq 3$, the function $f(x) = \frac{2x-2}{x+2} - \frac{2x-6}{x+1} - \frac{2}{x}$ is monotonicly decreasing on x.*

(2) For $x \geq 2$, the function $g(x) = \frac{2x-2}{x+2} - \frac{2x-4}{x+1}$ is monotonicly decreasing on x.

Proof. (1) We consider the derivative of $f(x)$. For $x \geq 3$, we have

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{6}{(x+2)^2} - \frac{8}{(x+1)^2} + \frac{2}{x^2} \\ &= \frac{6x^2(x+1)^2 + 2(x+2)^2(x+1)^2 - 8(x+2)^2x^2}{x^2(x+1)^2(x+2)^2} \\ &= \frac{-8x^3 + 24x + 8}{x^2(x+1)^2(x+2)^2} < 0.\end{aligned}$$

Thus $f(x)$ is monotonicly decreasing on x.

(2) Note that, for $x \geq 2$, we have

$$\frac{dg(x)}{dx} = \frac{6}{(x+2)^2} - \frac{6}{(x+1)^2} < 0.$$

Thus $g(x)$ is monotonicly decreasing on x. □

Lemma 2.5 *Let x, y be positive numbers with $1 \leq y \leq x - 1$. Let*

$$h(x,y) = \frac{2x-2y-2}{x+2} - \frac{2x-4y-4}{x+1} - \frac{2y}{x}.$$

Then $h(x, y)$ is monotonicly decreasing on x and y, respectively.

Proof. We consider some partial derivatives of $h(x, y)$. Note that

$$\begin{aligned}\frac{\partial h(x,y)}{\partial y} &= \frac{4}{x+1} - \frac{2}{x+2} - \frac{2}{x} \\ &= \frac{-4}{(x+2)(x+1)x} < 0.\end{aligned}$$

Thus, $h(x, y)$ is monotonicly decreasing on y. On the other hand,

$$\begin{aligned}\frac{\partial h(x,y)}{\partial x} &= \frac{2y+6}{(x+2)^2} + \frac{2y}{x^2} - \frac{4y+6}{(x+1)^2} \\ &= \frac{12x^2y + 24xy + 8y - 12x^3 - 18x^2}{(x+2)^2(x+1)^2x^2}.\end{aligned}$$

Note that $y \leq x - 1$. Then we have

$$\begin{aligned}\frac{\partial h(x,y)}{\partial x} &\leq \frac{12x^2(x-1) + 24x(x-1) + 8(x-1) - 12x^3 - 18x^2}{(x+2)^2(x+1)^2x^2} \\ &= \frac{-6x^2 - 16x - 8}{(x+2)^2(x+1)^2x^2} < 0.\end{aligned}$$

Thus $h(x, y)$ is monotonicly decreasing on x. □

3 Main results

Let n and m be positive integers and $n \geq 2m$. Let $T^0(n,m)$ be a tree of order n, which is obtained from a star S_{n-m+1} by attaching a pendant edge to each of certain $m-1$ non-central vertices of S_{n-m+1}. Obviously, $T^0(n,m)$ is a tree of order n with an m-matching. Let $f(n,m)$ be the harmonic index of $T^0(n,m)$. Then

$$f(n,m) = \frac{2(n-2m+1)}{n-m+1} + \frac{2(m-1)}{n-m+2} + \frac{2(m-1)}{3}.$$

Now, we give the following initial result.

Theorem 3.1 *Let T be a tree of order $n = 2m$ with a perfect matching. Then*

$$H(T) \geq f(2m,m),$$

the equality holds if and only if $T \cong T^0(2m,m)$.

Proof. We prove the theorem by induction on m. If $m = 1,2,3$, then the theorem holds clearly by the fact that there are at most two trees with $n = 2m$ vertices and a perfect matching for $m = 1,2,3$.

Let T be a tree of order $2m$ with a perfect matching ($m \geq 4$). Then T have a perfect matching M. If M contains a non-pendent edge xy of T, then let $\bar{T}$ be the tree obtained from T by performing separating an edge xy. Then $\bar{T}$ contains a perfect matching $M_0 = M \cup \{e_0\} \setminus \{xy\}$, where e_0 is the new edge added into T after performing separating. Repeat this procedure until there is no non-pendent edge in the most updated perfect matching. Let T' be the resulting tree and the corresponding perfect matching be M'. By Lemma 2.3, we have $H(T) \geq H(T')$. Note that the equality does not hold if at least one separating is preformed. Clearly, each edge in M' is a pendent edge.

