A Characterization On Potentially $K_{2,5}$-graphic Sequences *

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Abstract

For given a graph H, a graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be potentially H-graphic if there exists a realization of π containing H as a subgraph. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ where H is a subgraph of K_m. In this paper, we characterize potentially $K_{2,5}$-graphic sequences. This characterization implies a special case of a theorem due to Yin et al. [26].
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1 Introduction

We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. The set of all non-increasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G of order n; such a graph G is referred as a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. Let C_k and P_k denote a cycle on k

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vertices and a path on $k+1$ vertices, respectively. Let $\sigma(\pi)$ be the sum of all the terms of π, and let $[x]$ be the largest integer less than or equal to x. A graphic sequence π is said to be potentially H-graphic if it has a realization G containing H as a subgraph. Let $G-H$ denote the graph obtained from G by removing the edges set $E(H)$ where H is a subgraph of G. In the degree sequence, r^t means r repeats t times, that is, in the realization of the sequence there are t vertices of degree r.

Gould et al.[8] considered an extremal problem on potentially H-graphic sequences as follows: determine the smallest even integer $\sigma(H,n)$ such that every n-term positive graphic sequence π with $\sigma(\pi) \geq \sigma(H,n)$ has a realization G containing H as a subgraph. For $r \times s \times t$ complete 3-partite graph $K_{r,s,t}$, Yin [23] and Lai [17] independently determined $\sigma(K_{1,1,3},n)$. Chen [6] determined $\sigma(K_{1,1,t},n)$ for $t \geq 3$, $n \geq 2[\frac{(t+5)^2}{4}]+3$. For the graph $K_m - G$, Lai [16] determined $\sigma(K_4-e,n)$ and Yin et al.[27] determined $\sigma(K_5-e,n)$. In [29], Yin determined $\sigma(K_{r+1}-K_3,n)$ for $r \geq 3$, $n \geq 3r+5$.

A harder question is to characterize potentially H-graphic sequences without zero terms. Yin and Li [24] gave two sufficient conditions for $\pi \in GS_n$ to be potentially K_r-e-graphic. Luo [20] characterized potentially C_k-graphic sequences for each $k=3,4,5$. Chen [2] characterized potentially C_6-graphic sequences. Chen et al.[3] characterized potentially ${}_kC_l$-graphic sequences for each $3 \leq k \leq 5$, $l=6$. Recently, Luo and Warner [21] characterized potentially K_4-graphic sequences. Eschen and Niu [7] characterized potentially K_4-e-graphic sequences. Yin et al.[25] characterized potentially ${}_3C_4$, ${}_3C_5$ and ${}_4C_5$-graphic sequences. Yin and Chen [28] characterized potentially $K_{r,s}$-graphic sequences for $r=2, s=3$ and $r=2, s=4$. Yin et al.[30] characterized potentially K_5-e and K_6-graphic sequences. In [31], they characterized potentially K_6-K_3-graphic sequences. Moreover, Yin et al.[32] characterized potentially $K_{1,1,s}$-graphic sequences for $s=4$ and $s=5$. Chen and Li [5] characterized potentially $K_{1,t}+e$-graphic sequences. Chen [4] characterized potentially K_6-3K_2-graphic sequences. Hu, Lai and Wang [11] characterized potentially K_5-P_4 and K_5-Y_4 -graphic sequences, where Y_4 is a tree on 5 vertices and 3 leaves. Hu and Lai [9,12] characterized potentially K_5-C_4 and K_5-E_3-graphic sequences, where E_3 denotes graphs with 5 vertices and 3 edges. Besides, in [13,14], they characterized potentially $K_{3,3}$, K_6-C_6 and K_6-C_4-graphic sequences. Recently, Liu and Lai[19] characterized potentially $K_{1,1,2,2}$-graphic sequences. Xu and Lai [22] characterized potentially K_6-C_5-graphic sequences.

In this paper, we characterize potentially $K_{2,5}$-graphic sequences. This characterization implies a special case of a theorem due to Yin et al. [26].

2 Preparations

Let $\pi = (d_1, \cdots, d_n) \in NS_n, 1 \leq k \leq n$. Let

$$\pi_k'' = \begin{cases} (d_1 - 1, \cdots, d_{k-1} - 1, d_{k+1} - 1, \cdots, d_{d_k+1} - 1, d_{d_k+2}, \cdots, d_n), \\ \text{if } d_k \geq k, \\ (d_1 - 1, \cdots, d_{d_k} - 1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Denote $\pi_k' = (d_1', d_2', \cdots, d_{n-1}')$, where $d_1' \geq d_2' \geq \cdots \geq d_{n-1}'$ is a rearrangement of the $n-1$ terms of π_k''. Then π_k' is called the residual sequence obtained by laying off d_k from π. For simplicity, we denote π_n' by π' in this paper.

For a nonincreasing positive integer sequence $\pi = (d_1, d_2, \cdots, d_n)$, we write $m(\pi)$ and $h(\pi)$ to denote the largest positive terms of π and the smallest positive terms of π, respectively. We need the following results.

Theorem 2.1 [8] If $\pi = (d_1, d_2, \cdots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Theorem 2.2 [18] If $\pi = (d_1, d_2, \cdots, d_n)$ is a sequence of nonnegative integers with $1 \leq m(\pi) \leq 2$, $h(\pi) = 1$ and even $\sigma(\pi)$, then π is graphic.

Theorem 2.3 [28] Let $n \geq 6$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$. Then π is potentially $K_{2,4}$-graphic if and only if π satisfies the following conditions:

(1) $d_2 \geq 4$ and $d_6 \geq 2$;

(2) If $d_1 = n-1$ and $d_2 = 4$, then $d_3 = 4$ and $d_6 \geq 3$;

(3) $\pi \neq (4^3, 2^4), (4^2, 2^5), (4^2, 2^6), (5^2, 4, 2^4), (5^3, 3, 2^3), (6, 5^2, 2^5), (5^3, 2^4, 1), (6^3, 2^6), (n-1, 4^2, 3^4, 1^{n-7}), (n-1, 4^2, 3^5, 1^{n-8}), (n-2, 4^2, 2^3, 1^{n-6}), (n-2, 4^3, 2^2, 1^{n-6})$.

