# SUBSEQUENCE SUMS OF ZERO-SUM FREE SEQUENCES II

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ABSTRACT. Let $G$ be a finite abelian group, and let $S$ be a sequence over $G$. Let $f(S)$ denote the number of elements in $G$ which can be expressed as the sum over a nonempty subsequence of $S$. In this paper, we determine all the sequences $S$ that contains no zero-sum subsequences and $f(S) \le 2|S| - 1$.

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**Key words:** Zero-sum problems, Davenport's constant, zero-sum free sequences.

## 1. INTRODUCTION

Let $G$ be a finite abelian group (written additively)throughout the present paper. $\mathcal{F}(G)$ denotes the free abelian monoid with basis $G$, the elements of which are called *sequences* (over $G$). A sequence of not necessarily distinct elements from $G$ will be written in the form $S = g_1 \cdot \cdots \cdot g_k = \prod_{i=1}^{k} g_i = \prod_{g\in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G)$, where $\mathsf{v}_g(S) \ge 0$ is called the *multiplicity* of $g$ in $S$. Denote by $|S| = k$ the number of elements in $S$ (or the *length* of $S$) and let $\operatorname{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$ be the *support* of $S$.

We say that $S$ contains some $g \in G$ if $\mathsf{v}_g(S) \ge 1$ and a sequence $T \in \mathcal{F}(G)$ is a *subsequence* of $S$ if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for every $g \in G$, denoted by $T|S$. If $T|S$, then let $ST^{-1}$ denote the sequence obtained by deleting the terms of $T$ from $S$. Furthermore, by $\sigma(S)$ we denote the sum of $S$, (i.e. $\sigma(S) = \sum_{i=1}^{k} g_i = \sum_{g\in G} \mathsf{v}_g(S)g \in G$). By $\sum(S)$ we denote the set consisting of all elements which can be expressed as a sum over a nonempty subsequence of $S$, i.e.

$$\sum(S) = \{\sigma(T) : T \text{ is a nonempty subsequence of } S\}.$$

We write $f(S) = |\sum(S)|$, $< S >$ for the subgroup of $G$ generated by all the elements of $S$.

Let $S$ be a sequence over $G$. We call $S$ a *zero − sum sequence* if $\sigma(S) = 0$, a *zero − sum free sequence* if $\sigma(W) \neq 0$ for any subsequence $W$ of $S$, and *squarefree* if $\mathsf{v}_g(S) \leq 1$ for every $g \in G$. We denote by $\mathcal{A}^*(G)$ the set of all zero-sum free sequences in $\mathcal{F}(G)$.

Let $D(G)$ be the Davenport's constant of $G$, i.e., the smallest integer $d$ such that every sequence $S$ over $G$ with $|S| \geq d$ satisfies $0 \in \sum(S)$. For every positive integer $r$ in the interval $\{1, \ldots, D(G)-1\}$, let

$$f_G(r) = \min_{S, |S|=r} f(S), \tag{1.1}$$

where $S$ runs over all zero-sum free sequences of $r$ elements in $G$. How does the function $f_G$ behave?

In 2006, Gao and Leader proved the following result.

**Theorem A** [5] *Let $G$ be a finite abelian group of exponent $m$. Then*

*(i) If $1 \leq r \leq m-1$ then $f_G(r) = r$.*

*(ii) If $\gcd(6, m) = 1$ and $G$ is not cyclic then $f_G(m) = 2m - 1$.*

Recently, Sun[10] showed that $f_G(m) = 2m - 1$ still holds without the restriction that $\gcd(6, m) = 1$.

Using some techniques from the author [11], the author [12] proved the following two theorems.

**Theorem B**[12, 8] *Let $S$ be a zero-sum free sequence over $G$ such that $< S >$ is not a cyclic group, then $f(S) \geq 2|S| - 1$.*

**Theorem C** [12] *Let $S$ be a zero-sum free sequence over $G$ such that $< S >$ is not a cyclic group and $f(S) = 2|S| - 1$. Then $S$ is one of the following forms*

*(i) $S = a^x(a+g)^y$, $x \geq y \geq 1$, where $g$ is an element of order 2.*

*(ii) $S = a^x(a+g)^y g$, $x \geq y \geq 1$, where $g$ is an element of order 2.*

*(iii) $S = a^x b$, $x \geq 1$.*

However, Theorem B is an old theorem of Olson and White [8] which has been overlooked by the author. For more recent progress on this topic, see [4, 9, 13].

