Lower Bounds on some certain van der Waerden Functions *

Fang Tian[1†] Zi-Long Liu[2]
[1]Department of Applied Mathematics
Shanghai University of Finance and Economics, Shanghai, China
[2] School of Computer and Electronic Engineering
University of Shanghai for Science and Technology of China, Shanghai, China

Abstract

For positive integers r and $k_1, k_2, \cdots, k_r$, the van der Waerden number $W(k_1, k_2, \ldots, k_r; r)$ is the minimum integer N such that whenever set $\{1, 2, \ldots, N\}$ is partitioned into r sets $S_1, S_2, \ldots, S_r$, there is a k_i-term arithmetic progression contained in S_i for some i. This paper establishes an asymptotic lower bound for $W(k, m; 2)$ for fixed $m \geq 3$ which improves the result of T.C. Brown *et al*'s in [Bounds on some van der Waerden numbers. J. Combin. Theory, Ser.A 115 (2008), 1304-1309]. Some lower bounds on certain van der Waerden-like functions are also proposed.

Keywords: van der Waerden numbers, arithmetic progressions.

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1 Introduction

The theorem of van der Waerden [1, 2] asserts that for all positive integers r and $k_1, k_2, \ldots, k_r$, there exists a minimum positive integer $N = W(k_1, k_2, \ldots, k_r; r)$ such that for every r-coloring of set $[1, N] = \{1, 2, \ldots, N\}$ (i.e. $[1, N]$ is randomly partitioned into r sets), there is a k_i-term arithmetic progression of color i for some i, here $1 \leq i \leq r$.

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†Corresponding author: tianf@mail.shufe.edu.cn

The best known lower bound on $W(k,k;2)$ is $W(k,k;2) \geq (k-1)2^{(k-1)}$ for $k-1$ is prime, which is due to Berlekamp [3]. The best upper bound for $W(k,k;2)$ is

$$W(k,k;2) < 2^{2^{2^{2^{2^{(k+9)}}}}},$$

which is a striking result of Gowers [4]. Recently, Brown *et al* [5] propose an asymptotic lower bound on $W(k,m;2)$, that is,

$$W(k,m;2) > k^{m-1-\frac{1}{\log\log k}}.$$

for fixed $m \geq 3$ and all sufficiently large k, where $\log x$ is the natural logarithmic function.

On the other hand, Graham [6] and Brown *et al* [5] also investigate the bounds on van der Waerden-like function $\mathscr{W}(k,m;2)$, which defines as the least n such that every 2-coloring of $[1,n]$ (i.e. $[1,n]$ is randomly partitioned into 2 sets) gives either a k-term arithmetic progression in the first color or m consecutive integers in the second color. Clearly, $W(k,m;2) \leq \mathscr{W}(k,m;2)$. Graham [6] gives $m^{c\log m} < \mathscr{W}(3,m;2) < m^{dm^2}$ for all m and suitable constants $c,d > 0$. Brown *et al* prove $\mathscr{W}(4,m;2) < e^{m^{c\log m}}$ for all $m \geq 2$ and a suitable constant $c > 0$.

In this paper, we will prove an asymptotic lower bound for $W(k,m;2)$ for fixed $m \geq 3$ which improves the result of T.C. Brown *et al* in [5]. We also give a general lower bound on $\mathscr{W}(k,m;2)$.

2 Lower Bound for $W(k,m;2)$

Before proving Theorem 2.1, we state the form of Spencer local lemma [7] we use. In fact, the probabilistic method we adopted is a modification of the ingenious technique of Spencer [8]. By $f(x) \sim g(x)$, we mean that $\lim f(x)/g(x) = 1$ as $x \to \infty$.

Spencer Local Lemma *Let $A_1, A_2, \ldots, A_n$ be events in a probability space $(\Omega, \mathscr{F}, Pr)$. If there exist positive numbers $y_1, y_2, \cdots, y_n$ such that for $i = 1, 2, \cdots, n$,*

$$y_i Pr(A_i) < 1$$

and

$$\log y_i \geq - \sum_{ij \in E(D)} \log\left(1 - y_j Pr(A_j)\right),$$

then $Pr(\cap \overline{A_i}) > 0$, where D is the dependency graph for events $A_1, A_2, \ldots, A_n$.