Lemma 2.4 [10] Let $n \geq 4$ and $\pi = (d_1, d_2, \cdots, d_n)$ be a nonincreasing sequence of positive integers with even $\sigma(\pi)$. If $d_1 \leq 3$ and $\pi \neq (3^3, 1), (3^2, 1^2)$, then π is graphic.

Lemma 2.5 [31] Let $\pi = (4^x, 3^y, 2^z, 1^m)$ with even $\sigma(\pi)$, $x+y+z+m = n \geq 5$ and $x \geq 1$. Then $\pi \in GS_n$ if and only if $\pi \notin A$, where $A =$

$\{(4,3^2,1^2),(4,3,1^3),(4^2,2,1^2),(4^2,3,2,1),(4^3,1^2),(4^3,2^2),(4^3,3,1),(4^4,2),$ $(4^2,3,1^3),(4^2,1^4),(4^3,2,1^2),(4^4,1^2),(4^3,1^4)\}$.

Lemma 2.6 (Kleitman and Wang [15]) π is graphic if and only if π'_k is graphic.

The following corollary is obvious.

Corollary 2.7 Let H be a simple graph. If π' is potentially H-graphic, then π is potentially H-graphic.

In order to prove our main result, we need the following definitions and proposition in Yin [28].

Let $s \geq 2$ and $\pi = (d_1, d_2, \cdots, d_{s+2}, \cdots, d_n)$ be a non-increasing sequence of nonnegative integers, where $d_1 \leq n-2$ and $d_2 \geq s$. Denote

$$\rho'_s(\pi) = \begin{cases} (d_2-1, d_3-1, \cdots, d_{s+2}-1, d^{(1)}_{s+3}, \cdots, d^{(1)}_n), \\ \text{if } d_2 \geq s+1; \\ (d_2, d_3-1, \cdots, d_{s+2}-1, d_{s+3}-1, \cdots, d_{d_1+2}-1, d_{d_1+3}, \cdots, d_n), \\ \text{if } d_2 = s, \end{cases}$$

where $d^{(1)}_{s+3} \geq \cdots \geq d^{(1)}_n$ is a rearrangement of $d_{s+3}-1, \cdots, d_{d_1+1}-1, d_{d_1+2}, \cdots, d_n$.

From $\rho'_s(\pi)$, we construct the sequence

$$\rho_s(\pi) = \begin{cases} (d_3-2, \cdots, d_{s+2}-2, d^{(2)}_{s+3}, \cdots, d^{(2)}_n), \\ \text{if } d_2 \geq s+1; \\ (d_3-2, \cdots, d_{s+2}-2, d_{s+3}-1, \cdots, d_{d_1+2}-1, d_{d_1+3}, \cdots, d_n), \\ \text{if } d_2 = s, \end{cases}$$

where $d^{(2)}_{s+3} \geq \cdots \geq d^{(2)}_n$ is a rearrangement of $d^{(1)}_{s+3}-1, \cdots, d^{(1)}_{d_2+1}-1, d^{(1)}_{d_2+2}, \cdots, d^{(1)}_n$.

Proposition 2.8 (Yin [28]) Let $s \geq 2$ and $\pi = (d_1, d_2, \cdots, d_{s+2}, \cdots, d_n)$ be a non-increasing sequence of nonnegative integers, where $d_1 \leq n-2$ and $d_2 \geq s$. If $\rho_s(\pi)$ is graphic, then π is potentially $K_{2,s}$-graphic.

3 Main Theorems

Theorem 3.1 Let $n \geq 7$ and $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$. Then π is potentially $K_{2,5}$-graphic if and only if the following conditions hold:

(1) $d_2 \geq 5$, $d_7 \geq 2$;

(2) If $d_1 = n-1$ and $d_2 = 5$, then $d_3 = 5$ and $d_7 \geq 3$;

(3) $\pi = (n-l, 5^i, 4^j, 3^k, 2^t, 1^{n-7})$ implies $(3^{i-1}, 2^j, 1^{k+l-2})$ is graphic, where $n-l \geq 5$, $l = 2, 3, 4, i \geq 1$ and $i+j+k+t = 6$;

(4) π is not one of the following sequences:

$(n-1, 5^4, 3^2, 1^{n-7})$, $(n-1, 5^3, 3^3, 1^{n-7})$, $(n-1, 5^3, 3^4, 1^{n-8})$, $(n-1, 5^2, 3^5, 1^{n-8})$, $(n-1, 5^2, 3^6, 1^{n-9})$, $(n-2, 5^4, 2^3, 1^{n-8})$, $(n-2, 5^3, 3, 2^3, 1^{n-8})$, $(n-2, 5^2, 2^5, 1^{n-8})$, $(n-2, 5^2, 4, 2^4, 1^{n-8})$, $(n-3, 5^3, 2^4, 1^{n-8})$.

$n = 8:$ $(6^4, 3^4)$, $(6^3, 4^2, 2^3)$, $(6^3, 3^2, 2^3)$, $(6^2, 5, 3, 2^4)$, $(6^2, 4, 2^5)$, $(5^3, 3, 2^4)$, $(5^2, 4, 2^5)$, $(5^2, 2^6)$.

$n = 9:$ $(7, 6^2, 3, 2^5)$, $(7, 6, 5, 2^6)$, $(6^3, 4, 2^5)$, $(6^3, 2^6)$, $(6, 5^2, 2^6)$, $(5^2, 2^7)$, $(6^3, 3, 2^4, 1)$, $(6^2, 5, 2^5, 1)$, $(5^3, 2^5, 1)$.

$n = 10:$ $(8, 6^2, 2^7)$, $(7^3, 3, 2^6)$, $(7^2, 6, 2^7)$, $(6^3, 2^7)$, $(7, 6^2, 2^6, 1)$, $(6^3, 2^5, 1^2)$.

$n = 11:$ $(8, 7^2, 2^8)$, $(7^3, 2^7, 1)$.

$n = 12:$ $(8^3, 2^9)$.