The main purpose of the present paper is to determine all the sequences $S$ over a finite abelian group such that $S$ contains no zero-sum subsequences and $f(S) \leq 2|S| - 1$. To begin with, we need the notation of $g$-smooth.

**Definition 1.1.** [7, Definition 5.1.3] *A sequence $S \in \mathcal{F}(G)$ is called smooth if $S = (n_1 g)(n_2 g) \cdot \cdots \cdot (n_l g)$, where $|S| \in \mathbb{N}$, $g \in G$, $1 =$*

$n_1 \le \cdots \le n_l$, $n = n_1 + \cdots + n_l < ord(g)$ *and* $\sum(S) = \{g, \dots, ng\}$ *( in this case we say more precisely that* $S$ *is* $g$*-smooth).*

We have

**Theorem 1.1.** *Let* $G$ *be a finite abelian group and let* $S$ *be a zero-sum free sequence over* $G$ *with* $f(S) \le 2|S| - 1$. *Then* $S$ *has one of the following forms:*

*(i)* $S$ *is* $a$*-smooth for some* $a \in G$.

*(ii)* $S = a^k b$, *where* $k \in \mathbb{N}$ *and* $a, b \in G$ *are distinct.*

*(iii)* $S = a^k b^l$, *where* $k \ge l \ge 2$ *and* $a, b \in G$ *are distinct with* $2a = 2b$.

*(iv)* $S = a^k b^l (a - b)$, *where* $k \ge l \ge 2$ *and* $a, b \in G$ *are distinct with* $2a = 2b$.

For a sequence $S$ over $G$ we call

$$\mathsf{h}(S) = \max\{\mathsf{v}_g(S) | g \in G\} \in [0, |S|]$$

*the maximum of the multiplicities of* $S$.

Let $S = a^x b^y T$ with $x \ge y \ge \mathsf{h}(T)$, then Theorem 1.1(i) can be stated more precisely as that $S$ is $a$-smooth or $b$-smooth.

## 2. Some Lemmas

Let $\emptyset \ne G_0 \subseteq G$ be a subset of $G$ and $k \in \mathbb{N}$. Define

$$\mathsf{f}(G_0, k) = \min\{f(S) : S \in \mathcal{F}(G_0) \text{ zero-sumfree, squarefree and } |S| = k\}$$

and set $\mathsf{f}(G_0, k) = \infty$, if there are no sequences over $G_0$ of the above form.

**Lemma 2.1.** *Let* $G$ *be a finite abelian group.*

(1) *If* $k \in \mathbb{N}$ *and* $S = S_1 \cdot \cdots \cdot S_k \in \mathcal{A}^*(G)$, *then*

$$f(S) \ge f(S_1) + \cdots + f(S_k).$$

(2) *If* $G_0 \subset G$, $k \in \mathbb{N}$ *and* $\mathsf{f}(G_0, k) > 0$, *then*

$$\mathsf{f}(G_0, k) \begin{cases} = 1, & \text{if } k = 1, \\ = 3, & \text{if } k = 2, \\ \ge 5, & \text{if } k = 3, \\ \ge 6, & \text{if } k = 3 \text{ and } 2g \ne 0 \text{ for all } g \in G_0, \\ \ge 2k, & \text{if } k \ge 4. \end{cases}$$

*Proof.* 1. See [6, Theorem 5.3.1].

2. See [6, Corollary 5.3.4].

□

**Lemma 2.2.** *Let $a$, $b$ be two distinct elements in an abelian group $G$ such that $a^2b^2 \in \mathcal{A}^*(G)$, $2a \neq 2b, a \neq 2b$, and $b \neq 2a$. Then $f(a^2b^2) = 8$.*

*Proof.* It is easy to see that $a$, $2a$, $b$, $2b$, $a+b$, $a+2b$, $2a+b$, $2a+2b$ are all the distinct elements in $\sum(a^2b^2)$. We are done. □

**Lemma 2.3.** *Let $S = a^kb$ be a zero-sum free sequence over $G$. If $S = a^kb$ is not $a$-smooth, then $f(S) = 2k+1$.*