By Lemma 2.1, T' has a pendant vertex x_1 which is adjacent to a vertex x_2 of degree 2. Then $x_1x_2 \in E(T')$ and there is a unique vertex $x_3 \neq x_1$ such that $x_2x_3 \in E(T')$. Let $T^* = T' - x_1 - x_2$. Then T^* is a tree with $2(m-1)$ vertices and with an $(m-1)$-matching. Let $d(x_3) = d$ and $N(x_3) \setminus \{x_2\} = \{y_1, y_2, \dots, y_{d-1}\}$, then $d \geq 3$. Without loss of generality, we can assume $d(y_1) = 1$, $d(y_i) \geq 2$ for $i = 2, \dots, d-1$. By the induction assumption, we have

$$
\begin{aligned}
H(T') &= H(T^*) + \frac{2}{3} + \frac{2}{2+d} + \frac{2}{1+d} - \frac{2}{d} \\
&\quad + \sum_{i=2}^{d-1} \left(\frac{2}{d(y_i)+d} - \frac{2}{d(y_i)+d-1} \right) \\
&\geq f(2(m-1), m-1) + \frac{2}{3} + \frac{2}{2+d} + \frac{2}{1+d} - \frac{2}{d} \\
&\quad + \sum_{i=2}^{d-1} \left(\frac{2}{d(y_i)+d} - \frac{2}{d(y_i)+d-1} \right) \\
&= f(2m, m) + \frac{2}{m} - \frac{2}{m+1} + \frac{2(m-2)}{m+1} - \frac{2(m-1)}{m+2} \\
&\quad + \frac{2}{2+d} + \frac{2}{1+d} - \frac{2}{d} + \sum_{i=2}^{d-1} \left(\frac{2}{d(y_i)+d} - \frac{2}{d(y_i)+d-1} \right) \\
&\geq f(2m, m) + \frac{2}{m} + \frac{2m-6}{m+1} - \frac{2m-2}{m+2} + \frac{2}{2+d} + \frac{2}{1+d} - \frac{2}{d} \\
&\quad + (d-2) \left(\frac{2}{2+d} - \frac{2}{1+d} \right) \\
&= f(2m, m) + \frac{2}{m} + \frac{2m-6}{m+1} - \frac{2m-2}{m+2} + \frac{2d-2}{2+d} - \frac{2d-6}{1+d} - \frac{2}{d}.
\end{aligned}
$$

Moreover, note that $d \leq m$. Then Lemma 2.4(1) implies that

$$
\frac{2d-2}{2+d} - \frac{2d-6}{1+d} - \frac{2}{d} \geq \frac{2m-2}{2+m} - \frac{2m-6}{1+m} - \frac{2}{m}.
$$

That is,

$$
\begin{aligned}
H(T') &\geq f(2m, m) + \frac{2}{m} + \frac{2m-6}{m+1} - \frac{2m-2}{m+2} \\
&\quad + \frac{2m-2}{2+m} - \frac{2m-6}{1+m} - \frac{2}{m} \\
&= f(2m, m).
\end{aligned}
$$

The equality $H(T) = f(2m, m)$ holds if and only if separating is not preformed and equality holds throughout the above inequalities. That is, if and only if $T^* \cong T^0(2(m-1), m-1)$, $d(y_1) = 1, d(y_i) = 2$ for $i = 2, \ldots, d-1$ and $d = m$. Thus $T \cong T^0(2m, m)$. □

Another result of the present paper is to give a sharp lower bound on the harmonic index of trees with a given size of matching as follows.

Theorem 3.2 *Let T be a tree of order n with an m-matching, where $n > 2m$. Then*

$$H(T) \geq f(n,m),$$

the equality holds if and only if $T \cong T^0(n,m)$.

Proof. We prove the theorem by induction on n. Suppose that $n = 2m$. Then the result follows from Theorem 3.1. Now we suppose that $n > 2m$. Let T be a tree of order n with an m-matching. By Lemma 2.2, T has an m-matching M and a pendant vertex v which is M-unsaturated. Let $uv \in E(T)$ with $d(u) = d$ and $N(u) \setminus \{v\} = \{v_1, v_2, \ldots, v_{d-1}\}$. Obviously, $d \geq 2$. Let $T' = T - v$. Then T' is a tree with $n-1$ vertices and with an m-matching. By the induction assumption, we have

$$\begin{aligned}
H(T) &= H(T') + \frac{2}{d+1} + \sum_{i=1}^{d-1} \left(\frac{2}{d+d(v_i)} - \frac{2}{d+d(v_i)-1} \right) \\
&\geq f(n-1,m) + \frac{2}{d+1} + \sum_{i=1}^{d-1} \left(\frac{2}{d+d(v_i)} - \frac{2}{d+d(v_i)-1} \right) \\
&= f(n,m) + \frac{2(n-2m)}{n-m} - \frac{2(n-2m+1)}{n-m+1} + \frac{2(m-1)}{n-m+1} - \frac{2(m-1)}{n-m+2} \\
&\quad + \frac{2}{d+1} + \sum_{i=1}^{d-1} \left(\frac{2}{d+d(v_i)} - \frac{2}{d+d(v_i)-1} \right) \\
&= f(n,m) + \frac{2n-4m}{n-m} - \frac{2n-6m+4}{n-m+1} - \frac{2m-2}{n-m+2} \\
&\quad + \frac{2}{d+1} + \sum_{i=1}^{d-1} \left(\frac{2}{d+d(v_i)} - \frac{2}{d+d(v_i)-1} \right).
\end{aligned}$$

Now, we consider the following two cases.

