SOME CONVOLUTION SUMS AND THE REPRESENTATION NUMBERS

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Abstract

We evaluate the convolution sums
$\sum_{l+30m=n}\sigma(l)\sigma(m)$, $\sum_{3l+10m=n}\sigma(l)\sigma(m)$,
$\sum_{2l+15m=n}\sigma(l)\sigma(m)$, $\sum_{5l+6m=n}\sigma(l)\sigma(m)$,

$\sum_{l+33m=n}\sigma(l)\sigma(m)$, $\sum_{3l+11m=n}\sigma(l)\sigma(m)$,
$\sum_{l+39m=n}\sigma(l)\sigma(m)$, and $\sum_{3l+13m=n}\sigma(l)\sigma(m)$,
for all neN using the theory of quasimodular forms and use these convolution sums to determine the number of representations of a positive integer n by the form $x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2$
$+a\left(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2\right)$,
$a=10,11,13$.

Quasimodular forms, divisor functions, convolution sums, representation number 11A25,11F11,11F25,11F20

1 Introduction

Let ℕ denote the set of natural numbers and for n,k∈ ℕ

$$\sigma_k(n) := \sum_{d|n} d^k,$$

where d runs through positive divisors of n. If n∉ ℕ, we set $\sigma_k(n) := 0$.We write $\sigma(n)$ for $\sigma_1(n)$. Let

$$W_N(n) := \sum_{m<n/N} \sigma(m)\sigma(n-Nm), n, m, N \in \mathbb{N}$$

and

$$W_{a,b}(n) := \sum_{\substack{al+bm=n \\ l,m\in \mathbb{N}}} \sigma(l)\sigma(m)$$

for positive integers a,b. The convolution sums $W_N(n)$(for $1 \leq N \leq 24$ with a few exceptions) and $W_{a,b}(n)$ for $(a,b) \in \{(2,3),(3,4),(3,5),(3,8),(2,9)\}$ have been evaluated by using either elementary methods or analytic methods or algebraic methods.(see[[8],[14],[26],[18],[19],[23],[16],[1],[2],[6],[3],[5],[11],[12],[20],[29],[30],[25]]).The number of representations of integers by certain quadratic forms (see [[16],[1],[6],[3],[4],[29],[30],[25]]) have been found by evaluation of these convolution sums. In this article following the method of Royer and using Magma for the calculations, we evaluate the convolution sums

$$W_{30}(n), W_{3,10}(n), W_{2,15}(n), W_{5,6}(n),$$
$$W_{33}(n), W_{3,11}(n), W_{39}(n), W_{3,13}(n)$$

by using the theory of quasimodular forms. Then, by also using

$$W_3, W_{10}, W_{11}, W_{13}$$

we will determine the formulas for the number of representations of integers by the positive definite quadratic forms

$$\begin{aligned} Q &= x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 \\ &\quad + a\left(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2\right), \end{aligned}$$

for $a = 10, 11, 13$.

2 Evaluation of $W_{30}(n), W_{3,10}(n), W_{2,15}(n), W_{5,6}(n), W_{33}(n), W_{3,11}(n), W_{39}(n), W_{3,13}(n)$

Let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, c \mid N \right\}$$

for $N \in \mathbb{N}$.

In this section, we evaluate the convolution sums

$$W_{30}(n), W_{3,10}(n), W_{2,15}(n), W_{5,6}(n),$$
$$W_{33}(n), W_{3,11}(n), W_{39}(n), W_{3,13}(n)$$

by using the lemma 1.17

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(N)] = M_4[\Gamma_0(N)] \oplus DM_2[\Gamma_0(N)] \oplus \mathbb{C}DE_2,$$

in [27], about the structure of quasimodular forms. Here

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(N)]$$

is the space of quasimodular forms of weight 4, depth$\leq$ 2 on $\Gamma_0(N)$ and

$$M_4[\Gamma_0(N)], M_2[\Gamma_0(N)]$$

are the spaces of modular forms of weight 4 and weight 2 on $\Gamma_0(N)$ respectively.The differential operator D is defined by D:=$\frac{1}{2\pi i}\frac{d}{dz}$ and

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, q = e^{2\pi iz}, \operatorname{Im} z > 0.$$

As an application, we use these convolution sums together with the convolution sums

$$\begin{aligned} W_3(n) &= \frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{12}n\sigma(n) \\ &\quad -\frac{1}{4}n\sigma\left(\frac{n}{3}\right) + \frac{1}{24}\sigma(n) + \frac{1}{24}\sigma\left(\frac{n}{3}\right), \end{aligned}$$

$$\begin{aligned} W_{10}(n) &= \frac{1}{312}\sigma_3(n) + \frac{1}{78}\sigma_3\left(\frac{n}{2}\right) + \frac{25}{312}\sigma_3\left(\frac{n}{5}\right) \\ &\quad +\frac{25}{78}\sigma_3\left(\frac{n}{10}\right) - \frac{1}{40}n\sigma(n) - \frac{1}{4}n\sigma(\frac{n}{10}) \\ &\quad +\frac{1}{24}\sigma(n) + \frac{1}{24}\sigma(\frac{n}{10}) - \frac{1}{120}\tau_{4,10}(n) \\ &\quad -\frac{3}{260}\tau_{4,5}(n) - \frac{3}{65}\tau_{4,5}\left(\frac{n}{2}\right), \end{aligned}$$

$$W_{11}(n) =$$

$$\begin{aligned} &\frac{5}{1464}\sigma_3(n) + \frac{605}{1464}\sigma_3\left(\frac{n}{11}\right) - \frac{1}{44}n\sigma(n) \\ &-\frac{1}{4}n\sigma\left(\frac{n}{11}\right) + \frac{1}{24}\sigma(n) + \frac{1}{24}\sigma\left(\frac{n}{11}\right) \\ &-\frac{2w+43}{4026}\tau_{4,11,1}(n) + \frac{2w-47}{4026}\tau_{4,11,2}(n), \end{aligned}$$

$$w = 1 + \sqrt{3},$$

$$W_{13}(n) =$$

$$\begin{aligned} &\frac{1}{408}\sigma_3(n) + \frac{169}{408}\sigma_3\left(\frac{n}{13}\right) - \frac{1}{52}n\sigma(n) \\ &-\frac{1}{4}n\sigma\left(\frac{n}{13}\right) + \frac{1}{24}\sigma(n) + \frac{1}{24}\sigma\left(\frac{n}{13}\right) \\ &+\frac{s-6}{442}\tau_{4,13,2}(n) - \frac{s+5}{442}\tau_{4,13,3}(n), \end{aligned}$$

$$s = \frac{1+\sqrt{17}}{2},$$

see [27], to obtain formulas for the number of representations of a positive integer n by the quadratic form Q given by

$$\begin{aligned} Q &= x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 \\ &\quad + a\left(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2\right), \end{aligned}$$

for $a = 10, 11, 13$.