*Proof.* The assertion follows from the fact that $a, \ldots, ka, b, a+b, \ldots, ka+b$ are all the distinct elements in $\sum(a^kb)$. □

**Lemma 2.4.** [10, Lemma 4] *Let $S$ be a zero-sum free sequence over $G$. If there is some element $g$ in $S$ with order 2, then $f(S) \geq 2|S|-1$.*

**Lemma 2.5.** *Let $k \geq l \geq 2$ be two integers, and let $a$ and $b$ be two distinct elements of $G$ such that $a^kb^l \in \mathcal{A}^*(G)$ and $a^kb^l$ is not smooth. Then we have*

*(i) If $2a \neq 2b$, then $f(a^kb^l) \geq 2(k+l)$.*

*(ii) If $2a = 2b$, then $f(a^kb^l) = 2(k+l)-1$.*

*Proof.* If $nb \neq sa$ for any $n$ and $s$ with $1 \leq n \leq l$ and $1 \leq s \leq k$, then $ra+sb, r+s \neq 0, 0 \leq r \leq k, 0 \leq s \leq b$ are all the distinct elements in $\sum(a^kb^l)$, and so

$$f(a^kb^l) = kl+k+l \geq 2(k+l).$$

Now we assume that $nb = sa$ for some $n$ and $s$ with $1 \leq n \leq l$ and $1 \leq s \leq k$. Let $n$ be the least positive integer with $nb = sa, 1 \leq n \leq l, 1 \leq s \leq k$ . Then $n \geq 2$ and $s \geq 2$ by our assumptions. It is easy to see that

$$a, \ldots, ka, \ldots, (k+[\frac{l}{n}]s)a,$$

$$b, a+b, \ldots, b+ka, \ldots, b+(k+[\frac{l-1}{n}]s)a,$$

$$\cdots\cdots$$

$$(n-1)b, \ldots, (n-1)b+ka, \ldots, (n-1)b+(k+[\frac{l-n+1}{n}]s)a$$

are all the distinct elements in $\sum(a^kb^l)$, and so

$$f(a^kb^l) = k + [\frac{l}{n}]s + 1 + k + [\frac{l-1}{n}]s + \cdots + 1 + k + [\frac{l-n+1}{n}]s$$
$$= n(k-s+1) + ls + s - 1.$$

Since $n(k-s+1)+ls+s-1-2(k+l) = (n-2)(k-s)+(l-1)(s-2)+n-3$, we have $f(a^kb^l) \geq 2(k+l)-1$ and the equality holds if and only if $n = s = 2$, that is $2a = 2b$. This completes the proof. □

**Remark:** Note that if $a^kb^l \in \mathcal{A}^*(G)$, $k \geq l \geq 2$, then the conditions that $a^kb^l$ is smooth and $2a = 2b$ cannot hold simultaneously. Otherwise, we may suppose that $2a = 2b$ and $a^kb^l$ is $a$-smooth (the case that $a^kb^l$ is $b$-smooth is similar), then $b = ta$, $2 \leq t \leq (k+1)$. It follows that $b + (t-2)a = 2(t-1)a = 2b - 2a = 0$, $0 < t-2 \leq k-1$, which contradicts the fact that $a^kb^l \in \mathcal{A}^*(G)$.

**Lemma 2.6.** [12, Lemma 2.9]*Let $S = a^kb^lg$, $k \geq l \geq 1$ be a zero-sum free sequence over $G$ with $b - a = g$ and $\mathrm{ord}(g) = 2$, then $f(S) = 2(k+l)+1$.*

**Lemma 2.7.** *Let $S_1 \in \mathcal{F}(G)$ and $a$, $g \in G$ such that $S = S_1a \in \mathcal{A}^*(G)$, $S_1$ is $g$-smooth and $S$ is not $g$-smooth. Then $f(S) = 2f(S_1)+1$.*

*Proof.* If $a \notin < g >$, then $\sum(S) = \sum(S_1) \cup \{a\} \cup (\sum(S_1) + a)$, and so $f(S) = 2f(S_1) + 1$.