Proof: First we show the conditions (1)-(4) are necessary conditions for π to be potentially $K_{2,5}$-graphic. Assume that π is potentially $K_{2,5}$-graphic. (1) is obvious. If $d_1 = n-1$ and $d_2 = 5$, then the residual sequence $\pi_1' = (d_2 - 1, d_3 - 1, \cdots, d_n - 1)$ is potentially $K_{2,4}$-graphic, and hence $d_2 - 1 = d_3 - 1 = 4$ and $d_7 - 1 \geq 2$, i.e., $d_3 = 5$ and $d_7 \geq 3$. Hence, (2) holds. If $\pi = (n-2, 5^i, 4^j, 3^k, 2^t, 1^{n-7})$ is potentially $K_{2,5}$-graphic, then according to theorem 2.1, there exists a realization G of π containing $K_{2,5}$ as a subgraph so that the vertices of $K_{2,5}$ have the largest degrees of π. Therefore, the sequence $\pi_1 = (n-7, 0, 3^{i-1}, 2^j, 1^k, 0^t, 1^{n-7})$ obtained from $G - K_{2,5}$ is graphic. Since the edges of $K_{2,5}$ have been removed from the realization of π_1, thus, $(3^{i-1}, 2^j, 1^k)$ must be graphic. Similarly, with the same argument as above, one can show that $\pi = (n-l, 5^i, 4^j, 3^k, 2^t, 1^{n-7})$ implies $(3^{i-1}, 2^j, 1^{k+l-2})$ is graphic for the cases $l = 3$ and $l = 4$. Hence, (3) holds. Now it is easy to check that $(6^4, 3^4)$, $(6^3, 4^2, 2^3)$, $(6^3, 3^2, 2^3)$, $(6^2, 5, 3, 2^4)$, $(6^2, 4, 2^5)$, $(5^3, 3, 2^4)$, $(5^2, 4, 2^5)$, $(5^2, 2^6)$, $(7, 6^2, 3, 2^5)$, $(7, 6, 5, 2^6)$, $(6^3, 4, 2^5)$, $(6^3, 2^6)$, $(6, 5^2, 2^6)$, $(5^2, 2^7)$, $(6^3, 3, 2^4, 1)$, $(6^2, 5, 2^5, 1)$, $(5^3, 2^5, 1)$, $(8, 6^2, 2^7)$, $(7^3, 3, 2^6)$, $(7^2, 6, 2^7)$, $(6^3, 2^7)$, $(7, 6^2, 2^6, 1)$, $(6^3, 2^5, 1^2)$, $(8, 7^2, 2^8)$, $(7^3, 2^7, 1)$ and $(8^3, 2^9)$ are not potentially $K_{2,5}$-graphic. Since $\pi_1' = (4^4, 2^2)$, $(4^3, 2^3)$, $(4^3, 2^4)$, $(4^2, 2^5)$ and $(4^2, 2^6)$ are not potentially $K_{2,4}$(by theorem 2.3) or $K_{1,5}$-graphic, we have $\pi \neq (n-1, 5^4, 3^2, 1^{n-7})$, $(n-1, 5^3, 3^3, 1^{n-7})$, $(n-1, 5^3, 3^4, 1^{n-8})$, $(n-1, 5^2, 3^5, 1^{n-8})$ and $(n-1, 5^2, 3^6, 1^{n-9})$. If $\pi = (n-2, 5^4, 2^3, 1^{n-8})$ is potentially $K_{2,5}$-graphic, then according to theorem 2.1, there exists a realization G of π containing $K_{2,5}$ as a subgraph so that the vertices of $K_{2,5}$ have the largest degrees of π. Therefore, the sequence

$\pi_1 = (n-7, 0, 3^3, 0^2, 2, 1^{n-8})$ obtained from $G - K_{2,5}$ must be graphic. Since the edges of $K_{2,5}$ have been removed from the realization of π_1, $\pi_2 = (3^3, 1)$ is graphic, a contradiction. Thus, $\pi \neq (n-2, 5^4, 2^3, 1^{n-8})$. Similarly, one can show that $\pi \neq (n-2, 5^3, 3, 2^3, 1^{n-8})$, $(n-2, 5^2, 2^5, 1^{n-8})$, $(n-2, 5^2, 4, 2^4, 1^{n-8})$ and $(n-3, 5^3, 2^4, 1^{n-8})$. Hence, (4) holds.

Next, we will prove the sufficient conditions. Suppose $\pi = (d_1, d_2, \cdots, d_n) \in GS_n$ satisfies the conditions (1)-(4).

If $d_1 = n-1$, consider the residual sequence $\pi_1' = (d_1', d_2', \cdots, d_{n-1}')$ obtained by laying off d_1 form π. If $d_2 \geq 6$, then $d_1' = d_2 - 1 \geq 5$. Thus, π_1' is potentially $K_{1,5}$-graphic and so π is potentially $K_{2,5}$-graphic. If $d_2 = 5$, by π satisfies condition (2), we have $d_1' = d_2 - 1 = 4$, $d_2' = d_3 - 1 = 4$ and $d_6' = d_7 - 1 \geq 2$. Since π satisfies condition (4), $\pi_1' \neq (4^4, 2^2)$, $(4^3, 2^3)$, $(4^3, 2^4)$, $(4^2, 2^5)$ and $(4^2, 2^6)$. If $\pi_1' \notin A$, where $A = \{(5^2, 4, 2^4), (5^3, 3, 2^3), (6, 5^2, 2^5), (5^3, 2^4, 1), (6^3, 2^6), (n-2, 4^2, 3^4, 1^{n-8}), (n-2, 4^2, 3^5, 1^{n-9}), (n-3, 4^2, 2^3, 1^{n-7})(n \geq 8), (n-3, 4^3, 2^2, 1^{n-7})(n \geq 8)\}$, then π_1' is potentially $K_{2,4}$-graphic by theorem 2.3 and so π is potentially $K_{2,5}$-graphic. If $\pi_1' \in A$, then π_1' is potentially $K_{1,5}$-graphic, thus π is potentially $K_{2,5}$-graphic. Suppose $d_1 \leq n-2$.