Case 1. $d(v_i) \geq 2$ for $i = 1, 2, \ldots, d-1$.

In the case, we have

$$\begin{aligned}
H(T) &\geq f(n,m) + \frac{2n-4m}{n-m} - \frac{2n-6m+4}{n-m+1} - \frac{2m-2}{n-m+2} \\
&\quad + \frac{2}{d+1} + (d-1)\left(\frac{2}{d+2} - \frac{2}{d+1} \right) \\
&= f(n,m) + \frac{2n-4m}{n-m} - \frac{2n-6m+4}{n-m+1} - \frac{2m-2}{n-m+2} \\
&\quad + \frac{2d-2}{d+2} - \frac{2d-4}{d+1}.
\end{aligned}$$

Moreover, note that T has an m-matching and $d \le n-m$. Then Lemma 2.4(2) implies that

$$\frac{2d-2}{d+2} - \frac{2d-4}{d+1} \ge \frac{2n-2m-2}{n-m+2} - \frac{2n-2m-4}{n-m+1}.$$

That is,

$$\begin{aligned} H(T) &\ge f(n,m) + \frac{2n-4m}{n-m} - \frac{2n-6m+4}{n-m+1} - \frac{2m-2}{n-m+2} \\ &\quad + \frac{2n-2m-2}{n-m+2} - \frac{2n-2m-4}{n-m+1} \\ &= f(n,m) + \frac{2n-4m}{n-m} - \frac{4n-8m}{n-m+1} + \frac{2n-4m}{n-m+2} \\ &= f(n,m) + \frac{4n-8m}{(n-m)(n-m+1)(n-m+2)} \\ &> f(n,m). \end{aligned}$$

Case 2. There exists some i $(1 \le i \le d-1)$ such that $d(v_i) = 1$.

Without loss of generality, we assume that $d(v_1) = d(v_2) = \cdots = d(v_k) = 1$ and $d(v_i) \ge 2$ for $k+1 \le i \le d-1$, where $k \ge 1$. Then we have

$$\begin{aligned} H(T) &\ge f(n,m) + \frac{2n-4m}{n-m} - \frac{2n-6m+4}{n-m+1} - \frac{2m-2}{n-m+2} \\ &\quad + \frac{2}{d+1} + k\left(\frac{2}{d+1} - \frac{2}{d}\right) + \sum_{i=k+1}^{d-1}\left(\frac{2}{d+d(v_i)} - \frac{2}{d+d(v_i)-1}\right) \\ &\ge f(n,m) + \frac{2n-4m}{n-m} - \frac{2n-6m+4}{n-m+1} - \frac{2m-2}{n-m+2} \\ &\quad + \frac{2}{d+1} + \frac{2k}{d+1} - \frac{2k}{d} + (d-k-1)\left(\frac{2}{d+2} - \frac{2}{d+1}\right) \\ &= f(n,m) + \frac{2n-4m}{n-m} - \frac{2n-6m+4}{n-m+1} - \frac{2m-2}{n-m+2} \\ &\quad + \frac{2d-2k-2}{d+2} - \frac{2d-4k-4}{d+1} - \frac{2k}{d}. \end{aligned}$$

Note that T has an m-matching, $k \le n-2m$ and $d \le n-m$. Then Lemma 2.5 implies that

$$\frac{2d-2k-2}{d+2} - \frac{2d-4k-4}{d+1} - \frac{2k}{d} \ge \frac{2m-2}{n-m+2} + \frac{2n-6m+4}{n-m+1} - \frac{2n-4m}{n-m}.$$

That is,

$$\begin{aligned}H(T) &\geq f(n,m)+\frac{2n-4m}{n-m}-\frac{2n-6m+4}{n-m+1}-\frac{2m-2}{n-m+2}\\&+\frac{2m-2}{n-m+2}+\frac{2n-6m+4}{n-m+1}-\frac{2n-4m}{n-m}\\&= f(n,m).\end{aligned}$$

The equality $H(T) = f(n,m)$ holds if and only if equality holds throughout the above inequalities. That is, if and only if $T' \cong T^0(n-1,m)$, $d(v_i) = 1$ for $1 \leq i \leq n-2m$, $d(y_i) = 2$ for $n-2m+1 \leq i \leq d-1$ and $d = n-m$. Thus $T \cong T^0(n,m)$. □

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