If $a \in < g >$, we let $\sum(S_1) = \{g, \ldots, ng\}$, $a = tg$, $t \in \mathbb{N}$, then $t \geq n+2$ by our assumptions. It follows that $\sum(S) = \{g, \ldots, ng, tg, (t+1)g, \ldots, (t+n)g\}$, and so $f(S) = 2f(S_1) + 1$. □

**Lemma 2.8.** *Let $k \geq 2$ be a positive integer and $a$, $b$, $c$ three distinct elements in $G$ such that $a^kbc \in \mathcal{A}^*(G)$ and $a^kbc$ is not $a$-smooth. Then $f(a^kbc) \geq 2k+4$.*

*Proof.* Observe that $f(a^kbc) \geq 2k+4$ when $a^kbc$ is $b$ or $c$-smooth. We consider first the case that $a^kb$ is $a$-smooth (the case that $a^kc$ is $a$-smooth is similar). It is easy to see $f(a^kb) \geq k+2$, and so $f(a^kbc) = 2f(a^kb) + 1 \geq 2k+5$ by Lemma 2.7. Therefore we may assume that both $a^kb$ and $a^kc$ are not $a$-smooth in the remaining arguments. We divide the proof into three cases.

(i) If $a^k(b+c)$ is not $a$-smooth, then $a, \ldots, ka, b, b+a, \ldots, b+c, b+c+a, \ldots, b+c+ka$ are distinct elements in $\sum(a^kbc)$, and so

$$f(a^kbc) \geq k+k+1+k+1 \geq 2k+4.$$

(ii) If neither $a^k(b-c)$ nor $a^k(c-b)$ is $a$-smooth, then $a, \ldots, ka, b, b+a, \ldots, c, c+a, \ldots, c+ka, b+c+ka$ are distinct elements in $\sum(a^kbc)$, and so

$$f(a^kbc) \geq k+k+1+k+1+1 \geq 2k+5.$$

(iii) If $a^k(b+c)$ is $a$-smooth and $a^k(b-c)$ (or $a^k(c-b)$ ) is $a$-smooth, then we have

$$b+c = sa, \quad b-c = ta, \quad 1 \leq s, t \leq k+1, \quad s \neq t.$$

It is easy to see that $a, \ldots, ka, (k+1)a, \ldots, (k+s)a, c, c+a, \ldots, c+(k+t)a$ are all distinct elements in $\sum(a^kbc)$, and so

$$f(a^kbc) = k+s+k+t+1 \geq 2k+4.$$

The second equality holds if and only if $(s, t) = (1, 2)$ or $(2, 1)$. We are done. □

The following corollary follows immediately from Lemmas 2.1, and 2.7 and the proof of Lemma 2.9.

**Corollary 2.1.** *Let $k \geq 1$ be a positive integer and $a, b, c, d$ four distinct elements in $G$ such that $a^kbcd \in \mathcal{A}^*(G)$ and $a^kbcd$ is not $a$-smooth. Then $f(a^kbcd) \geq 2k+6$.*

**Lemma 2.9.** *Let $a, b, x$ be three distinct elements in $G$ such that $a^kb^lx \in \mathcal{A}^*(G)$, $k \geq l \geq 1$, $2a = 2b$, and $x \neq a-b$, then $f(a^kb^lx) \geq 2(k+l+1)+1$.*

*Proof.* If there are no distinct pairs $(m, n) \neq (0, 0), (m_1, n_1) \neq (0, 0)$, $0 \leq m, m_1 \leq k$, $0 \leq n, n_1 \leq l$ such that $ma+nb = m_1a+n_1b+x$, then $\sum(a^kb^lx) = \sum(a^kb^l) \cup \{x\} \cup (\sum(a^kb^l)+x)$, and so $f(a^kb^lx) = 2f(a^kb^l)+1 = 4(k+l)-1 \geq 2(k+l+1)+1$.

If there are two distinct pairs $(m, n) \neq (0, 0), (m_1, n_1) \neq (0, 0)$, $0 \leq m, m_1 \leq k$, $0 \leq n, n_1 \leq l$ such that $ma+nb = m_1a+n_1b+x$, then $x = a-b$ or $x = ua+b$, $1 \leq u \leq (k+l-1)$ or $x = vb$, $v \geq 2$ or $x = ta$, $t \geq 2$.