2.1 Evaluation of $W_{30}(n)$, $W_{3,10}(n)$, $W_{2,15}(n)$, $W_{5,6}(n)$

The vector space $M_4[\Gamma_0(30)]$ has dimension 22 and is spanned by the 8 linearly independent Eisenstein forms

$$E_4, E_{4,2}, E_{4,3}, E_{4,5},$$
$$E_{4,6}, E_{4,10}, E_{4,15}, E_{4,30}$$

(see theorem 4.5.2 in [13]),

12 old cusp forms (see theorem 5.8.3 in [13])

$$\Delta_{4,5}(z) = \sum_{n=1}^{\infty} \tau_{4,5}(n) q^n$$

$$= (\Delta(z)\Delta(5z))^{1/6} = q - 4q^2 + \cdots + 115q^{45} + O(q^{46}),$$

$$\Delta_{4,5}(2z), \Delta_{4,5}(3z), \Delta_{4,5}(6z),$$

$$\Delta_{4,6}(z) = q - 2q^2 + \cdots + 54q^{45} + O(q^{46}),$$

$$\Delta_{4,6}(5z),$$

$$\Delta_{4,10}(z) = q + 2q^2 + \cdots + 185q^{45} + O(q^{46}),$$

$$\Delta_{4,10}(3z),$$

$$\Delta_{4,15,1} = q + q^2 + \cdots + 45q^{45} + O(q^{46}),$$

$$\Delta_{4,15,2} = q + 3q^2 + \cdots + q^{45} + O(q^{46}),$$

$$\Delta_{4,15,1}(2z), \Delta_{4,15,2}(2z),$$

and two newforms in $S_4[\Gamma_0(30)]$

$$\Delta_{4,30,1} = q + 2q^2 + \cdots + (-45)q^{45} + O(q^{46}),$$

$$\Delta_{4,30,2} = q - 2q^2 + \cdots + 45q^{45} + O(q^{46}).$$

The vector space

$$M_2[\Gamma_0(30)]$$

has dimension 10 (see theorem 4.6.2 in [13]) and is spanned by the 7 linearly independent Eisenstein series:

$$\Phi_{1,2}, \Phi_{1,3}, \Phi_{1,5}, \Phi_{1,6}, \Phi_{1,10}, \Phi_{1,15}, \Phi_{1,30},$$

1 newform (see theorem 5.8.3 in [13])

$$\Delta_{2,30} = q - q^2 + \cdots + (-45)\, q^{45} + O(q^{46})$$

and 2 oldforms

$$\Delta_{2,15} = q - q^2 + \cdots + q^{45} + O(q^{46}),$$

$\Delta_{2,15}\,(2z)$. By taking the suitable coefficients, we can get the following equality.

$$\begin{aligned}
E_2\,(z)\,E_2\,(30z) &= 1 - 24\sum_{n=1}^{+\infty}\left[\sigma(n) + \sigma\left(\frac{n}{30}\right)\right] e^{2\pi i n z} \\
&\quad +576\sum_{n=1}^{+\infty} W_{30}(n) e^{2\pi i n z}
\end{aligned}$$

$$\begin{aligned}
= \;& 1/1300 E_4\,(z) + 1/325 E_{4,2}\,(z) + 9/1300 E_{4,3}\,(z) \\
& +1/52 E_{4,5}\,(z) + 9/325 E_{4,6}\,(z) + 1/13 E_{4,10}\,(z) \\
& +9/52 E_{4,15}\,(z) + 9/13 E_{4,30}\,(z) - 24/5 \Delta_{4,30,1}\,(z) \\
& -12/7 \Delta_{4,30,2} - 108/35 \Delta_{4,15,1}\,(z) - 432/35 \Delta_{4,15,1}\,(2z) \\
& -16/7 \Delta_{4,15,2}\,(z) - 64/7 \Delta_{4,15,2}(2z) - 24/5 \Delta_{4,10}\,(z) \\
& -216/5 \Delta_{4,10}\,(3z) - 4/5 \Delta_{4,6}\,(z) - 20 \Delta_{4,6}\,(5z) \\
& -864/455 \Delta_{4,5}\,(z) - 3456/455 \Delta_{4,5}\,(2z) \\
& -7776/455 \Delta_{4,5}\,(3z) - 31104/455 \Delta_{4,5}\,(6z) \\
& +2/5 D E_2 + 29/5 D \Phi_{1,30}(z)
\end{aligned}$$

After simplification we get,

$$W_{30}(n) =$$

$$
\begin{aligned}
- \quad & 1/120 n\sigma(n) + 1/24\sigma(n) - 1/4 n\sigma\left(\frac{n}{30}\right) \\
& +1/24\sigma\left(\frac{n}{30}\right) + 1/3120\sigma_3(n) + 1/780\sigma_3(\frac{n}{2}) \\
& +3/1040\sigma_3(\frac{n}{3}) + 5/624\sigma_3(\frac{n}{5}) + 3/260\sigma_3(\frac{n}{6}) \\
& +5/156\sigma_3(\frac{n}{10}) + 15/208\sigma_3(\frac{n}{15}) + 15/52\sigma_3(\frac{n}{30}) \\
& -1/120\tau_{4,30,1}(n) - 1/336\tau_{4,30,2}(n) - 3/560\tau_{4,15,1}(n) \\
& -3/140\tau_{4,15,1}\left(\frac{n}{2}\right) - 1/252\tau_{4,15,2}(n) \\
& -1/63\tau_{4,15,2}\left(\frac{n}{2}\right) - 3/910\tau_{4,5}(n) - 6/455\tau_{4,5}(\frac{n}{2}) \\
& -27/910\tau_{4,5}(\frac{n}{3}) - 54/455\tau_{4,5}\left(\frac{n}{6}\right) + 1/720\tau_{4,6}(n) \\
& -5/144\tau_{4,6}(\frac{n}{5}) - 1/120\tau_{4,10}(n) - 3/40\tau_{4,10}(\frac{n}{3}).
\end{aligned}
$$