Theorem 2.1 *Let $m \geq 3$ be a fixed integer, for sufficiently large k,*

$$W(k, m; 2) > (2mk \log k)^m.$$

Proof Let us 2-color the set $[1, N] = \{1, 2, \ldots, N\}$ randomly and independently, where each integer between 1 and N is colored red or blue with probability p or $q = 1 - p$, respectively. For each k-term arithmetic progression S in $[1, N]$, let A_S be the event that S is a monochromatic red arithmetic progression. For each m-term arithmetic progression T in $[1, N]$, let B_T be the event that T is a monochromatic blue arithmetic progression. Then $Pr(A_S) = p^k$ and $Pr(B_T) = (1 - p)^m$. Obviously, two events are dependent if and only if the corresponding subsets in $[1, N]$ have some integers in common.

Take any event A_S or B_T, and let x be any integer in the corresponding subset $S \subseteq [1, N]$ or $T \subseteq [1, N]$. The number of k-term arithmetic progressions $\mathscr{S}$ in $[1, N]$ that contain x is bounded by $k\frac{N}{k-1}$, since there are k positions that x may occupy in $\mathscr{S}$ and the gap size in $\mathscr{S}$ is no greater than $\frac{N}{k-1}$. Similarly, the number of m-term arithmetic progressions $\mathscr{T}$ in $[1, N]$ that contain x is bounded by $m\frac{N}{m-1}$. Hence, each A_S event is dependent of at most $k \cdot k\frac{N}{k-1} = \frac{k^2N}{k-1}$ other A_S events and dependent of at most $k \cdot m\frac{N}{m-1} = \frac{kmN}{m-1}$ of the B_T events; each B_T event is dependent of at most $m \cdot k\frac{N}{k-1} = \frac{kmN}{k-1}$ of the A_S events and dependent of at most $m \cdot m\frac{N}{m-1} = \frac{m^2N}{m-1}$ other B_T events.

We aim to prove that there exist positive numbers a and b satisfying the form of Spencer local lemma, namely, $ap^k < 1$ and $bq^m = b(1-p)^m < 1$ hold with $y_i = a$ for each A_S event, $y_i = b$ for each B_T event,

$$\log a \geq -\frac{k^2N}{k-1}\log(1 - ap^k) - \frac{kmN}{m-1}\log(1 - bq^m) \tag{1}$$

$$\log b \geq -\frac{kmN}{k-1}\log(1 - ap^k) - \frac{m^2N}{m-1}\log(1 - bq^m) \tag{2}$$

If such a and b are available, then there exists a 2-coloring of $[1, N]$ in which there is neither red k-term arithmetic progression nor blue m-term arithmetic progression, that is, $W(k, m; 2) > N$.

To this end, set $\beta = \frac{1}{m}$ and

$$\begin{aligned}
k &= \tfrac{1}{2}{c_1}^{\beta+1}N^\beta(\log N)^{-1},\\
a &= \exp\left\{c_2\left[\log(c_1N)\right]^{-\beta-1}N^\beta\right\},\\
b &= c_3N^{-1-\beta},\\
p &= 1 - q = 1 - c_4\left[\log(c_1N)\right]^{-\beta-1}N^\beta \cdot k^{-1}\\
&= 1 - 2c_1^{-\beta-1}c_4\left[\log(c_1N)\right]^{-\beta-1}(\log N)
\end{aligned}$$

where c_1, c_2, c_3 and c_4 are positive constants to be chosen.

Firstly, by $1 - x < \exp(-x)$ for $x > 0$, we have

$$\begin{aligned} p^k &= (1-q)^k \\ &< \exp(-kq) \\ &= \exp\left\{-c_4\left[\log(c_1 N)\right]^{-\beta-1} N^\beta\right\}. \end{aligned}$$

Thus

$$ap^k \quad < \quad \exp\left\{(c_2 - c_4)\left[\log(c_1 N)\right]^{-\beta-1} N^\beta\right\}.$$

If we choose the constants c_2 and c_4 such that $c_2 - c_4 < 0$, then $ap^k \to 0$ as $k \to \infty$ hence as $N \to \infty$. By the fact that $\log(1-x) \sim x$ as $x \to 0$, the first term in the right side of the inequality (1)

$$\begin{aligned} & \tfrac{k^2 N}{k-1} \log(1 - ap^k) \\ \sim\ & \tfrac{k^2 N}{k-1} ap^k \\ <\ & 2kNap^k \qquad (\text{by } k - 1 > k/2) \\ <\ & \exp\left\{(\beta+1)\log(c_1 N) + (c_2 - c_4)\left[\log(c_1 N)\right]^{-\beta-1} N^\beta\right\} \\ =\ & \exp\left\{(\beta+1)\log(c_1 N)\left[1 + \tfrac{(c_2 - c_4)}{(\beta+1)}\left[\log(c_1 N)\right]^{-\beta-2} N^\beta\right]\right\} \\ \to\ & 0 \end{aligned}$$