Let $x = ua+b, 1 \leq u \leq (k+l-1)$, then $a, \ldots, (k+l+u)a, b, \cdots, b+(k+l+u)a$ are all distinct elements in $\sum(a^kb^lx)$, and so $f(a^kb^lx) = 2(k+l+u)+1 \geq 2(k+l+1)+1$.

Let $x = vb$, $2 \le v \le (k+l)$ (the case that $x = ta$, $t \ge 2$ is similar). If $k$ is even, then $b, \dots, (k+l+v)b, a, a+b, \dots, a+(k+l-2+v)b$ are all distinct elements in $\sum(a^k b^l x)$, and so $f(a^k b^l x) = 2(k+l+v-1)+1 \ge 2(k+l+1)+1$. If $k$ is odd, then $b, \dots, (k+l+v-1)b, a, a+b, \dots, a+(k+l-1+v)b$ are all distinct elements in $\sum(a^k b^l x)$, and so $f(a^k b^l x) = 2(k+l+v-1)+1 \ge 2(k+l+1)+1$. We are done. □

**Lemma 2.10.** *Let $a, b, x$ be three distinct elements in $G$ such that $a^k b^2 x \in \mathcal{A}^*(G)$, $k \ge 2$ and $a^k b^2 x$ is not a-smooth or b-smooth, then $f(a^k b^2 x) = 2k+5$ if and only if $2a = 2b$ and $x = b - a$.*

*Proof.* We divide the proof into four cases.

**Case 1** $a^k b^2$ is not smooth and $2b = sa$, $2 \le s \le k$. If $x = b - a$, then $a, \dots, (k+s)a, b-a, b, \dots, b+(k+s-1)a$ are all the distinct elements in $\sum(a^k b^2 x)$, and so $f(a^k b^2 x) = 2(k+s)+1$. If $x = ta$, $2 \le t \le k$, then $a, \dots, (k+s+t)a, b, \dots, b+(k+t)a$ are all the distinct elements in $\sum(a^k b^2 x)$, and so $f(a^k b^2 x) = 2(k+t)+s+1$. If $x = ta+b$, $1 \le t \le k$, then $a, \dots, (k+s+t)a, b, \dots, b+(k+t+s)a$ are all the distinct elements in $\sum(a^k b^2 x)$, and so $f(a^k b^2 x) = 2(k+t+s)+1$. Therefore $f(a^k b^2 x) = 2k+5$ if and only if $2a = 2b$ and $x = b - a$ in this case.

**Case 2** $a^k b^2$ is not smooth and $2b = sa$, $s > k$ or $2b \notin < a >$, then $f(a^k b^2) = 3k+2$. If $k \ge 3$, then $f(a^k b^2 x) \ge f(a^k b^2) + 1 = 3k+3 > 2k+5$. If $k = 2$ and $f(abx) = 7$, then $f(a^2 b^2 x) \ge f(abx) + f(ab) = 7+3 > 2k+5$. If $k = 2$ and $f(abx) = 6$ (i.e., $x = a+b$ or $x = a-b$ or $x = b-a$), then it is easy to check that $f(a^2 b^2 x) > 2k+5$.

**Case 3** $a^k b^2$ is smooth and $a^k b^2 x$ is not smooth. If $a^k b^2$ is $a$-smooth, then $f(a^k b^2 x) = 2f(a^k b^2) + 1 \ge 2(k + 2 \times 2) + 1 > 2k+5$. If $a^k b^2$ is $b$-smooth, then $f(a^k b^2 x) = 2f(a^k b^2) + 1 \ge 2(2+2k)+1 > 2k+5$.

**Case 4** $a^k b^2 x$ is $x$-smooth. We have $f(a^k b^2 x) \ge 1 + 2k + 2 \times 3 > 2k+5$.

This completes the proof of the lemma.

□

## 3. Proofs of the Main Theorems

To prove the main theorem of the present paper, we still need the following two obviously facts on smooth sequences.

**Fact 1** Let $r$ be a positive integer and $a \in G$. If $WT_i \in \mathcal{A}^*(G)$ is $a$-smooth for all $i = 1, \dots, r$, then $S = T_1 \cdot \dots \cdot T_r W$ is $a$-smooth.