Similarly, from the equality

$$E_2(3z)\, E_2(10z) =$$

$$
\begin{aligned}
& 1 - 24\sum_{n=1}^{+\infty}\left[\sigma(\frac{n}{10}) + \sigma\left(\frac{n}{3}\right)\right] e^{2\pi i n z} \\
& +576\sum_{n=1}^{+\infty} W_{3,10}(n) e^{2\pi i n z}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 1/1300 E_4(z) + 1/325 E_{4,2}(z) + 9/1300 E_{4,3}(z) \\
& +1/52 E_{4,5}(z) + 9/325 E_{4,6}(z) + 1/13 E_{4,10}(z) \\
& +9/52 E_{4,15}(z) + 9/13 E_{4,30}(z) + 24/5\Delta_{4,30,1}(z) \\
& +12/7\Delta_{4,30,2}(z) + 108/35\Delta_{4,15,1}(z) \\
& +432/35\Delta_{4,15,1}(2z) - 16/7\Delta_{4,15,2}(z) \\
& -64/7\Delta_{4,15,2}(2z) - 24/5\Delta_{4,10}(z) \\
& -216/5\Delta_{4,10}(3z) - 4/5\Delta_{4,6}(z) \\
& -20\Delta_{4,6}(5z) - 864/455\Delta_{4,5}(z) \\
& -3456/455\Delta_{4,5}(2z) - 7776/455\Delta_{4,5}(3z) \\
& -31104/455\Delta_{4,5}(6z) + 2/5 D E_2 \\
& +2/5 D\Phi_{1,3}(z) + 9/5 D\Phi_{1,10}(z).
\end{aligned}
$$

we get

$$W_{3,10}(n) =$$

$$
\begin{aligned}
&-1/40n\sigma\left(\frac{n}{3}\right)+1/24\sigma\left(\frac{n}{3}\right)-1/12n\sigma\left(\frac{n}{10}\right)\\
&+1/24\sigma\left(\frac{n}{10}\right)+1/3120\sigma_3\left(n\right)+1/780\sigma_3\left(\frac{n}{2}\right)\\
&+3/1040\sigma_3\left(\frac{n}{3}\right)+5/624\sigma_3\left(\frac{n}{5}\right)\\
&+3/260\sigma_3\left(\frac{n}{6}\right)+5/156\sigma_3\left(\frac{n}{10}\right)\\
&+15/208\sigma_3\left(\frac{n}{15}\right)+15/52\sigma_3\left(\frac{n}{30}\right)\\
&+1/120\tau_{4,30,1}\left(n\right)+1/336\tau_{4,30,2}\left(n\right)\\
&+3/560*\tau_{4,15,1}\left(n\right)+3/140\tau_{4,15,1}\left(\frac{n}{2}\right)\\
&-1/252\tau_{4,15,2}\left(n\right)-1/63\tau_{4,15,2}\left(\frac{n}{2}\right)\\
&-3/910\tau_{4,5}\left(n\right)-6/455\tau_{4,5}\left(\frac{n}{2}\right)\\
&-27/910\tau_{4,5}\left(\frac{n}{3}\right)-54/455\tau_{4,5}\left(\frac{n}{6}\right)\\
&-1/720\tau_{4,6}\left(n\right)-5/144\tau_{4,6}\left(\frac{n}{5}\right)\\
&-1/120\tau_{4,10}\left(n\right)-3/40\tau_{4,10}\left(\frac{n}{3}\right).
\end{aligned}
$$

By the following equality

$$E_2\left(2z\right)E_2\left(15z\right) =$$

$$
\begin{aligned}
&1-24\sum_{n=1}^{+\infty}\left[\sigma(\frac{n}{2})+\sigma\left(\frac{n}{15}\right)\right]e^{2\pi inz}\\
&+576\sum_{n=1}^{+\infty}W_{2,15}(n)e^{2\pi inz}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 1/1300E_4(z) + 1/325E_{4,2}(z) + 9/1300E_{4,3}(z) \\
& +1/52E_{4,5}(z) + 9/325E_{4,6}(z) + 1/13E_{4,10}(z) \\
& +9/52E_{4,15}(z) + 9/13E_{4,30}(z) + 24/5\Delta_{4,30,1}(z) \\
& -12/7\Delta_{4,30,2}(z) - 108/35\Delta_{4,15,1} \\
& -432/35\Delta_{4,15,1}(2z) - 16/7\Delta_{4,15,2}(z) \\
& -64/7\Delta_{4,15,2}(2z) + 24/5\Delta_{4,10}(z) \\
& +216/5\Delta_{4,10}(3z) - 4/5\Delta_{4,6}(z) - 20\Delta_{4,6}(5z) \\
& -864/455\Delta_{4,5}(z) - 3456/455\Delta_{4,5}(2z) \\
& -7776/455\Delta_{4,5}(3z) - 31104/455\Delta_{4,5}(6z) \\
& +2/5DE_2 + 1/5D\Phi_{1,2}(z) + 14/5D\Phi_{1,15}(z).
\end{aligned}
$$

we get

$$W_{2,15}(n) =$$

$$
\begin{aligned}
& -1/60n\sigma\left(\frac{n}{2}\right) + 1/24\sigma\left(\frac{n}{2}\right) \\
& -1/8n\sigma\left(\frac{n}{15}\right) + 1/24\sigma\left(\frac{n}{15}\right) \\
& +1/3120\sigma_3(n) + 1/780\sigma_3\left(\frac{n}{2}\right) \\
& +3/1040\sigma_3\left(\frac{n}{3}\right) + 5/624\sigma_3\left(\frac{n}{5}\right) \\
& +3/260\sigma_3\left(\frac{n}{6}\right) + 5/156\sigma_3\left(\frac{n}{10}\right) \\
& +15/208\sigma_3\left(\frac{n}{15}\right) + 15/52\sigma_3\left(\frac{n}{30}\right) \\
& +1/120\tau_{4,30,1}(n) - 1/336\tau_{4,30,2}(n) \\
& -3/560\tau_{4,15,1}(n) - 3/140\tau_{4,15,1}\left(\frac{n}{2}\right) \\
& -1/252\tau_{4,15,2}(n) - 1/63\tau_{4,15,2}\left(\frac{n}{2}\right) \\
& -3/910\tau_{4,5}(n) - 6/455\tau_{4,5}\left(\frac{n}{2}\right) \\
& -27/910\tau_{4,5}\left(\frac{n}{3}\right) - 54/455\tau_{4,5}\left(\frac{n}{6}\right) \\
& -1/720\tau_{4,6}(n) - 5/144\tau_{4,6}\left(\frac{n}{5}\right) \\
& +1/120\tau_{4,10}(n) + 3/40\tau_{4,10}\left(\frac{n}{3}\right).
\end{aligned}
$$

