# A Note on Chromatic Sum

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**Abstract**

The chromatic sum $\Sigma(G)$ of a graph $G$ is the smallest sum of colors among of proper coloring with the natural number. In this paper, we introduce a necessary condition for the existence of graph homomorphisms. Also, we present $\Sigma(G) < \chi_f(G)|G|$ for every graph $G$.

**Key words:** chromatic sum, graph homomorphism, Fractional chromatic number.

**Subject classification:** 05C .

## 1 Introduction and Preliminaries

We consider finite undirected graphs with no loops and multiple edges and use [4] for the notions and notations not defined here. Let $G$ be a graph and $c$ be a proper coloring of it, define $\Sigma_c(G) = \sum_{\{v \in V(G)\}} c(v)$. The vertex-chromatic sum of $G$, denoted by $\Sigma(G)$, is defined as $\min\{\Sigma_c(G)|\ c \text{ is a proper coloring of } G\}$.

The chromatic sum first appeared in 1987 from two different sources. In theoretical graph theory, Kubika in her Ph.D. thesis introduced the chromatic sum of a graph with the above notation. Supowit introduced the optimum cost chromatic partition problem, from its application in VLSI design [10]. The vertex-strength of G denoted by $s(G)$, or briefly by $s$, is the smallest number $s$ such that there is a proper coloring $c$ with $s$ colors where $\Sigma_c(G) = \Sigma(G)$. Clearly, $s(G) \geq \chi(G)$ and equality does not always hold. In fact, for

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every positive integer $k$, almost all trees satisfy $s > k$; see [7]. Chromatic sum has been investigated in literature [1, 2, 3, 5, 6, 7, 10, 11].

In [11], Thomassen et al. obtained several bounds for chromatic sum for general graphs. The first is a rather natural result of an application of a greedy algorithm: $\Sigma(G) \leq n + e$, where $n$ and $e$ are the number of vertices and edges of $G$, respectively. Also, they presented an upper and lower limit for the chromatic sum in terms of $e$. They showed that $\sqrt{8e} \leq \Sigma(G) \leq \frac{3}{2}(e+1)$ and these bounds are sharp.

Let $G$ and $H$ be two graphs. A homomorphism $\sigma$ from a graph $G$ to a graph $H$ is a map $\sigma : V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ implies $\sigma(u)\sigma(v) \in E(H)$. The set of all homomorphisms from $G$ to $H$ is denoted by $\mathrm{Hom}(G, H)$. An isomorphism of $G$ to $H$ is a homomorphism $f : G \to H$ which is a vertex and edge bijective homomorphism. An isomorphism $f : G \to G$ is called an automorphism of $G$, and the set of all automorphism of $G$ is denoted by $\mathrm{Aut}(G)$.

Suppose $m \geq 2n$ are positive integers. We denote by $[m]$ the set $\{1, 2, \cdots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all $n$-subsets of $[m]$. The Kneser graph $KG(m, n)$ has vertex set $\binom{[m]}{n}$, in which $A \sim B$ if and only if $A \cap B = \emptyset$. The graph $KG(5, 2)$ is named Petersen graph that is denoted by $P$. It was conjectured by Kneser in 1955 and proved by Lovász [8] in 1978 that $\chi(KG(m, n)) = m - 2n + 2$.

The fractional chromatic number of a graph $G$, denoted by $\chi_f(G)$, is the infimum of the ratios $\frac{m}{n}$ such that there is a homohomrphism from $G$ to $KG(m, n)$. It is known [9] that the infimum in the definition can be attained, and hence can be replaced by the minimum. It is easy to see $\chi_f(G) \leq \chi(G)$. On the other hand, the ratio $\frac{\chi(G)}{\chi_f(G)}$ can be arbitrary large, see [9].

In next section we present a necessary condition for existence of graph homomorphisms in terms of chromatic sum. Next, we introduce an upper bound for chromatic sum based on fractional chromatic number.

## 2 Graph Homomorphism and Chromatic Sum

Graph homomorphism is a fundamental concept in graph theory, where it is related to many important concepts and problems in the field. It is well-known that in general it is a hard problem to decide whether there exists a homomorphism from a given graph $G$ to a given graph $H$, and consequently, it is interesting to obtain necessary conditions for the existence of such mappings. In this regard, we have the following theorem.

**Theorem 1.** *Let $G$ and $H$ be two graphs such that $H$ is a vertex transitive graph. If $\sigma : G \longrightarrow H$ is a homomorphism, then*

$$\frac{\Sigma(G)}{|G|} \leq \frac{\Sigma(H)}{|H|}.$$

**Proof.** Let Aut(H) $= \{f_1, f_2, \ldots, f_t\}$ and $\tilde{G} = \bigcup_{i=1}^{t} G_i$ that $G_i$ is an isomorphic copy of $G$. Define $\tilde{\sigma} : \tilde{G} \longrightarrow H$ such that its restriction to $G_i$ is $f_i \circ \sigma$. Since $H$ is a vertex transitive graph, one can easily show that for every $v \in V(H)$, $|\tilde{\sigma}^{-1}(v)| = t\frac{|G|}{|H|}$ and it is independent of $v$. Now, suppose $c$ is a proper coloring of $H$ such that $\Sigma_c(H) = \Sigma(H)$. For any vertex $v \in V(\tilde{G})$, set $\tilde{c}(v) = c(\tilde{\sigma}(v))$. Obviously, $\tilde{c}$ is a proper coloring of $\tilde{G}$ and also $\Sigma_{\tilde{c}}(\tilde{G}) = \frac{t|G|}{|H|} \times \Sigma(H)$. Therefore, there is an $i$ such that $\Sigma_{\tilde{c}|_{G_i}}(G_i) \leq \frac{|G|}{|H|} \times \Sigma(H)$ and since $G = G_i$, $\Sigma(G) \leq \frac{|G|}{|H|} \times \Sigma(H)$ which is the desired conclusion. ■