Our proof is by induction on n. We first prove the base case where $n = 7$. In this case, $d_1 = d_2 = 5$, i.e., $\pi = (5^i, 4^j, 3^k, 2^{7-i-j-k})$ where $i \geq 2$. Then $\rho_5(\pi) = (3^{i-2}, 2^j, 1^k, 0^{7-i-j-k})$. By π satisfies (3), $\rho_5(\pi)$ is graphic, and so π is potentially $K_{2,5}$-graphic by proposition 2.8. Now suppose that the sufficiency holds for $n-1 (n \geq 8)$, we will show that π is potentially $K_{2,5}$-graphic in terms of the following cases:

Case 1: $d_n \geq 5$. Consider $\pi' = (d_1', d_2', \cdots, d_{n-1}')$. Clearly, π' satisfies (1) and (4). If π' also satisfies (2)-(3), then by the induction hypothesis, π' is potentially $K_{2,5}$-graphic, and hence so is π.

If $d_1' = n-2$ and $d_2' = 5$, by $d_1 \leq n-2$, then $d_1 = d_2 = \cdots = d_6 = n-2$ and $n = 8$. Thus, $\pi' = (6, 5^6)$ which satisfies condition (2).

If $\pi' = (n-1-l, 5^i, 4^{6-i})$, then $l = 2$ and $n = 8$. Thus, $\pi' = (5^2, 4^5)$, $(5^4, 4^3)$ or $(5^6, 4)$. Since (2^5), $(3^2, 2^3)$ and $(3^4, 2)$ are graphic, π' satisfies condition (3).

Case 2: $d_n = 4$. Consider $\pi' = (d_1', d_2', \cdots, d_{n-1}')$ where $d_{n-3}' \geq 4$ and $d_{n-1}' \geq 3$. Clearly, π' satisfies (4). If π' also satisfies (1)-(3), then by the induction hypothesis, π' is potentially $K_{2,5}$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d_2' = 4$, then $d_2 = 5$. We will proceed with the following two cases: $d_1 = 5$ and $d_1 \geq 6$.

Subcase 1: $d_1 = 5$. Then $\pi = (5^k, 4^{n-k})$ where $2 \le k \le 5$. Since $\sigma(\pi)$ is even, we have $k = 2$ or $k = 4$, i.e., $\pi = (5^2, 4^{n-2})$ or $(5^4, 4^{n-4})$. Then $\rho_5(\pi) = (2^5, 4^{n-7})$ or $(3^2, 2^3, 4^{n-7})$. By lemma 2.5, $\rho_5(\pi)$ is graphic, and so $\pi = (5^k, 4^{n-k})$ is potentially $K_{2,5}$-graphic by proposition 2.8.

Subcase 2: $d_1 \ge 6$. Then $\pi = (d_1, 5^k, 4^{n-1-k})$ where $1 \le k \le 3$, and, d_1 and k have the same parity.

If $k = 1$, then $\pi = (d_1, 5, 4^{n-2})$ and $\rho_5(\pi) = (2^5, 3^{d_1-5}, 4^{n-2-d_1})$. By lemma 2.4 and lemma 2.5, $\rho_5(\pi)$ is graphic, and so $\pi = (d_1, 5, 4^{n-2})$ is potentially $K_{2,5}$-graphic by proposition 2.8.

If $k = 2$, then $\pi = (d_1, 5^2, 4^{n-3})$ and $\rho_5(\pi) = (3, 2^4, 3^{d_1-5}, 4^{n-2-d_1})$. By lemma 2.4 and lemma 2.5, $\rho_5(\pi)$ is graphic, and so $\pi = (d_1, 5^2, 4^{n-3})$ is potentially $K_{2,5}$-graphic by proposition 2.8. Similarly, with the same argument as above, one can show that $\pi = (d_1, 5^3, 4^{n-4})$ is also potentially $K_{2,5}$-graphic.

If $d'_1 = n-2$ and $d'_2 = 5$, by $d_1 \le n-2$, then $d_1 = d_2 = d_3 = d_4 = d_5 = n-2$ and $n = 8$. Thus, $\pi' = (6, 5^6)$ or $(6, 5^4, 4^2)$. Therefore, π' satisfies condition (2).

If $\pi' = (n-1-l, 5^i, 4^j, 3^{6-i-j})$, then $n = 8$, $l = 2$ and $i+j \ge 4$. Hence, π' is one of the following: $(5^6, 4)$, $(5^4, 4^3)$, $(5^2, 4^5)$, $(5^4, 4, 3^2)$, $(5^2, 4^3, 3^2)$, $(5^5, 4, 3)$, $(5^3, 4^3, 3)$. By lemma 2.4, $(3^4, 2)$, $(3^2, 2^3)$, (2^5), $(3^2, 2, 1^2)$, $(2^3, 1^2)$, $(3^3, 2, 1)$ and $(3, 2^3, 1)$ are graphic. Thus, π' satisfies condition (3).

Case 3: $d_n = 3$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_2 \ge 4$, $d'_{n-2} \ge 3$ and $d'_{n-1} \ge 2$. If π' satisfies (1)-(4), then by the induction hypothesis, π' is potentially $K_{2,5}$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_2 = 4$, then $d_2 = 5$. We will proceed with the following two cases: $d_1 = 5$ and $d_1 \ge 6$.

Subcase 1: $d_1 = 5$. Then $\pi = (5^i, 4^j, 3^{n-i-j})$ where $2 \le i \le 4$, $n-i-j \ge 1$ and $n-j$ is even. If $i = 2$, i.e., $\pi = (5^2, 4^j, 3^{n-2-j})$. Then $\rho_5(\pi) = (2^5, 4^{j-5}, 3^{n-2-j})(j \ge 5)$ or $\rho_5(\pi) = (2^j, 1^{5-j}, 3^{n-7})(j < 5)$. By lemma 2.4 and lemma 2.5, $\rho_5(\pi)$ is graphic, and so $\pi = (5^2, 4^j, 3^{n-2-j})$ is potentially $K_{2,5}$-graphic by proposition 2.8. Similarly, with the same argument as above, one can show that $\pi = (5^i, 4^j, 3^{n-i-j})$ is also potentially $K_{2,5}$-graphic for the cases $i = 3$ and $i = 4$.