as $k \to \infty$ hence as $N \to \infty$. The first term in the right side of the inequality (2)

$$\begin{aligned} & \tfrac{kmN}{k-1} \log(1 - ap^k) \\ \sim\ & \tfrac{kmN}{k-1} ap^k \\ <\ & 2mNap^k \qquad (\text{by } k - 1 > k/2) \\ <\ & \exp\left\{\log(2mN) + (c_2 - c_4)\left[\log(c_1 N)\right]^{-\beta-1} N^\beta\right\} \\ =\ & \exp\left\{\log(2mN)\left[1 + (c_2 - c_4)\left[\log(c_1 N)\right]^{-\beta-1} \log(2mN)^{-1} N^\beta\right]\right\} \\ \to\ & 0 \end{aligned}$$

as $N \to \infty$.

Similarly,

$$\begin{aligned} b(1-p)^m &= c_3 N^{-1-\beta}\left\{2c_1^{-\beta-1} c_4\left[\log(c_1 N)\right]^{-\beta-1}(\log N)\right\}^m \\ &= 2^m c_1^{-1-m} c_3 N^{-1-\beta}\left[\log(c_1 N)\right]^{-1-m}(\log N)^m \\ &\to 0 \end{aligned}$$

as $N \to \infty$. Thus, the second term in the right side of the inequality (1)

$$\begin{aligned} \frac{kmN}{m-1}\log(1-bq^m) &\sim \frac{kmN}{m-1}bq^m \\ &< 2kNbq^m \qquad (\text{by } m-1 > m/2) \\ &< 2^m c_1^{\beta-m} c_3 \big[\log(c_1 N)\big]^{-1-m} (\log N)^{m-1} \\ &\to 0 \end{aligned}$$

as $N \to \infty$. The second term in the right side of the inequality (2)

$$\begin{aligned} \frac{m^2 N}{m-1}\log(1-bq^m) &\sim \frac{m^2 N}{m-1}bq^m \\ &\to 0 \end{aligned}$$

as $N \to \infty$.

Thus, (1) and (2) are satisfied for large k hence for all large N if c_1, c_3 are any fixed positive constants and c_2, c_4 are fixed positive constants satisfying $c_2 - c_4 < 0$. We then conclude $W(k,m;2) > N$ for sufficiently large k.

Take $c_1 = 1$. Since $k = \frac{1}{2}{c_1}^{\beta+1}N^{\beta}(\log N)^{-1}$, we have $\log k \sim \beta \log N$. Thus, by $\beta = \frac{1}{m}$,

$$\begin{aligned} N &= (2k\log N)^{\frac{1}{\beta}} \\ &\sim (2mk\log k)^m. \end{aligned}$$

This completes the proof. ∎

Remark 2.2 *Theorem 2.1 improves the main result in [5], that is,*

$$W(k,m;2) > k^{m-1-\frac{1}{\log\log k}}$$

for fixed $m \geq 3$ and sufficiently large k.

3 Lower Bounds for $\mathscr{W}(k,m;2)$

Let $\gamma_k(n) = \max\limits_{F\subseteq[1,n]} \Big\{|F| : F \text{ has no } k\text{-term arithmetic progression}\Big\}$.

Lemma 3.1 *(Rankin [9]) There exists a constant $c > 0$ such that*

$$\gamma_k(n) > n\exp\Big(-c(\log n)^{\frac{1}{\lfloor\log_2 k\rfloor+1}}\Big),$$

for all $n \geq 3$.

Lemma 3.2 *(Brown et al [5]) Let $k \geq 3$ and $t = \lfloor \log_2 k \rfloor$. There exists a constant $d > 0$ such that*

$$W(\underbrace{k, k, \cdots, k}_{s}; s) > s^{d(\log s)^t},$$

for all sufficiently large s.

Theorem 3.3 *Let $n \geq 1$ and $k \geq 3$ be fixed integers,*

$$\mathscr{W}(k, kn; 2) > (k-1)\Big(W(\underbrace{k, k, \cdots, k}_{\gamma}; \gamma) - 1\Big)n + n,$$

where, $\gamma = \gamma_k(n)$.

Proof Let $\mathscr{N} = W(k, k, \cdots, k; \gamma) - 1$. Then there is a γ-coloring of $[1, \mathscr{N}]$,

$$\chi : [1, \mathscr{N}] \to [1, \gamma]$$

in which there is no monochromatic k-term arithmetic progressions. By the definition of $\gamma = \gamma_k(n)$, there also exists a subset $F = \{f(1), f(2), \cdots, f(\gamma)\} \subseteq [1, n]$ in which there is no k-term arithmetic progressions.