Theorem 1 provides a necessary condition for the existence of graph homomorphisms. Here we show that the Petersen graph $P$ has the same chromatic number and circular chromatic number. To see this, it is sufficient to show that there is no homomorphism from the Petersen graph $P$ to the circular complete graph $K_{\frac{8}{3}}$. One can check that $\Sigma(P) = 19$ and $\Sigma(K_{\frac{8}{3}}) = 15$. Therefore, as an application of the previous theorem, there is no homomorphism from $P$ to $K_{\frac{8}{3}}$.

It is well-known that the chromatic sum is an NP-complete problem [7]. In this regard, finding upper and lower bounds for chromatic sum is useful. It was shown in [3] that $\Sigma(G) \leq (\frac{\chi(G)+1}{2})|G|$. Since $\Sigma(K_n) = \frac{n(n+1)}{2}$, if we set $H = K_{\chi(G)}$, then Theorem 1 implies this

bound. Here we obtain an upper bound for the chromatic sum in terms of fractional chromatic number.

For an independent set $S$ in a graph $G$ the following inequality is an immediate consequence of the definition of the chromatic sum [2],

$$\Sigma(G) \leq |G| + \Sigma(G \setminus S). \tag{1}$$

**Theorem 2.** *For every graph $G$, we have*

$$\Sigma(G) < \chi_f(G)|G|.$$

**Proof.** Assume that $\chi_f(G) = \frac{m}{n}$ and $\mathrm{Hom}(G, KG(m,n)) \neq \emptyset$. In view of equation 1, we have $\Sigma(KG(m,n)) \leq \binom{m}{n} + \Sigma(KG(m-1,n))$. Hence $\Sigma(KG(m,n)) \leq \sum_{i=0}^{m-2n-1} \binom{m-i}{n} + \Sigma(KG(2n,n))$. On the other hand, $\sum_{i=0}^{m-2n-1} \binom{m-i}{n} = \binom{m+1}{n+1} - \binom{2n+1}{n+1}$ and $\Sigma(KG(2n,n)) = \frac{3}{2}\binom{2n}{n}$. Therefore, $\Sigma(KG(m,n)) \leq \binom{m+1}{n+1} - (\frac{n-1}{2n+2})\binom{2n}{n}$. Now, since $\mathrm{Hom}(G, KG(m,n)) \neq \emptyset$, Theorem 1 implies that

$$\Sigma(G) \leq \left( \frac{m+1}{n+1} - (\frac{n-1}{2n+2}) \frac{\binom{2n}{n}}{\binom{m}{n}} \right) |G|.$$

Furthermore, $\frac{m+1}{n+1} - (\frac{n-1}{2n+2}) \frac{\binom{2n}{n}}{\binom{m}{n}} \leq \frac{m+1}{n+1} < \frac{m}{n} = \chi_f(G)$, as desired. ■

In particular, if $G$ is a vertex transitive graph, $\chi_f(G) = \frac{|G|}{\alpha(G)}$ and hence $\Sigma(G) < \frac{|G|^2}{\alpha(G)}$. Furthermore, $e(G) = \frac{\Delta(G)|G|}{2}$. Also, if $\chi_f(G) \leq \frac{3}{4}\Delta(G)$, then $\chi_f(G)|G| < \frac{3}{2}(e(G)+1)$. Therefore, in this case, the bound in Theorem 2 is better than the upper bound $\frac{3}{2}(e(G)+1)$ (see [11]).

On the other hand, in view of Theorem 1 we have

$$\Sigma(G) \geq \frac{\omega(G)+1}{2}|G|$$

where $G$ is a vertex transitive graph and $\omega(G)$ is the size of the largest clique in it.

Also, it is a known result that the ratio $\frac{\chi(G)}{\chi_f(G)}$ can be arbitrary large (see [9]). Let $\mathcal{G} = \{G_i\}_{i \in \mathbb{N}}$ such that $\frac{\chi(G_n)}{\chi_f(G_n)} \to \infty$. We can

assume that $G_n$ is critical for all $n$ ($G$ is critical if $\chi(G \setminus v) < \chi(G)$ for every $v \in V(G)$). Thus, $e(G_n) \geq \frac{|G_n|(\chi(G_n)-1)}{2}$ and we also have $\frac{\frac{3}{2}(e(G_n)+1)}{\chi_f(G_n)|G_n|} \longrightarrow \infty$. It means the bound in Theorem 2 is better than the upper bound $\frac{3}{2}(e(G)+1)$ for the graphs in $\mathcal{G}$.

In Theorem 2 we used an upper bound of $\Sigma(KG(m,n))$, but we do not know the exact value of $\Sigma(KG(m,n))$. The improvement of this upper bound yields an improvement in Theorem 2.

**Question 1** *What is the exact value of* $\Sigma(KG(m,n))$*? Is it true that* $\Sigma(KG(m,n)) = \binom{m}{n}(\frac{m+1}{n+1} - (\frac{n-1}{2n+2})\frac{\binom{2n}{n}}{\binom{m}{n}})$*?*

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