Subcase 2: $d_1 \ge 6$. Then $\pi = (d_1, 5^i, 4^j, 3^{n-1-i-j})$ where $1 \le i \le 2$, $n-1-i-j \ge 1$, and, d_1 and $n-1-j$ have the same parity. If $i = 1$, then $\pi = (d_1, 5, 4^j, 3^{n-2-j})$. If $j < 5$, then $\rho_5(\pi) = (2^j, 1^{5-j}, 2^{d_1-5}, 3^{n-2-d_1})$. If $5 \le j < d_1$, then $\rho_5(\pi) = (2^5, 3^{j-5}, 2^{d_1-j}, 3^{n-2-d_1})$. If $j \ge d_1$, then

$\rho_5(\pi) = (2^5, 3^{d_1-5}, 4^{j-d_1}, 3^{n-2-j})$. By lemma 2.4 and lemma 2.5, $\rho_5(\pi)$ is graphic, and so $\pi = (d_1, 5^i, 4^j, 3^{n-1-i-j})$ is potentially $K_{2,5}$-graphic by proposition 2.8. Similarly, with the same argument as above, one can show that $\pi = (d_1, 5^2, 4^j, 3^{n-3-j})$ is also potentially $K_{2,5}$-graphic.

If $d_1' = n-2$ and $d_2' = 5$, by $d_1 \leq n-2$, then $d_1 = d_2 = d_3 = d_4 = n-2$ and $n = 8$. In this case, $\pi' = (6, 5^3, d_5', d_6', d_7')$ where $3 \leq d_7' \leq d_6' \leq d_5' \leq 5$ and $\sigma(\pi')$ is even. Clearly, π' satisfies condition (2).

If $\pi' = (n-1-l, 5^i, 4^j, 3^k, 2^{6-i-j-k})$, then $n = 8$, $l = 2$ and $i+j+k \geq 5$. If $\pi' = (5, 5^i, 4^j, 3^{6-i-j})$, then it is easy to see that $(3^{i-1}, 2^j, 1^{6-i-j})$ is graphic by lemma 2.4(since $(i-1)+j+(6-i-j) = 5 > 4$). If $\pi' = (5, 5^i, 4^j, 3^{5-i-j}, 2)$, then $d_3 = \cdots = d_7 = 3$. We have $\pi' = (5^2, 3^4, 2)$. It follows $(3^{i-1}, 2^j, 1^{5-i-j}) = (1^4)$, which is also graphic. In other words, π' satisfies (3).

If π' does not satisfy (4), since $d_1 \leq n-2$ and $\pi \neq (6^4, 3^4)$, then $\pi' = (6, 5^4, 3^2)$ or $(6^4, 3^4)$. Thus, $\pi = (6^4, 5, 3^3)$ or $(7^3, 6, 3^5)$. It is easy to check that both of them are potentially $K_{2,5}$-graphic.

Case 4: $d_n = 2$. Consider $\pi' = (d_1', d_2', \cdots, d_{n-1}')$ where $d_2' \geq 4$ and $d_{n-1}' \geq 2$. If π' satisfies (1)-(4), then by the induction hypothesis, π' is potentially $K_{2,5}$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d_2' = 4$, then $d_2 = 5$. We will proceed with the following two cases: $d_1 = 5$ and $d_1 \geq 6$.

Subcase 1: $d_1 = 5$. Then $\pi = (5^k, 4^i, 3^j, 2^{n-k-i-j})$ where $2 \leq k \leq 3$, $n-k-i-j \geq 1$ and $k+j$ is even. If $k = 2$, i.e., $\pi = (5^2, 4^i, 3^j, 2^{n-2-i-j})$, then $\rho_5(\pi) = (d_3 - 2, \cdots, d_7 - 2, d_8, \cdots, d_n)$. If $m(\rho_5(\pi)) = 4$, then $i \geq 6$ and so $\rho_5(\pi) = (2^5, 4^{i-5}, 3^j, 2^{n-2-i-j})$. By lemma 2.5, $\rho_5(\pi)$ is graphic. If $m(\rho_5(\pi)) = 3$, then $i \leq 5$, $i+j \geq 6$ and so $\rho_5(\pi) = (2^i, 1^{5-i}, 3^{i+j-5}, 2^{n-2-i-j})$, it follows from lemma 2.4 that $\rho_5(\pi)$ is also graphic. If $m(\rho_5(\pi)) = 2$, then $i+j \leq 5$ and so $\rho_5(\pi) = (2^i, 1^j, 0^{5-i-j}, 2^{n-7})$. In this case, $\rho_5(\pi)$ is not graphic if and only if $\rho_5(\pi) = (2)$ or (2^2) which is impossible since $\pi \neq (5^2, 2^6)$, $(5^2, 2^7)$ and $(5^2, 4, 2^5)$. Thus, $\pi = (5^2, 4^i, 3^j, 2^{n-2-i-j})$ is potentially $K_{2,5}$-graphic by proposition 2.8. Similarly, one can show that $\pi = (5^3, 4^i, 3^j, 2^{n-3-i-j})$ is also potentially $K_{2,5}$-graphic.

Subcase 2: $d_1 \geq 6$. Then $\pi = (d_1, 5, 4^i, 3^j, 2^{n-2-i-j})$ where $n-2-i-j \geq 1$, and, d_1 and $j+1$ have the same parity. Thus, $\rho_5(\pi) = (d_3 - 2, \cdots, d_7 - 2, d_8 - 1, \cdots, d_{d_1+2} - 1, d_{d_1+3}, \cdots, d_n)$. If $i+j < 5$, then $\rho_5(\pi) = (2^i, 1^j, 0^{5-i-j}, 1^{d_1-5}, 2^{n-2-d_1})$. If $i \geq d_1$, then $\rho_5(\pi) = (2^5, 3^{d_1-5}, 4^{i-d_1}, 3^j, 2^{n-2-i-j})$. If $5 \leq i < d_1$ and $i+j \geq d_1$, then $\rho_5(\pi) =$

$(2^5, 3^{i-5}, 2^{d_1-i}, 3^{i+j-d_1}, 2^{n-2-i-j})$. If $5 \le i+j < d_1$, then $\rho_5(\pi) = (2^5, 3^{i-5}, 2^j, 1^{d_1-i-j}, 2^{n-2-d_1})(i \ge 5)$ or $\rho_5(\pi) = (2^i, 1^{5-i}, 2^{i+j-5}, 1^{d_1-i-j}, 2^{n-2-d_1})$ $(i < 5)$. By lemma 2.4 and lemma 2.5, in the above cases, $\rho_5(\pi)$ is graphic, and so π is potentially $K_{2,5}$-graphic by proposition 2.8.