**Fact 2** Let $r$, $k$, $l$ be three positive integers and $a$, $b$ two distinct elements in $G$. If $S \in \mathcal{A}^\star(G)$ is $a$-smooth and $a^k b^l T_i \in \mathcal{A}^\star(G)$ is $a$-smooth or $b$-smooth for all $i = 1, \dots, r$, then the sequence $Sa^k b^l T_1 \cdot \dots \cdot T_r$ is $a$-smooth or $b$-smooth.

**Proof of Theorem 1.1:**

We start with the trivial case that $S = a^k$ with $k \in \mathbb{N}$ and $a \in G$. Then $\sum(S) = \{a, \dots, ka\}$, and since $S$ is zero-sum free, it follows that $k < ord(a)$. Thus $S$ is $a$-smooth.

If $S = S_1 g \in \mathcal{A}^\star(G)$, where $g$ is an element of order 2, then $f(S) \geq 2|S| - 1$ by Lemma 2.4, and $f(S) \geq f(S_1) + 2$ since $\sum(S) \supseteq \sum(S_1) \cup \{g, g + \sigma(S_1)\}$. If $S = S_1 g_1 g_2 \in \mathcal{A}^\star(G)$, where $g_1$ and $g_2$ are two elements of order 2, then $f(S) \geq 2|S|$ since $\sum(S) \supseteq \sum(S_1 g_1) \cup \{g_2, g_1 + g_2, g_1 + g_2 + \sigma(S_1)\}$. Therefore it suffices to determine all $S \in \mathcal{A}^\star(G)$ such that $S$ does not contain any element of order 2 and $f(S) \leq 2|S| - 1$, and when $f(S) \leq 2|S| - 1$, determine all $Sg \in \mathcal{A}^\star(G)$ such that $g$ is an element of order 2 and $f(Sg) = 2|S| + 1$.

To begin with, we determine all $S \in \mathcal{A}^\star(G)$ such that $S$ does not contain any element of order 2 and $f(S) \leq 2|S| - 1$. Let $S = a^x b^y c^z T$ with $x \geq y \geq z \geq \mathsf{h}(T)$ and $a, b, c \notin \text{supp}(T)$. The case that $|\text{supp}(S)| = 2$ follows from Lemmas 2.3 and 2.5 and the remark after Lemma 2.5. Therefore we may assume that $|\text{supp}(S)| \geq 3$ and $S$ does not contain any element of order 2 in the following arguments.

If $x = y = z$, then $S$ allows the product decomposition

$$S = S_1 \cdot \dots \cdot S_x,$$

where $S_i = abc \cdot \dots$ , $i = 1, \dots, x$ are squarefree of length $|S_i| \geq 3$. By Lemma 2.1, we obtain

$$f(S) \geq \sum_{i=1}^{x} f(S_i) \geq 2 \sum_{i=1}^{x} |S_i| = 2|S|.$$

If $x \geq y > z \geq \mathsf{h}(T)$, or $x > y \geq z \geq \mathsf{h}(T)$, then $S$ allows a product decomposition

$$S = T_1 \cdot \dots \cdot T_r W$$

having the following properties:

- $r \geq 1$ and, for every $i \in [2, r]$, $S_i \in \mathcal{F}(G)$ is squarefree of length $|S_i| = 3$.
- $W \in \mathcal{F}(G)$ has the form $W = a^k$, $k \geq 1$ or $W = a^k b$, $k \geq 1$ or $W = a^k b^l$, $k \geq l \geq 2$.

We choose a product decomposition such that $k$ is the largest integer in $W = a^k$ (or $a^k b$ or $a^k b^l$, $k \geq l \geq 2$) among all such product decompositions. We divide the remaining proof into three cases.