Take $\mathscr{N}$ intervals $\{(k-1)sn - (k-1)n + 1, (k-1)sn - (k-1)n + 2, \cdots, (k-1)sn - (k-2)n\}$ of $[1, (k-1)\mathscr{N}n + n]$, denoted as $\mathscr{I}_s$, where $1 \leq s \leq \mathscr{N}$. Select $\ell_s = (k-1)sn - (k-1)n + f(\chi(s))$ from $\mathscr{I}_s$. We will show that the subset $L = \{\ell_1, \ell_2, \cdots, \ell_{\mathscr{N}}\}$ has no k-term arithmetic progressions.

Suppose that $\ell_{i_1}, \ell_{i_2}, \cdots, \ell_{i_k}$ form a k-term arithmetic progression in L, where $1 \leq i_1 < i_2 < \cdots < i_k \leq \mathscr{N}$. Thus, $(k-1)(i_{(j+1)} - i_j)n - n + 1 \leq \ell_{i_{(j+1)}} - \ell_{i_j} \leq (k-1)(i_{(j+1)} - i_j)n + n - 1$ and $\ell_{i_{(j+1)}} - \ell_{i_j}$ equals a nonzero constant, where $1 \leq j \leq k-1$. So we have $\{i_1, i_2, \cdots, i_k\}$ forms a k-term arithmetic progression. Since $\ell_{i_{(j+1)}} - \ell_{i_j} = (k-1)(i_{(j+1)} - i_j)n + f(\chi(i_{(j+1)})) - f(\chi(i_j))$, we have $\{f(\chi(i_1)), f(\chi(i_2)), \cdots, f(\chi(i_k))\}$ form a k term arithmetic progression or $f(\chi(i_1)) = f(\chi(i_2)) = \cdots = f(\chi(i_k))$. The former contradicts with F containing no k-term arithmetic progressions. The latter implies $\chi(i_1) = \chi(i_2) = \cdots = \chi(i_k)$ which contradicts with the definition of $\mathscr{N}$. Thus $L = \{\ell_1, \ell_2, \cdots, \ell_{\mathscr{N}}\}$ contains no k-term arithmetic progressions.

Let $[1, (k-1)\mathscr{N}n + n]$ be partitioned into L and $[1, (k-1)\mathscr{N}n + n] - L$. Since there is no k-term arithmetic progression in L and there is no consecutive kn integers in $[1, (k-1)\mathscr{N}n + n] - L$, we have

$$\mathscr{W}(k, kn; 2) > (k-1)\mathscr{N}n + n.$$

This completes the proof. ∎

Theorem 3.4 *For $k \geq 3$, $t = \lfloor \log_2 k \rfloor$ and sufficiently large m, there exists a constant $\mathscr{D} > 0$ such that*

$$\mathscr{W}(k,m;2) > \frac{m}{k}\exp\left(\mathscr{D}\left(\log\left(\frac{m}{k}\right)\right)^{t+1}\right).$$

Proof By Lemma 3.1 and 3.2, we have there exist constants $c, d > 0$ and $t = \lfloor \log_2 k \rfloor$, for sufficiently large $\gamma = \gamma_k(n)$ hence for sufficiently large n,

$$\begin{aligned} W(k,k,\cdots,k;\gamma) &> \gamma^{d(\log\gamma)^t} \\ &= \exp\left(d(\log\gamma)^{t+1}\right) \\ &> \exp\left(d\left(\log n - c(\log n)^{\frac{1}{t+1}}\right)^{t+1}\right) \\ &> \exp\left(d\,(d'\log n)^{t+1}\right) \\ &= n^{\mathscr{D}(\log n)^t} \qquad (\mathscr{D} = dd'^{t+1}), \end{aligned}$$

where the second last inequality comes from the fact that there must exist a constant $d' > 0$ such that $\log n - c(\log n)^{\frac{1}{t+1}} > d' \log n$.

By Theorem 3.3 and take $n = \frac{m}{k}$, for sufficiently m hence for sufficiently n, we have

$$\begin{aligned} \mathscr{W}(k,m;2) &> (k-1)\left(\left(\frac{m}{k}\right)^{\mathscr{D}(\log(\frac{m}{k}))^t} - 1\right)\cdot\frac{m}{k} + \frac{m}{k} \\ &> \frac{m}{k}\exp\left(\mathscr{D}\left(\log\left(\frac{m}{k}\right)\right)^{t+1}\right). \end{aligned}$$

∎

At last, we give a general lower bound for $\mathscr{W}(k,m;2)$ using Lovász Local Lemma [2].