If $d'_1 = n-2$ and $d'_2 = 5$, by $d_1 \le n-2$, we have $d_1 = d_2 = d_3 = n-2$ and $n = 8$. If $d_7 \ge 3$, then π' satisfies (2). If $d_7 = 2$, then $\pi = (6^3, d_4, d_5, d_6, 2^2)$ where $2 \le d_6 \le d_5 \le d_4 \le 5$. Since $\pi \ne (6^3, 4^2, 2^3)$ and $(6^3, 3^2, 2^3)$, then $\pi = (6^3, 5^2, 4, 2^2)$, $(6^3, 5, 4, 3, 2^2)$, $(6^3, 4^3, 2^2)$ or $(6^3, 4, 3^2, 2^2)$. It is easy to check that all of these are potentially $K_{2,5}$-graphic.

If $\pi' = (n-1-l, 5^i, 4^j, 3^k, 2^{6-i-j-k})$, then $n = 8$ and $l = 2$, i.e., $\pi' = (5, 5^i, 4^j, 3^k, 2^{6-i-j-k})$. If $(3^{i-1}, 2^j, 1^k)$ is graphic, then π' satisfies (3). If $(3^{i-1}, 2^j, 1^k)$ is not graphic, then $(3^{i-1}, 2^j, 1^k) \in \{(3^3, 1), (3^2, 1^2), (3^2, 2), (3, 2, 1), (3, 1), (2^2), (2)\}$. By $\pi \ne (6^2, 5, 3, 2^4)$, $(6^2, 4, 2^5)$, $(6, 5^2, 2^5)$, then π' is one of the following: $(5^5, 3, 2)$, $(5^4, 3^2, 2)$, $(5^4, 4, 2^2)$, $(5^3, 4, 3, 2^2)$, $(5^2, 4^2, 2^3)$. Since $\pi \ne (6, 5^4, 2^3)$,$(6, 5^3, 3, 2^3)$, $(6, 5^2, 4, 2^4)$ and $(5^4, 2^4)$, then π is one of the following: $(6^2, 5^3, 3, 2^2)$, $(6^2, 5^2, 3^2, 2^2)$, $(6^2, 5^2, 4, 2^3)$, $(6^2, 5, 4, 3, 2^3)$, $(6^2, 4^2, 2^4)$. It is easy to check that all of these are potentially $K_{2,5}$-graphic.

If π' does not satisfy (4), since $d_1 \le n-2$ and $\pi \ne (7, 6, 5, 2^6)$, $(6^3, 2^6)$, $(7^2, 6, 2^7)$, then π' is one of the following:

$n-1 = 7 : (6, 5^4, 3^2)$, $(6, 5^3, 3^3)$,

$n-1 = 8 : (6, 5^4, 2^3)$, $(6, 5^3, 3, 2^3)$, $(6, 5^2, 4, 2^4)$, $(5^4, 2^4)$, $(6^4, 3^4)$, $(6^3, 4^2, 2^3)$, $(6^3, 3^2, 2^3)$, $(6^2, 5, 3, 2^4)$, $(6^2, 4, 2^5)$, $(5^3, 3, 2^4)$, $(5^2, 4, 2^5)$, $(5^2, 2^6)$,

$n-1 = 9 : (7, 6^2, 3, 2^5)$, $(7, 6, 5, 2^6)$, $(6^3, 4, 2^5)$, $(6, 5^2, 2^6)$, $(5^2, 2^7)$,

$n-1 = 10 : (8, 6^2, 2^7)$, $(7^3, 3, 2^6)$, $(7^2, 6, 2^7)$, $(6^3, 2^7)$,

$n-1 = 11 : (8, 7^2, 2^8)$,

$n-1 = 12 : (8^3, 2^9)$,

Since $\pi \ne (6^3, 4, 2^5)$, $(7, 6^2, 3, 2^5)$, $(6, 5^2, 2^6)$, $(7^3, 3, 2^6)$, $(8, 6^2, 2^7)$, $(7^2, 6, 2^7)$, $(6^3, 2^7)$, $(8, 7^2, 2^8)$, $(8^3, 2^9)$, then π is one of the following:

$n = 8 : (6^3, 5^2, 3^2, 2)$, $(6^3, 5, 3^3, 2)$.

$n = 9 : (7, 6, 5^3, 2^4)$, $(6^3, 5^2, 2^4)$, $(7, 6, 5^2, 3, 2^4)$, $(6^3, 5, 3, 2^4)$, $(7, 6, 5, 4, 2^5)$, $(6^2, 5^2, 2^5)$, $(7^2, 6^2, 3^4, 2)$, $(7^2, 6, 4^2, 2^4)$, $(7^2, 6, 3^2, 2^4)$, $(7^2, 5, 3, 2^5)$, $(7^2, 4, 2^6)$, $(6^2, 5, 3, 2^5)$, $(6^2, 4, 2^6)$, $(6^2, 2^7)$.

$n = 10 : (8, 7, 6, 3, 2^6)$, $(8, 7, 5, 2^7)$, $(7^2, 6, 4, 2^6)$, $(7, 6, 5, 2^7)$, $(6^2, 2^8)$.

$n = 11 : (9, 7, 6, 2^8)$, $(8^2, 7, 3, 2^7)$, $(8^2, 6, 2^8)$, $(7^2, 6, 2^8)$.

$n = 12 : (9, 8, 7, 2^9)$.

$n = 13 : (9^2, 8, 2^{10})$.

It is easy to check that all of these are potentially $K_{2,5}$-graphic.