Similarly, from the equality

$$E_2(5z)\,E_2(6z) =$$

$$1 - 24\sum_{n=1}^{+\infty}\left[\sigma(\frac{n}{5}) + \sigma\left(\frac{n}{6}\right)\right]e^{2\pi inz}$$

$$+576\sum_{n=1}^{+\infty} W_{5\_6}(n)e^{2\pi inz}$$

$$\begin{aligned}
= \quad & 1/1300E_4\,(z) + 1/325E_{4,2}\,(z) + 9/1300E_{4,3}\,(z) \\
& +1/52E_{4,5}\,(z) + 9/325E_{4,6}\,(z) + 1/13E_{4,10}\,(z) \\
& +9/52E_{4,15}\,(z) + 9/13E_{4,30}\,(z) - 24/5\Delta_{4,30,1}\,(z) \\
& +12/7\Delta_{4,30,2}\,(z) + 108/35\Delta_{4,15,1}\,(z) \\
& +432/35\Delta_{4,15,1}\,(2z) - 16/7\Delta_{4,15,2}\,(z) \\
& -64/7\Delta_{4,15,2}\,(2z) + 24/5\Delta_{4,10}\,(z) + 216/5\Delta_{4,10}\,(3z) \\
& -4/5\Delta_{4,6}\,(z) - 20\Delta_{4,6}\,(5z) - 864/455\Delta_{4,5}\,(z) \\
& -3456/455\Delta_{4,5}\,(2z) - 7776/455\Delta_{4,5}\,(3z) \\
& -31104/455\Delta_{4,5}\,(6z) + 2/5DE_2 \\
& +4/5D\Phi_{1,5}(z) + D\Phi_{1,6}(z),
\end{aligned}$$

we get

$$W_{5,6}(n) =$$

$$\begin{aligned}
& -1/24n\sigma\left(\frac{n}{5}\right) + 1/24\sigma\left(\frac{n}{5}\right) - 1/20n\sigma\left(\frac{n}{6}\right) \\
& +1/24\sigma\left(\frac{n}{6}\right) + 1/3120\sigma_3\,(n) + 1/780\sigma_3\left(\frac{n}{2}\right) \\
& +3/1040\sigma_3\left(\frac{n}{3}\right) + 5/624\sigma_3\left(\frac{n}{5}\right) \\
& +3/260\sigma_3\left(\frac{n}{6}\right) + 5/156\sigma_3\left(\frac{n}{10}\right) \\
& +15/208\sigma_3\left(\frac{n}{15}\right) + 15/52\sigma_3\left(\frac{n}{30}\right) \\
& -1/120\tau_{4,30,1}\,(n) + 1/336\tau_{4,30,2}\,(n) \\
& +3/560\tau_{4,15,1}\,(n) + 3/140\tau_{4,15,1}\left(\frac{n}{2}\right) \\
& -1/252\tau_{4,15,2}\,(n) - 1/63\tau_{4,15,2}\left(\frac{n}{2}\right) \\
& -3/910\tau_{4,5}\,(n) - 6/455\tau_{4,5}\left(\frac{n}{2}\right) \\
& -27/910\tau_{4,5}\left(\frac{n}{3}\right) - 54/455\tau_{4,5}\left(\frac{n}{6}\right) \\
& -1/720\tau_{4,6}\,(n) - 5/144\tau_{4,6}\left(\frac{n}{5}\right) \\
& +1/120\tau_{4,10}\,(n) + 3/40\tau_{4,10}\left(\frac{n}{3}\right).
\end{aligned}$$

In this example, we did not leave the field of rational numbers, but in the next examples, we will work in number fields of degree 8 and 48 respectively.

2.2 Evaluation of $W_{33}(n)$, $W_{3,11}(n)$

Let $\{B_k\}_{k\in\{0\}\cup\mathbb{N}}$ be the sequence of rational numbers determined by

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}$$

and

$$E_{k,m}(z):=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\,q^{mn},$$

the Eisenstein series of weight

$$k=4,6,8,\ldots,m\in\mathbb{N}$$

(see [13] for details). Since the positive divisors of 33 are 1,3,11,33, the Eisenstein

subspace of $M_4[\Gamma_0(33)]$ is spanned by the 4 linearly independent Eisenstein forms

$$E_4,E_{4,3},E_{4,11},E_{4,33}$$

(see theorem4.5.2 in [13]). Using the dimension formula (see for example [13], [24]) it follows that the vector space $M_4[\Gamma_0(33)]$ has dimension 14.We can easily see by Magma that the cusp forms in $M_4[\Gamma_0(33)]$

is spanned by 4 old cusp forms (see theorem 5.8.3 in [13])

$$\Delta_{4,11,2}(z)=q+wq^2+\cdots+(-44w+55)\,q^{33}+O(q^{34}),$$

$$\Delta_{4,11,2}(3z)\,,\Delta_{4,11,1}(z)$$

(the Galois conjugate of $\Delta_{4,11,2}(z)$ by the polynomial

$$x^2-2x-2,w=1+\sqrt{3}),$$

$$\Delta_{4,11,1}(3z)$$

and 6 newforms

$$\Delta_{4,33,1}(z)\,,\Delta_{4,33,2}(z)\,,\Delta_{4,33,3}(z)\,,$$
$$\Delta_{4,33,4}(z)\,,\Delta_{4,33,5}(z)\,,\Delta_{4,33,6}(z)\,,$$

where

$$\Delta_{4,33,1}=$$

$$\sum_{n=1}^{\infty} \tau_{4,33,1} q^n = q - q^2 + \cdots + (-33)\, q^{33} + O(q^{34}),$$

$$\Delta_{4,33,2}(z) =$$

$$q - 5q^2 + \cdots + (-33)\, q^{33} + O(q^{34}),$$

$$\Delta_{4,33,3} =$$

$$q + tq^2 + \cdots + 33q^{33} + O(q^{34}), t = \frac{1+\sqrt{33}}{2},$$

$$\Delta_{4,33,4}$$

(the Galois conjugate of $\Delta_{4,33,3}$ by the polynomial x^2-x-8),

$$\Delta_{4,33,5} =$$

$$q + uq^2 + \cdots + 33q^{33} + O(q^{34}), u = \frac{1+\sqrt{97}}{2},$$

$$\Delta_{4,33,6}$$

(the Galois conjugate of $\Delta_{4,33,5}$ by the polynomial $x^2 - x - 24$).