Theorem 3.5 *For fixed $k, m \geq 3$, $\mathscr{W}(k,m;2) > \frac{(2^k-e)(k-1)}{2ek^2} + m(k-1)$.*

Proof Let $n(> k)$ be a fixed positive integer and $\mathscr{F}$ denote the set of all sequences $\beta = b_1 b_2 \cdots b_n$, where $b_i \in \{(i-1)(m-1)+1, (i-1)(m-1)+2\}$ for $1 \leq i \leq n$. That is to say, each β in $\mathscr{F}$ contains exactly one of the two elements in each of the intervals $[1,2]$, $[m, m+1]$, $[2m-1, 2m]$, $\cdots$, $[(n-1)(m-1)+1, (n-1)(m-1)+2]$. Obviously, the common differences of arithmetic progressions contained in every β of $\mathscr{F}$ must be greater than m.

For each k-term arithmetic progression S in $[1, n(m-1)+1]$, let A_S denote the event that there exists a $\beta \in \mathscr{F}$ such that $S \subseteq \beta$. Since $|\mathscr{F}| = 2^n$ and $|\beta \in \mathscr{F} : S \subseteq \beta| = 2^{n-k}$, we have $P(A_S) = \frac{2^{n-k}}{2^n} = \frac{1}{2^k}$ by classical probability.

The event A_S is dependent of all of the other events A_T in which T have at least a common interval $\{(i-1)(m-1)+1, (i-1)(m-1)+2\}$ with S for some $1 \leq i \leq n$. Since S meets k intervals in $\Big\{[1,2], [m, m+1], [2m-1, 2m], \cdots, [(n-1)(m-1)+1, (n-1)(m-1)+2]\Big\}$, T must contain at least one of these $2k$ elements, denoted this element as x. The number of k-term arithmetic progressions T in $[1, n(m-1)+1]$ that contain x is bounded by $k\big(\frac{n(m-1)+1}{k-1} - m\big)$, since there are k positions that x may occupy in T and the gap size in T is between $m+1$ and $\frac{n(m-1)+1}{k-1}$. Thus, A_S is dependent of at most $2k^2\big(\frac{n(m-1)+1}{k-1} - m\big)$ events A_T.

By Lovász Local Lemma, if $ep\Big(2k^2(\frac{n(m-1)+1}{k-1} - m) + 1\Big) < 1$, then $P(\cap\overline{A_S}) > 0$. Thus, there exists a $\beta = b_1 b_2 \cdots b_n \in \mathscr{F}$ that does not contain any k-term arithmetic progressions. Decompose $[1, n(m-1)+1]$ as $\{b_1, b_2, \cdots, b_n\}$ which does not contain k-term arithmetic progression and $[1, n(m-1)+1] - \{b_1, b_2, \cdots, b_n\}$ which does not contain m consecutive integers. So we have $\mathscr{W}(k, m; 2) > n(m-1)+1$.

Take $n \leq \frac{(2^k - e)(k-1)}{2ek^2(m-1)} + \frac{m(k-1)}{m-1}$, we have $ep\Big(2k^2(\frac{n(m-1)+1}{k-1} - m) + 1\Big) < 1$ and $\mathscr{W}(k, m; 2) > \frac{(2^k - e)(k-1)}{2ek^2} + m(k-1)$. ∎

References

[1] B. L. van der Waerden, Beweis einer baudetschen vermutung. *Nieuw Arch. Wiskunde*, **15** (1927), 212-216.

[2] R. L. Graham, B. L. Rothschild and J. H. Spencer, Ramsey Theory (Second ed.). *Wiley, New York*, 1990.

[3] E. Berlekamp, A construction for partitions which avoid long arithmetic progressions. *Canad. Math. Bull.*, **11**, (1968), 409-414.

[4] W. T. Gowers, A new proof of Szemerédi's theorem. *Geom. Funct. Anal.*, **11**(3) (2001), 465-588.

[5] T. Brown, M. Landman and A. Robertson, Bounds on some van der Waerden numbers. *J. Combin. Theory, Ser. A*, **15** (2008), 1304-1309.

[6] R. Graham, On the growth of a van der Waerden-like function. *Integers: Electron. J. Combin. Number Theory*, **6** (2006), *A*29.

[7] J. Spencer, Ten lectures on the probabilistic method (Second ed.). *SIAM,Philadelphia*, 1994.

[8] J. Spencer, Asympotic lower bounds for Ramsey functions. *Discrete Math.*, **20** (1977), 69-76.

[9] R. Rankin, Sets of integers containing not more than a given number of terms in arithmetic progression. *Proc. Roy. Soc. Edinburgh Sect.*, A65 (1960/61), 332-344.