Case 5: $d_n = 1$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_1 \geq 5$, $d'_2 \geq 4$ and $d'_7 \geq 2$. If π' satisfies (1)-(4), then by the induction hypothesis, π' is potentially $K_{2,5}$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_2 = 4$. Then $d_1 = d_2 = 5$ and $d_3 \leq 4$, i.e., $\pi = (5^2, d_3, \cdots, d_n)$ where $d_7 \geq 2$ and $d_n = 1$. Consider $\rho_5(\pi) = (d_3 - 2, \cdots, d_7 - 2, d_8, \cdots, d_n)$. If $m(\rho_5(\pi)) = 4$, then $d_3 = \cdots = d_8 = 4$, i.e., $\rho_5(\pi) = (2^5, 4, d_9, \cdots, d_{n-1}, 1)$. By lemma 2.5, $\rho_5(\pi)$ is graphic. If $m(\rho_5(\pi)) = 3$, then $d_8 = 3$, i.e., $\rho_5(\pi) = (d_3-2, \cdots, d_7-2, 3, d_9, \cdots, d_{n-1}, 1)$ where $1 \leq d_7 - 2 \leq d_6 - 2 \leq \cdots \leq d_3 - 2 \leq 2$. By lemma 2.4, $\rho_5(\pi)$ is also graphic. If $1 \leq m(\rho_5(\pi)) \leq 2$, it follows from $h(\rho_5(\pi)) = 1$ and theorem 2.2 that $\rho_5(\pi)$ is graphic. Hence, π is potentially $K_{2,5}$-graphic by proposition 2.8.

If $d'_1 = n - 2$ and $d'_2 = 5$, by $d_1 \leq n - 2$, we have $d_1 = d_2 = n - 2$ and $n = 8$, i.e., $\pi = (6^2, d_3, \cdots, d_7, 1)$. If $d_3 = 5$ and $d_7 \geq 3$, then π' satisfies (2). If $d_3 \leq 4$, then π is one of the following: $(6^2, 4^4, 3, 1)$, $(6^2, 4^3, 3, 2, 1)$, $(6^2, 4^2, 3^3, 1)$, $(6^2, 4^2, 3, 2^2, 1)$, $(6^2, 4, 3^3, 2, 1)$, $(6^2, 4, 3, 2^3, 1)$, $(6^2, 3^5, 1)$, $(6^2, 3^3, 2^2, 1)$, $(6^2, 3, 2^4, 1)$. If $d_3 = 5$ and $d_7 = 2$, then π is one of the following: $(6^2, 5^3, 4, 2, 1)$, $(6^2, 5^2, 4, 3, 2, 1)$, $(6^2, 5, 4^3, 2, 1)$, $(6^2, 5, 4^2, 2^2, 1)$, $(6^2, 5, 4, 3^2, 2, 1)$, $(6^2, 5, 3^2, 2^2, 1)$. It is easy to check that all of the above sequences are potentially $K_{2,5}$-graphic.

If $\pi' = (n - 1 - l, 5^i, 4^j, 3^k, 2^t, 1^{n-8})$, then there are three subcases:

Subcase 1: $n - 1 - l = 5$. If $j = 0$, then $\pi' = (5, 5^i, 3^k, 2^t, 1^{n-8})$ and $\pi = (6, 5^i, 3^k, 2^t, 1^{n-7})$. By π satisfies (3), then $(3^{i-1}, 1^{k+n-8})$ is graphic. Thus, π' satisfies (3). If $j \geq 1$, then $\pi = (6, 5^i, 4^j, 3^k, 2^t, 1^{n-7})$ or $(5^{i+2}, 4^{j-1}, 3^k, 2^t, 1^{n-7})$. If $\pi = (6, 5^i, 4^j, 3^k, 2^t, 1^{n-7})$, then with the same argument as above, one can show that π' satisfies (3). If $\pi = (5^{i+2}, 4^{j-1}, 3^k, 2^t, 1^{n-7})$, then $\rho_5(\pi) = (3^i, 2^{j-1}, 1^k, 0^t, 1^{n-7})$. Since π satisfies (3), $\rho_5(\pi)$ is graphic. Thus, π is potentially $K_{2,5}$-graphic by proposition 2.8.

Subcase 2: $n - 1 - l = 6$. Then $\pi = (7, 5^i, 4^j, 3^k, 2^t, 1^{n-7})$ or $(6^2, 5^{i-1}, 4^j, 3^k, 2^t, 1^{n-7})$. If $\pi = (7, 5^i, 4^j, 3^k, 2^t, 1^{n-7})$, with the same argument as subcase1, we have π' satisfies (3). If $\pi = (6^2, 5^{i-1}, 4^j, 3^k, 2^t, 1^{n-7})$, then $\rho_5(\pi) = (3^{i-1}, 2^j, 1^k, 0^t, 1^{n-7})$ where $n \geq 9$. In this case, $\rho_5(\pi)$ is not graphic if and only if $\rho_5(\pi) = (3^2, 1^2)$ which is impossible since $\pi = (6^2, 5^2, 2^3, 1^2)$ is not graphic. Thus, $\pi = (6^2, 5^{i-1}, 4^j, 3^k, 2^t, 1^{n-7})$ is potentially $K_{2,5}$-graphic by proposition 2.8.

Subcase 3: $n-1-l \geq 7$. Then $\pi = (n-l, 5^i, 4^j, 3^k, 2^t, 1^{n-7})$. By π satisfies (3), then $(3^{i-1}, 2^j, 1^{k+l-2})$ is graphic. In other words, π' satisfies (3).

If π' does not satisfy (4), since $\pi \neq (n-1, 5^3, 3^4, 1^{n-8})$, $(n-1, 5^2, 3^5, 1^{n-8})$, $(n-1, 5^2, 3^6, 1^{n-9})$, $(n-2, 5^2, 2^5, 1^{n-8})$, $(n-3, 5^3, 2^4, 1^{n-8})$, $(n-3, 5^2, 3, 2^4, 1^{n-8})$, $(6^2, 5, 2^5, 1)$, $(7, 6^2, 2^6, 1)$, then π' is one of the following:

$n-1=7$: $(6, 5^4, 3^2)$, $(6, 5^3, 3^3)$,

$n-1=8$: $(6, 5^4, 2^3)$, $(6, 5^3, 3, 2^3)$, $(6, 5^2, 4, 2^4)$, $(6^4, 3^4)$, $(6^3, 4^2, 2^3)$, $(6^3, 3^2, 2^3)$, $(6^2, 5, 3, 2^4)$, $(6^2, 4, 2^5)$, $(5^3, 3, 2^4)$, $(5^2, 4, 2^5)$, $(5^2, 2^6)$,

$n-1=9$: $(7, 6^2, 3, 2^5)$, $(7, 6, 5, 2^6)$, $(6^3, 4, 2^5)$, $(6, 5^2, 2^6)$, $(5^2, 2^7)$, $(6^3, 3, 2^4, 1)$, $(6^2, 5, 2^5, 1)$, $(5^3, 2^5, 1)$,

$n-1=10$: $(8, 6^2, 2^7)$, $(7^3, 3, 2^6)$, $(7^2, 6, 2^7)$, $(6^3, 2^7)$, $(7, 6^2, 2^6, 1)$, $(6^3, 2^5, 1^2)$,

$n-1=11$: $(8, 7^2, 2^8)$, $(7^3, 2^7, 1)$,

$n-1=12$: $(8^3, 2^9)$.