The vector space $M_2[\Gamma_0(33)]$ has dimension 6 (see theorem 4.6.2 in [13]) and is spanned by one newform (see theorem 5.8.3 in [13]

$$\Delta_{2,33} = \sum_{n=1}^{\infty} \tau_{2,33} q^n = q + q^2 + \cdots + (-1)\, q^{33} + O(q^{34}),$$

three Eisenstein forms

$$\Phi_{1,3}, \Phi_{1,11}, \Phi_{1,33},$$

where

$$\Phi_{a,b} = \frac{1}{b-a}\left[bE_2(bz) - aE_2(az)\right], a|b$$

and two oldforms

$$\Delta_{2,11}(z) =$$

$$q - 2q^2 + \cdots + (-1)\, q^{33} + O(q^{34}),$$

$$\Delta_{2,11}(3z).$$

Consequently, we get

$$E_2(z)\, E_2(33z)$$

$$= \quad 1-24\sum_{n=1}^{+\infty}\left[\sigma(n)+\sigma\left(\frac{n}{33}\right)\right]e^{2\pi inz}$$
$$+576\sum_{n=1}^{+\infty}W_{33}(n)q^n.$$

So

$$1+576\sum_{n=1}^{+\infty}W_{33}(n)q^n$$
$$= \quad 24\sum_{n=1}^{+\infty}\left[\sigma(n)+\sigma\left(\frac{n}{33}\right)\right]q^n$$

$$\begin{aligned}
= \quad & 1/1220E_4\left(z\right)+9/1220E_{4,3}\left(z\right) \\
& +121/1220E_{4,11}\left(z\right)+1089/1220E_{4,33}\left(z\right) \\
& +\left(-12/11t-48/11\right)\Delta_{4,33,3}(z) \\
& +(12/11t-60/11)\Delta_{4,33,4}\left(z\right)+(-1248/7081u \\
& -8688/7081)\Delta_{4,33,5}\left(z\right)+(1248/7081u \\
& -9936/7081)\Delta_{4,33,6}\left(z\right)+(1/48983(-60672w \\
& -120192))\Delta_{4,11,1}\left(z\right)+(1/48983(-546048w \\
& -1081728))\Delta_{4,11,1}\left(3z\right)+1/48983(60672w \\
& -241536)\Delta_{4,11,2}\left(z\right)+1/48983(546048w \\
& -2173824)\Delta_{4,11,2}\left(3z\right)+64/11D\Phi_{1,33}\left(z\right) \\
& +4/11DE_2.
\end{aligned}$$

So for n= $1,2,3,\ldots.$we have

$$W_{33}(n)=$$

$$
\begin{aligned}
&-1/132n\sigma(n) + 1/24\sigma(n) - 1/4n\sigma\left(\frac{n}{33}\right)\\
&+1/24\sigma\left(\frac{n}{33}\right) + 1/2928\sigma_3(n)\\
&+3/976\sigma_3\left(\frac{n}{3}\right) + 121/2928\sigma_3\left(\frac{n}{11}\right)\\
&+363/976\sigma_3\left(\frac{n}{33}\right) - 1/528t\tau_{4,33,3}(n)\\
&-1/132\tau_{4,33,3}(n) + 1/528t\tau_{4,33,4}(n)\\
&-5/528\tau_{4,33,4}(n) - 13/42486u\tau_{4,33,5}(n)\\
&-181/84972\tau_{4,33,5}(n) + 13/42486u\tau_{4,33,6}(n)\\
&-69/28324\tau_{4,33,6}(n)\\
&-316/146949w\tau_{4,11,1}(n) - 626/146949\tau_{4,11,1}(n)\\
&-948/48983w\tau_{4,11,1}\left(\frac{n}{3}\right) - 1878/48983\tau_{4,11,1}\left(\frac{n}{3}\right)\\
&+316/146949w\tau_{4,11,2}(n) - 1258/146949\tau_{4,11,2}(n)\\
&+948/48983w\tau_{4,11,2}\left(\frac{n}{3}\right) - 3774/48983\tau_{4,11,2}\left(\frac{n}{3}\right).
\end{aligned}
$$

Similarly, from the equality

$$
\begin{aligned}
&E_2(3z)\,E_2(11z) =\\
&1 + \sum_{n=1}^{+\infty}(-24\left(\sigma(\frac{n}{11}) + \sigma\left(\frac{n}{3}\right)\right)\\
&+576W_{3,11}(n))q^n
\end{aligned}
$$

$$
\begin{aligned}
= \;& 1/1220E_4(z) + 9/1220E_{4,3}(z) + 121/1220E_{4,11}(z)\\
&+1089/1220E_{4,33}(z) + (12/11t + 48/11)\Delta_{4,33,3}(z)\\
&+(-12/11 * t + 60/11)\Delta_{4,33,4}(z) + (-1248/7081u\\
&-8688/7081)\Delta_{4,33,5}(z) + (1248/7081u\\
&-9936/7081)\Delta_{4,33,6}(z)\\
&+(1/48983(-60672w - 120192))\,\Delta_{4,11,1}(z)\\
&+(1/48983(-546048w - 1081728))\Delta_{4,11,1}(3z)\\
&+(1/48983(60672w - 241536))\,\Delta_{4,11,2}(z)\\
&+(1/48983(546048w - 2173824))\Delta_{4,11,2}(3z)\\
&+4/11D\Phi_{1,3}(z) + 20/11D\Phi_{1,11}(z) + 4/11DE_2,
\end{aligned}
$$

we get

$$W_{3,11}(n) =$$

$$-1/44n\sigma\left(\frac{n}{3}\right)+1/24\sigma\left(\frac{n}{3}\right)-1/12n\sigma\left(\frac{n}{11}\right)$$
$$+1/24\sigma\left(\frac{n}{11}\right)+1/2928\sigma_3(n)+3/976\sigma_3\left(\frac{n}{3}\right)$$
$$+121/2928\sigma_3\left(\frac{n}{11}\right)+363/976\sigma_3\left(\frac{n}{33}\right)$$
$$+1/528t\tau_{4,33,3}(n)+1/132\tau_{4,33,3}(n)$$
$$-1/528t\tau_{4,33,4}(n)+5/528\tau_{4,33,4}(n)$$
$$-13/42486u\tau_{4,33,5}(n)-181/84972\tau_{4,33,5}(n)$$
$$+13/42486u\tau_{4,33,6}(n)-69/28324\tau_{4,33,6}(n)$$
$$-316/146949w\tau_{4,11,1}(n)-626/146949\tau_{4,11,1}(n)$$
$$-948/48983w\tau_{4,11,1}\left(\frac{n}{3}\right)$$
$$-1878/48983\tau_{4,11,1}\left(\frac{n}{3}\right)$$
$$+316/146949w\tau_{4,11,2}(n)$$
$$-1258/146949\tau_{4,11,2}(n)$$
$$+948/48983w\tau_{4,11,2}\left(\frac{n}{3}\right)$$
$$-3774/48983\tau_{4,11,2}\left(\frac{n}{3}\right).$$