**Case 1** $W = a^k$, $k \geq 1$. If $T_i = xyz$ with $a \notin \{x, y, z\}$ for some $i$, $1 \leq i \leq r$ such that $a^k xyz$ is not $a$-smooth whenever $k > 1$, then $S$ admits the product decomposition

$$S = T_1 \cdot \cdots \cdot T_{i-1} T_i' T_{i+1} \cdot \cdots \cdot T_r,$$

where $T_i$, $i = 1, \ldots, r$ have the properties described above and $T_i' = a^k xyz$. By Lemma 2.1, and Corollary 2.1, we get

$$f(S) \geq \sum_{j \neq i}^{r} f(T_j) + f(T_i') \geq \sum_{j \neq i}^{r} 2|T_j| + 2|T_i'| = 2|S|.$$

If $T_i = axy$ for some $i$, $1 \leq i \leq r$ such that $a^{k+1}xy$ is not $a$-smooth, then $S$ admits the product decomposition

$$S = T_1 \cdot \cdots \cdot T_{i-1} T_i' T_{i+1} \cdot \cdots \cdot T_r,$$

where $T_i$, $i = 1, \ldots, r$ have the properties described above and $T_i' = a^{k+1}xy$. By Lemmas 2.1 and 2.8, we get

$$f(S) \geq \sum_{j \neq i}^{r} f(T_j) + f(T_i') \geq \sum_{j \neq i}^{r} 2|T_j| + 2|T_i'| = 2|S|.$$

Therefore we have proved that if $S$ is not $a$-smooth and $W = a^k$, then $f(S) \geq 2|S|$.

**Case 2** $W = a^k b$, $k \geq 1$.

Let $T_i = xyz$ with $a \notin \{x, y, z\}$ for some $i$, $1 \leq i \leq r$. If $k = 1$, then $T_i W = abxyz$. If $k = 2$, then $T_i W = abx \cdot ayz$. If $k \geq 3$ and one sequence among three sequences $a^{k-1}yz$, $a^{k-1}xz$, and $a^{k-1}xy$, say, $a^{k-1}yz$ is not $a$-smooth, then $T_i W = abx \cdot a^{k-1}yz$. It follows from Lemmas 2.1 and 2.8 that $f(T_i W) \geq 2|T_i| + 2|W|$, and so $f(S) \geq 2|S|$.

Let $T_i = bxy$ for some $i$, $1 \leq i \leq r$, then $k \geq 2$. If $k = 2$, then $T_i W = abx \cdot aby$. If $k > 2$ and $a^{k-1}by$ (or $a^{k-1}bx$) is not $a$-smooth, then $T_i W = abx \cdot a^{k-1}by$ (or $T_i W = aby \cdot a^{k-1}bx$). It follows from Lemmas 2.1 and 2.8 that $f(T_i W) \geq 2|T_i| + 2|W|$, and so $f(S) \geq 2|S|$.

Let $T_i = abx$ for some $i$, $1 \leq i \leq r$, then $T_i W = a^{k+1}b^2 x$. If $a^{k+1}b^2 x$ is not $a$-smooth or $b$-smooth, then by Lemma 2.10 we have $f(T_i W) \geq 2|T_i| + 2|W|$, and so $f(S) \geq 2|S|$.

Therefore we have proved that if $S$ is not $a$-smooth or $b$-smooth, then $f(S) \geq 2|S|$ in this case.

**Case 3** $W = a^k b^l$, $k \geq l \geq 2$. If $2a \neq 2b$ and $a^k b^l$ is not smooth, then by Lemma 2.5 we have $f(W) \geq 2|W|$ and we are done. Note that the conditions that $2a = 2b$ and $a^k b^l$ is smooth cannot hold simultaneous. Here we omit the similar arguments as we have done in Case 1.

**Subcase 1** $2a = 2b$.

Let $T_i = xyz$ with $a \notin \{x, y, z\}$ for some $i$, $1 \leq i \leq r$, then $T_i W = abxy \cdot a^{k-1}b^{l-1}z$. It follows from Lemmas 2.1 and 2.9 that $f(T_i W) \geq 2|T_i| + 2|W|$, and so $f(S) \geq 2|S|$.

Let $T_i = byz$ for some $i$, $1 \leq i \leq r$, then $k \geq l + 1$, $T_i W = aby \cdot a^{k-1}b^l z$. It follows from Lemmas 2.1 and 2.9 that $f(T_i W) \geq 2|T_i| + 2|W|$, and so $f(S) \geq 2|S|$.

Let $T_i = abx$ for some $i$, $1 \leq i \leq r$, then $T_i W = a^{k+1}b^{l+1}x$. If $a^{k+1}b^2 x$ is not $a$-smooth or $b$-smooth, then by Lemma 2.10 we have $f(T_i W) \geq 2|T_i| + 2|W|$, and so $f(S) \geq 2|S|$.