Since $\pi \neq (n-1, 5^4, 3^2, 1^{n-7})$, $(n-1, 5^3, 3^3, 1^{n-7})$, $(n-2, 5^4, 2^3, 1^{n-8})$, $(n-2, 5^3, 3, 2^3, 1^{n-8})$, $(n-2, 5^2, 4, 2^4, 1^{n-8})$, $(6^3, 3, 2^4, 1)$, $(5^3, 2^5, 1)$, $(6^3, 2^5, 1^2)$ and $(7^3, 2^7, 1)$, then π is one of the following:

$n=8$: $(6^2, 5^3, 3^2, 1)$, $(6^2, 5^2, 3^3, 1)$,

$n=9$: $(6^2, 5^3, 2^3, 1)$, $(6^2, 5^2, 3, 2^3, 1)$, $(6^2, 5, 4, 2^4, 1)$, $(7, 6^3, 3^4, 1)$, $(7, 6^2, 4^2, 2^3, 1)$, $(7, 6^2, 3^2, 2^3, 1)$, $(7, 6, 5, 3, 2^4, 1)$, $(7, 6, 4, 2^5, 1)$, $(6, 5^2, 3, 2^4, 1)$, $(6, 5, 4, 2^5, 1)$, $(6, 5, 2^6, 1)$,

$n=10$: $(8, 6^2, 3, 2^5, 1)$, $(7^2, 6, 3, 2^5, 1)$, $(8, 6, 5, 2^6, 1)$, $(7^2, 5, 2^6, 1)$, $(7, 6^2, 4, 2^5, 1)$, $(7, 5^2, 2^6, 1)$, $(6^2, 5, 2^6, 1)$, $(6, 5, 2^7, 1)$, $(7, 6^2, 3, 2^4, 1^2)$, $(7, 6, 5, 2^5, 1^2)$, $(6, 5^2, 2^5, 1^2)$,

$n=11$: $(9, 6^2, 2^7, 1)$, $(8, 7^2, 3, 2^6, 1)$, $(8, 7, 6, 2^7, 1)$, $(7, 6^2, 2^7, 1)$, $(8, 6^2, 2^6, 1^2)$, $(7^2, 6, 2^6, 1^2)$, $(7, 6^2, 2^5, 1^3)$,

$n=12$: $(9, 7^2, 2^8, 1)$, $(8^2, 7, 2^8, 1)$, $(8, 7^2, 2^7, 1^2)$,

$n=13$: $(9, 8^2, 2^9, 1)$.

It is easy to check that all of the above sequences are potentially $K_{2,5}$-graphic.

4 Application

In the remaining of this section, we will use theorem 3.1 to find the exact value of $\sigma(K_{2,5}, n)$. Note that the value of $\sigma(K_{2,5}, n)$ was a special case of theorem 3.1 in [26] so another proof is given here.

Theorem (Yin et al. [26]) If $n \geq 37$, then

$$\sigma(K_{2,5}, n) = \begin{cases} 5n-3, & \text{if } n \text{ is odd,} \\ 5n-2, & \text{if } n \text{ is even.} \end{cases}$$

Proof: First we claim that for $n \geq 37$,

$$\sigma(K_{2,5}, n) \geq \begin{cases} 5n-3, & \text{if } n \text{ is odd,} \\ 5n-2, & \text{if } n \text{ is even.} \end{cases}$$

If n is odd, take $\pi_1 = ((n-1), 5, 4^{n-3}, 3)$, then $\sigma(\pi_1) = 5n-5$, and it is easy to see that π_1 is not potentially $K_{2,5}$-graphic by theorem 3.1. If n is even, take $\pi_2 = (n-1, 5, 4^{n-2})$, then $\sigma(\pi_2) = 5n-4$ and π_2 is not potentially $K_{2,5}$-graphic by theorem 3.1. Thus,

$$\sigma(K_{2,5}, n) \geq \begin{cases} \sigma(\pi_1)+2 = 5n-3, & \text{if } n \text{ is odd,} \\ \sigma(\pi_2)+2 = 5n-2, & \text{if } n \text{ is even.} \end{cases}$$

Now we show that if π is an n-term ($n \geq 37$) graphical sequence with $\sigma(\pi) \geq 5n-3$, then there exists a realization of π containing $K_{2,5}$. Hence, it suffices to show that π is potentially $K_{2,5}$-graphic.

If $d_2 \leq 4$, then $\sigma(\pi) \leq d_1+4(n-1) \leq n-1+4(n-1) = 5n-5 < 5n-3$, a contradiction. Hence, $d_2 \geq 5$.

If $d_7 = 1$, then $\sigma(\pi) = d_1+d_2+d_3+d_4+d_5+d_6+(n-6) \leq 30+(n-6)+(n-6) = 2n+18 < 5n-3$, a contradiction. Hence, $d_7 \geq 2$.

If $d_1 = n-1$, $d_2 = 5$ and $d_3 \leq 4$, then $\sigma(\pi) \leq (n-1)+5+4(n-2) = 5n-4 < 5n-3$, a contradiction. If $d_1 = n-1$, $d_2 = 5$ and $d_7 \leq 2$, then $\sigma(\pi) \leq (n-1)+5\times 5+2(n-6) = 3n+12 < 5n-3$, a contradiction. Hence, π satisfies condition (2) in theorem 3.1.

Since $\sigma(\pi) \geq 5n-3$, it is easy to check that π satisfies condition (4) in theorem 3.1. Therefore, π is potentially $K_{2,5}$-graphic.

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