2.3 Evaluation of $W_{39}(n)$ $W_{3,13}(n)$

The vector space $M_4[\Gamma_0(39)]$ has dimension 16 and is spanned by the 4 linearly independent Eisenstein forms

$$E_4, E_{4,3}, E_{4,13}, E_{4,39}$$

(see theorem 4.5.2 in [13]) , 6 oldforms (see theorem 5.8.3 in [13])

$$\Delta_{4,13,1}=q-5q^2+\cdots+(-91)q^{39}+O(q^{40})$$
$$\Delta_{4,13,1}(3z),$$

$$\Delta_{4,13,3}=q+sq^2+\cdots+(-39s-13)q^{39}+O(q^{40}),$$
$$s=\frac{1+\sqrt{17}}{2},$$

$$\Delta_{4,13,2}(3z), \Delta_{4,13,2}$$

(the Galois conjugate of $\Delta_{4,13,3}$ by the polynomial x^2-x-4),

$$\Delta_{4,13,3}(3z)$$

and 6 newforms

$$\Delta_{4,39,1} = q + \cdots + (-39)q^{39} + O(q^{40}),$$

$$\begin{aligned}\Delta_{4,39,2} &= q + rq^2 + \cdots + 39q^{39} + O(q^{40}),\\ &r = 1 + \sqrt{14},\end{aligned}$$

$$\Delta_{4,39,3}$$

(the Galois conjugate of $\Delta_{4,39,2}$ by the polynomial x^2-2x-13),

$$\Delta_{4,39,4} = q + uq^2 + \cdots + 39q^{39} + O(q^{40}),$$
$$u^3 - 2u^2 - 15u + 24 = 0,$$

$$\Delta_{4,39,5}$$

(the Galois conjugate of $\Delta_{4,39,4}$),

$$\Delta_{4,39,6}$$

(the other Galois conjugate of $\Delta_{4,39,4}$),

The vector space $M_2[\Gamma_0(39)]$ has dimension 6 and is spanned by the Eisenstein series (see theorem 4.6.2 in [13])

$$\Phi_{1,3}, \Phi_{1,13}, \Phi_{1,39},$$

and 3 newforms (see theorem 5.8.3 in [13])

$$\Delta_{2,39,1} = q + q^2 + \cdots + (-1)\, q^{39} + O(q^{40}),$$

$$\Delta_{2,39,2} = q + tq^2 + \cdots + (-1)\, q^{39} + O(q^{16}),$$
$$t = -1 + \sqrt{2},$$

$$\Delta_{2,39,3}$$

(the Galois conjugate of $\Delta_{4,39,2}$ by the polynomial x^2+2x-1). Similarly, from the equality

$$E_2\,(z)\,E_2\,(39z) =$$

$$1 - 24\sum_{n=1}^{+\infty}\left[\sigma(\frac{n}{39}) + \sigma\,(n)\right]e^{2\pi i n z}$$
$$+576\sum_{n=1}^{+\infty} W_{39}(n)e^{2\pi i n z} = 1/1700E_4\,(z)$$

$$
\begin{aligned}
&+9/1700E_{4,3}(z)+169/1700E_{4,13}(z)\\
&+1521/1700E_{4,39}+76/13D\Phi_{1,39}(z)\\
&+4/13DE_2(z)+(-8928/9503s\\
&-22176/9503)\Delta_{4,13,2}(z)+(-80352/9503s\\
&-199584/9503)\Delta_{4,13,2}(3z)+(8928/9503s\\
&-31104/9503)\Delta_{4,13,3}(z)+(80352/9503s\\
&-279936/9503)\Delta_{4,13,3}(3z)\\
&(-576/1505r-2952/1505)\Delta_{4,39,2}(z)\\
&+(576/1505r-4104/1505)\Delta_{4,39,3}(z)\\
&+(45/1703u^2-951/1703u\\
&-5640/1703)\Delta_{4,39,4}(z)\\
&+(-45/1703u-861/1703)p-45/1703u^2\\
&+90/1703u-4965/1703)\Delta_{4,39,5}(z)\\
&+((45/1703u+861/1703)p\\
&+861/1703u-6687/1703)\Delta_{4,39,6}(z)
\end{aligned}
$$

we get

$$W_{39}(n)=$$

$$
\begin{aligned}
&-1/156n\sigma(n)+1/24\sigma(n)-1/4n\sigma\left(\frac{n}{39}\right)\\
&+1/24\sigma\left(\frac{n}{39}\right)+1/4080\sigma_3(n)\\
&+3/1360\sigma_3\left(\frac{n}{3}\right)+169/4080\sigma_3\left(\frac{n}{13}\right)\\
&+507/1360\sigma_3\left(\frac{n}{39}\right)+(-1/1505r\\
&-41/12040)\tau_{4,39,2}(n)+(1/1505r\\
&-57/12040)\tau_{4,39,3}(n)+(5/108992u^2\\
&-317/326976u-235/40872)\tau_{4,39,4}(n)\\
&+((-5/108992u-287/326976)p\\
&+(-5/108992u^2+5/54496u\\
&-1655/326976))\tau_{4,39,5}(n)+((5/108992u\\
&++287/326976)p+(287/326976u-\\
&743/108992))\tau_{4,39,6}(n)+(-31/19006s\\
&-77/19006)\tau_{4,13,2}(n)+(-279/19006s\\
&-693/19006)\tau_{4,13,2}\left(\frac{n}{3}\right)+(31/19006s\\
&-54/9503)\tau_{4,13,3}(n)+(279/19006s\\
&-486/9503)\tau_{4,13,3}\left(\frac{n}{3}\right).
\end{aligned}
$$

Similarly, from the equality

$$E_2\left(3z\right)E_2\left(13z\right)=$$

$$1-24\sum_{n=1}^{+\infty}\left[\sigma(\frac{n}{13})+\sigma\left(\frac{n}{3}\right)\right]e^{2\pi inz}$$
$$+576\sum_{n=1}^{+\infty}W_{3,13}(n)e^{2\pi inz}=1/1700E_4\left(z\right)$$

$$\begin{aligned}
&+9/1700E_{4,3}\left(z\right)+169/1700E_{4,13}(z)\\
&+1521/1700E_{4,39}+4/13D\Phi_{1,3}\left(z\right)\\
&+24/13D\Phi_{1,13}\left(z\right)+4/13DE_2\left(z\right)\\
&+(-8928/9503s-22176/9503)\Delta_{4,13,2}\left(z\right)\\
&+(-80352/9503s\\
&-199584/9503)\Delta_{4,13,2}\left(3z\right)+(8928/9503s\\
&-31104/9503)\Delta_{4,13,3}\left(z\right)+(80352/9503s\\
&-279936/9503)\Delta_{4,13,3}\left(3z\right)\\
&(-576/1505r-2952/1505)\,\Delta_{4,39,2}\left(z\right)\\
&+(576/1505r-4104/1505)\,\Delta_{4,39,3}\left(z\right)\\
&+(45/1703u^2-951/1703u\\
&-5640/1703)\Delta_{4,39,4}\left(z\right)\\
&+(-45/1703u-861/1703)p\\
&-45/1703u^2+90/1703u\\
&-4965/1703)\Delta_{4,39,5}\left(z\right)+((45/1703u\\
&+861/1703)p+861/1703u\\
&-6687/1703)\Delta_{4,39,6}\left(z\right)
\end{aligned}$$