**Subcase 2** $a^k b^l$ is smooth, $a \neq 2b$, and $b \neq 2a$. Then $W = (a^2b^2)^s W_1$, $W_1 = a^{k_1}$ or $W_1 = a^{k_1}b$. If $S_1 = SW^{-1}W_1$ is not $a$-smooth or $b$-smooth, then $f(S_1) \geq 2|S_1|$, and so by Lemmas 2.1 and 2.2 $f(S) \geq sf(a^2b^2) + f(S_1) \geq 8s + 2|S_1| = 2|S|$. If $S_1 = SW^{-1}W_1$ is $a$-smooth or $b$-smooth, then $S$ is $a$-smooth or $b$-smooth.

**Subcase 3** $a = 2b$.

Let $T_i = xyz$ with $a, b \notin \{x, y, z\}$ for some $i$, $1 \leq i \leq r$, then it is easy to see that $f(T_i W) = f(a^k b^l xyz) = f(b^{2k+l}xyz)$. It follows from Corollary 2.1 that $b^{2k+l}xyz$ is $b$-smooth or $f(T_i W) \geq 2(|T_i| + |W|)$.

Let $T_i = bxy$ with $a, b \notin \{x, y\}$ for some $i$, $1 \leq i \leq r$, then $f(T_i W) = f(a^k b^{l+1}xy) = f(b^{2k+l+1}xyz)$. It follows from Lemma 2.8 that $b^{2k+l+1}xy$ is $b$-smooth or $f(T_i W) \geq 2(|T_i| + |W|)$.

Let $T_i = abx$ with $a \neq x$, $b \neq x$ for some $i$, $1 \leq i \leq r$, then $f(T_i W) = f(a^{k+1}b^{l+1}x) = f(b^{2k+l+3}xyz)$. It follows from Lemma 2.3 that $b^{2k+l+3}x$ is $b$-smooth or $f(T_i W) \geq 2(|T_i| + |W|)$.

**Subcase 4** $b = 2a$. Similar to Subcase 3.

Therefore we have proved that if $S$ is not $a$-smooth or $b$-smooth, then $f(S) \geq 2|S| - 1$ and $f(S) = 2|S| - 1$ if and only if $S = a^k b$ or $S = a^k b^l$, $2a = 2b$, $k \geq l \geq 2$.

Finally, when $f(S) \leq 2|S| - 1$, we will determine all $Sg \in \mathcal{A}^*(G)$ such that $g$ is an element of order 2 and $f(Sg) = 2|S| + 1$.

(i) If $S$ is $a$-smooth (the case that $S$ is $b$-smooth is similar), we set $\sum(S) = \{a, \ldots, na\}$, $n \leq 2|S| - 1$, then $g \notin \sum(S)$ since $g$ is an element of order 2 and $Sg \in \mathcal{A}^*(G)$. It follows that $\sum(Sg) = \sum(S) \cup$

$\{g\} \cup \{g + \sum(S)\}$, and so $f(Sg) = 2n+1$. Therefore $f(Sg) \leq 2|S|+1$ if and only if $S = a^k$.

(ii) $S = a^k b$ is not smooth, by Lemma 2.8, $f(a^k bg) \leq 2k+1$ only if $a^k bg$ is $a$-smooth, which is impossible since $g$ is an element of order 2 and $a^k bg \in \mathcal{A}^*(G)$.

(iii) $S = a^k b^l$, $2a = 2b$, $k \geq l \geq 2$. The result follows from Lemmas 2.5 and 2.9.

Therefore we have proved that if $S = a^x b^y \cdot \cdots \in \mathcal{A}^*(G)$, $x \geq y \geq \cdots$, where $a, b, \ldots$ are distinct elements of $G$ and $f(S) \leq 2|S| - 1$, then $S$ is $a$-smooth or $b$-smooth or $S = a^k b$, $b \notin \sum(a^k)$ or $S = a^k b^l$, $k \geq l \geq 2, 2a = 2b$ or $S = a^k b^l$, $k \geq l \geq 2, 2a = 2b$, $g = a - b$. Theorem 1.1 is proved.

□

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