we get

$$W_{3\_13}\left(n\right)=$$

$$
\begin{aligned}
&-1/52n\sigma\left(\frac{n}{3}\right)+1/24\sigma\left(\frac{n}{3}\right)-1/12n\sigma\left(\frac{n}{13}\right)\\
&+1/24\sigma\left(\frac{n}{13}\right)+1/4080\sigma_3(n)+3/1360\sigma_3\left(\frac{n}{3}\right)\\
&+169/4080\sigma_3\left(\frac{n}{13}\right)+507/1360\sigma_3\left(\frac{n}{39}\right)\\
&+(-1/1505r-41/12040)\tau_{4,39,2}(n)+(1/1505r\\
&-57/12040)\tau_{4,39,3}(n)+(-5/108992u^2\\
&+317/326976u+235/40872)\tau_{4,39,4}(n)\\
&+((5/108992u+287/326976)p\\
&+(5/108992u^2-5/54496u\\
&+1655/326976))\tau_{4,39,5}(n)+((-5/108992u\\
&-287/326976)p+(-287/326976u\\
&+743/108992))\tau_{4,39,6}(n)\\
&+(-31/19006s-77/19006)\tau_{4,13,2}(n)\\
&+(-279/19006s\\
&-693/19006)\tau_{4,13,2}\left(\frac{n}{3}\right)+(31/19006s\\
&-54/9503)\tau_{4,13,3}(n)\\
&+(279/19006s-486/9503)\tau_{4,13,3}\left(\frac{n}{3}\right).
\end{aligned}
$$

The formulas are not complicated, even if we are living in a number field of degree 48.

3 Application to the number of representations

Theorem 1 *The number $N_i(n)$ of representations of a positive integer n by the quadratic forms*

$$
\begin{aligned}
Q_i &= x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2\\
&\quad +a_i\left(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2\right),
\end{aligned}
$$

$$a_1=10, a_2=11, a_3=13,$$

are equal to

$$N_1(n)=$$

$$12/65\sigma_3(n) + 48/65\sigma_3\left(\frac{n}{2}\right)$$
$$+108/65\sigma_3\left(\frac{n}{3}\right) + 60/13\sigma_3\left(\frac{n}{5}\right)$$
$$+432/65\sigma_3\left(\frac{n}{6}\right) + 240/13\sigma_3\left(\frac{n}{10}\right)$$
$$+540/13\sigma_3\left(\frac{n}{15}\right) + 2160/13\sigma_3\left(\frac{n}{30}\right)$$
$$+24/7\tau_{4,15,2}(n) + 96/7\tau_{4,15,2}\left(\frac{n}{2}\right)$$
$$+108/91\tau_{4,5}(n) + 432/91\tau_{4,5}(\frac{n}{2})$$
$$+972/91\tau_{4,5}(\frac{n}{3}) + 3888/91\tau_{4,5}(\frac{n}{6})$$
$$+6/5\tau_{4,6}(n) + 30\tau_{4,6}(\frac{n}{5}) + 6\tau_{4,10}(n)$$
$$+54\tau_{4,10}(\frac{n}{3}),$$

$$N_2(n) =$$

$$12/61\sigma_3(n) + 108/61\sigma_3\left(\frac{n}{3}\right)$$
$$+1452/61\sigma_3\left(\frac{n}{11}\right) + 13068/61\sigma_3\left(\frac{n}{33}\right)$$
$$+1872/7081u\tau_{4,33,5}(n) + 13032/7081\tau_{4,33,5}(n)$$
$$-1872/7081u\tau_{4,33,6}(n) + 14904/7081\tau_{4,33,6}(n)$$
$$+8592/4453w\tau_{4,11,1}(n) + 8904/4453\tau_{4,11,1}(n)$$
$$+77328/4453w\tau_{4,11,1}(\frac{n}{3})$$
$$+80136/4453\tau_{4,11,1}(\frac{n}{3})$$
$$-8592/4453w\tau_{4,11,2}(n)$$
$$+26088/4453\tau_{4,11,2}(n)$$
$$+77328/4453w\tau_{4,11,2}(\frac{n}{3})$$
$$+234792/4453\tau_{4,11,2}(\frac{n}{3}),$$

$$N_3(n) =$$

$$12/85\sigma_3(n) + 108/85\sigma_3\left(\frac{n}{3}\right)$$
$$+2028/85\sigma_3\left(\frac{n}{13}\right) + 18252/85\sigma_3\left(\frac{n}{39}\right)$$
$$+(864/1505r + 4428/1505)\tau_{4,39,2}(n)$$
$$+(-864/1505r + 6156/1505)\tau_{4,39,3}(n)$$
$$+(792/731s + 1368/731)\tau_{4,13,2}(n)$$
$$+(7128/731s + 12312/731)\tau_{4,13,2}\left(\frac{n}{3}\right)$$
$$+(-792/731s + 2160/731)\tau_{4,13,3}(n)$$
$$+(-7128/731s + 19440/731)\tau_{4,13,3}\left(\frac{n}{3}\right),$$

respectively.

Proof. Let

$$r(l) =$$
$$\#\left\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 = l\right\}$$

for l$\in \{0\} \cup \mathbb{N}$. Since

$$M_2(\Gamma_0(3))$$

is generated by the Eisenstein series (see theorem 4.6.2 in [13])

$$\begin{aligned}\Phi_{1,3} &= \frac{1}{2}[3E_2(3z) - E_2(z)] \\ &= 1 + \sum_{n=1}^{\infty}\left(12\sigma(n) - 36\sigma\left(\frac{n}{3}\right)\right)q^n.\end{aligned}$$

So we have

$$r(l) = 12\sigma(l) - 36\sigma\left(\frac{l}{3}\right).$$

Now for

$$\begin{aligned}Q_1 &= x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 \\ &\quad +10\left(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2\right),\end{aligned}$$

$$N_1(n) =$$
$$r(0)\,r\left(\frac{n}{11}\right) + r(n)\,r(0) + \sum_{\substack{l,m\in\mathbb{N}\\ l+11m=n}} r(l)\,r(m)$$

$$
\begin{aligned}
&= 12\sigma\left(\frac{n}{11}\right) - 36\sigma\left(\frac{n}{33}\right) + 12\sigma(n) \\
&\quad -36\sigma\left(\frac{n}{3}\right) + \sum_{\substack{l,m\in\mathbf{N}\\ l+11m=n}} \left(12\sigma(l) - 36\sigma\left(\frac{l}{3}\right)\right) \\
&\quad \cdot(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)) \\
&= 12\sigma\left(\frac{n}{11}\right) - 36\sigma\left(\frac{n}{33}\right) + 12\sigma(n) \\
&\quad -36\sigma\left(\frac{n}{11}\right) + 144 \sum_{\substack{l,m\in\mathbf{N}\\ l+11m=n}} \sigma(l)\,\sigma(m) \\
&\quad -432 \sum_{\substack{l,m\in\mathbf{N}\\ l+11m=n}} \sigma(l)\,\sigma\left(\frac{m}{3}\right) \\
&\quad -432 \sum_{\substack{l,m\in\mathbf{N}\\ l+11m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m) \\
&\quad +1296 \cdot \sum_{\substack{l,m\in\mathbf{N}\\ l+11m=n}} \sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right) \\
&= 12\sigma\left(\frac{n}{11}\right) - 36\sigma\left(\frac{n}{33}\right) + 12\sigma(n) \\
&\quad -36\sigma\left(\frac{n}{3}\right) + 144W_{11}(n) - 432W_{33}(n) \\
&\quad -432W_{3,11}(n) + 1296W_{11}\left(\frac{n}{3}\right).
\end{aligned}
$$

Similarly,for

$$
\begin{aligned}
Q_2 &= x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 \\
&\quad +11\left(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2\right)
\end{aligned}
$$

$$N_2(n) =$$

$$
\begin{aligned}
&12\sigma\left(\frac{n}{10}\right) - 36\sigma\left(\frac{n}{30}\right) + 12\sigma(n) \\
&-36\sigma\left(\frac{n}{3}\right) + 144W_{10}(n) - 432W_{30}(n) \\
&-432W_{3,10}(n) + 1296W_{10}\left(\frac{n}{3}\right),
\end{aligned}
$$

and for

$$
\begin{aligned}
Q_3 &= x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 \\
&\quad +13\left(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2\right),
\end{aligned}
$$

$$N_3(n) =$$

$$12\sigma\left(\frac{n}{13}\right) - 36\sigma\left(\frac{n}{39}\right) + 12\sigma(n)$$
$$-36\sigma\left(\frac{n}{3}\right) + 144W_{13}(n) - 432W_{39}(n)$$
$$-432W_{3,13}(n) + 1296W_{13}\left(\frac{n}{3}\right).$$

■

3.1 Numerical Examples

Here the calculations have been done by the formulas obtained in this article.

n	$W_{10}(n)$	$W_{30}(n)$	$W_{3,10}(n)$	$W_{5,6}(n)$	$W_{2,15}(n)$	$N_1(n)$
1	0	0	0	0	0	12
2	0	0	0	0	0	36
3	0	0	0	0	0	12
4	0	0	0	0	0	84
5	0	0	0	0	0	72
6	0	0	0	0	0	36
7	0	0	0	0	0	96
8	0	0	0	0	0	180
9	0	0	0	0	0	12
10	0	0	0	0	0	228
11	1	0	0	1	0	288
12	3	0	0	0	0	516
13	4	0	1	0	0	312
14	7	0	0	0	0	1296
15	6	0	0	0	0	936
16	12	0	3	3	0	804
17	8	0	0	3	1	1368
18	15	0	0	0	0	2196
19	13	0	4	0	3	384
20	18	0	0	0	0	3132
21	15	0	0	4	4	2256
22	37	0	7	9	0	2736
23	26	0	3	4	7	2736
24	45	0	0	0	0	6660
25	42	0	6	0	6	3828

n	$W_{10}(n)$	$W_{30}(n)$	$W_{3,10}(n)$	$W_{5,6}(n)$	$W_{2,15}(n)$	$N_1(n)$
26	84	0	9	7	0	6264
27	59	0	0	12	12	6060
28	96	0	12	12	0	7584
29	72	0	12	7	8	3672
30	132	0	0	0	0	14052
31	82	1	8	6	15	6864
32	160	3	21	21	3	9396
33	127	4	4	16	13	9792
34	183	7	15	21	9	14184
35	126	6	18	6	18	8496
36	233	12	12	12	12	19956
37	142	8	13	18	12	9528
38	270	15	36	28	21	12240
39	183	13	16	28	28	13272
40	304	18	18	18	18	26112
41	204	12	24	20	14	11304
42	379	28	28	36	36	28944
43	279	14	19	24	24	15648
44	425	24	45	49	24	25776
45	286	24	24	24	24	28692
46	489	31	49	51	45	27504
47	317	18	39	32	35	17136
48	600	39	48	48	39	48756
49	356	20	42	42	30	19548

n	$W_{33}(n)$	$W_{3,11}(n)$	$N_2(n)$	$W_{39}(n)$	$W_{3,13}(n)$	$N_3(n)$
1	0	0	12	0	0	12
2	0	0	36	0	0	36
3	0	0	12	0	0	12
4	0	0	84	0	0	84
5	0	0	72	0	0	72
6	0	0	36	0	0	36
7	0	0	96	0	0	96
8	0	0	180	0	0	180
9	0	0	12	0	0	12
10	0	0	216	0	0	216
11	0	0	156	0	0	144
12	0	0	228	0	0	84
13	0	0	600	0	0	180
14	0	1	432	0	0	432
15	0	0	1080	0	0	504
16	0	0	1236	0	1	516
17	0	3	648	0	0	1224
18	0	0	1188	0	0	900
19	0	0	2400	0	3	672
20	0	4	648	0	0	1656
21	0	0	2688	0	0	2256
22	0	0	2196	0	4	576
23	0	7	1728	0	0	2880
24	0	0	3492	0	0	1908
25	0	3	4260	0	7	1380

n	$W_{33}(n)$	$W_{3,11}(n)$	$N_2(n)$	$W_{39}(n)$	$W_{3,13}(n)$	$N_3(n)$
26	0	6	4392	0	0	2556
27	0	0	7068	0	0	3900
28	0	9	4560	0	6	2832
29	0	12	4248	0	3	5256
30	0	0	9576	0	0	5832
31	0	12	6864	0	12	3408
32	0	8	9684	0	9	4932
33	0	0	10524	0	0	9648
34	1	21	7272	0	8	8280
35	3	15	9216	0	12	6192
36	4	4	15060	0	0	11316
37	7	18	10104	0	15	7800
38	6	13	15120	0	21	8200
39	12	12	16440	0	0	12276
40	8	36	7848	1	13	11736
41	15	18	13896	3	18	11592
42	13	16	23184	4	4	18144
43	18	24	15648	7	18	11904
44	12	12	20100	6	36	7776
45	28	28	20232	12	12	18216
46	14	45	19296	8	12	21888
47	24	35	19440	15	24	13968
48	24	24	31908	13	16	22980
49	31	39	20268	18	28	14652

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