A note on the vertex-distinguishing proper edge coloring of graphs*

Enqiang Zhu, Chanjuan Liu

Peking University; Key Laboratory of High Confidence Software Technologies (Peking University), Ministry of Education, CHINA
zhuenqiang@163.com

Abstract. The number of colors, required to color properly the edges of a simple graph G in such a way that any two vertices are incident with different sets of colors, is referred to as vertex-distinguishing edge chromatic, denoted by $\chi'_{vd}(G)$. An interesting phenomenon on vertex-distinguishing proper edge coloring is concerned in this paper. We prove that for every integer $m \geq 3$, there is always a graph G of maximum degree m with $\chi'_{vd}(G) < \chi'_{vd}(H)$, where H is a proper subgraph of G.

Keywords. vertex-distinguishing edge coloring, vertex-distinguishing edge chromatic number, subgraph, cycle, fan

1 Introduction

All graphs considered in this paper are simple and finite. We use [1] as a general reference, and the notation and terminology are standard.

A proper edge k-coloring of a graph G is called a *vertex-distinguishing proper edge coloring*, or simply *k-vdec*, if for any two distinct vertices u and v of G, the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v. It is easy to see that if G contains no more than one isolated vertex and no isolated edges, then there is a k-vdec of G and G is called a *vdec-graph*. The *vertex-distinguishing*

*This project is supported by National Natural Science Foundation of China Equipment(61127005); National Natural Science Foundation of China under grant 61309015.

edge chromatic number of G, denoted by $\chi'_{vd}(G)$, is the minimum number k such that G admits a k-vdec of G.

The concept of vertex-distinguishing edge coloring was introduced independently by Aigner et al.[2], by Burris and Schelp[3] and by Horňák and Soták [6], and has been considered in several papers [4],[5],[7],[8],[9],[10].

For convenience, we here use the notation $[a,b]$ to denote an integer set $\{a, a+1, a+2, \cdots, b\}$, where a, b are two integer with $0 \le a \le b$. Let $n_d(G)$ denote the number of vertices of degree d in a vdec-graph G. It is clear that $\binom{\chi'_{vd}(G)}{d} \ge n_d$ for all of d in $[\delta(G), \Delta(G)]$, where $\delta(G), \Delta(G)$ denote the minimum and maximum degrees of G, respectively. In [3], Burris and Schelp posed the following conjecture.

Conjecture 1. Let G be a vdec-graph and k be the minimum integer such that $\binom{k}{d} \ge n_d(G)$ for all d with $\delta(G) \le d \le \Delta(G)$. Then $\chi'_{vd}(G) = k$, or $k+1$.

The above conjecture is rather difficult to verify, even for regular graphs. One of the reasons is that there may exist some vdec-subgraph H of a vdec-graph G such that $\chi'_{vd}(H) > \chi'_{vd}(G)$, which indicates the induction can not be used to attack the above conjecture. However, according to our knowledge, there have been no papers discussing this problem and constructing such graphs up to now. In this paper, we will show the theorem as follow.

Theorem *For every integer $m \ge 3$, there exists a simple graph G of maximum degree m such that $\chi'_{vd}(H) > \chi'_{vd}(G)$ for some proper subgraph H of G.*

2 The proof of Theorem

In [3], the vertex-distinguishing proper edge coloring is also computed for some families of graphs, such as complete graphs K_n, bipartite complete graphs $K_{m,n}$, paths P_n, and cycles C_n. Here, in order to prove the theorem, we introduce one of the results as follow.

Lemma 2.1 [3] Let $n \ge 3$ and let k be the minimum integer such that $\binom{k}{2} \ge n$. Then

$$\chi'_{vd}(C_n) = \begin{cases} k+1 & \text{if } k \text{ is odd and } \binom{k}{2} - 2 \le n \le \binom{k}{2} - 1 \text{ or} \\ & k \text{ is even and } n > (k^2 - 2k)/2, \text{ and} \\ k & \text{otherwise.} \end{cases}$$

In [9], some regular graphs are studied concerning vertex-distinguishing proper edge coloring. For the proof of the theorem of this paper, we need the following lemma.

Lemma 2.2 [9] *Let G be an union of cycles $C_{m_1}, \cdots, C_{m_t}$ and let $L = \sum_{i=1}^{t} m_i, m_i \geq 3$ for $i = 1, \cdots, t$. Then $\chi'_{vd}(G) \leq k$ if and only if either*

1. *k is odd, $L = \binom{k}{2}$ or $L \leq \binom{k}{2} - 3$, or*

2. *k is even, $L \leq \binom{k}{2} - \frac{k}{2}$.*

A *fan* on $m+1$ vertices, denoted by F_{m+1}, is a graph with vertex-set $V(F_{m+1}) = \{u_i : i \in [0,m]\}$ and edge-set $E(F_{m+1}) = \{u_0u_i : i \in [1,m]\} \cup \{u_iu_{i+1} : i \in [1, m-1]\}$. A *cycle* on n vertices is denoted by $C_n = v_1v_2 \cdots v_nv_1$.

In what follows, we use $C(u)$ to denote the set of colors assigned to the edges incident to vertex u, and $\binom{[m]}{2}$ to denote all of 2-subsets of $[m]$, where $[m] = \{1, 2, \cdots, m\}$.

Lemma 2.3 *For a $(m+1)$-vertex$(m \geq 5)$ fan F_{m+1} defined as above, let $P = u_2u_1u_0u_mu_{m-1}$ is the subgraph of F_{m+1} induced by $\{u_2, u_1, u_0, u_m, u_{m-1}\}$. Then any proper edge m-coloring of P satisfying $C(u_1) \neq C(u_m)$ can be extended to a m-vdec of G.*

Proof. Let f is a proper edge m-coloring of P with $C(u_1) \neq C(u_m)$.

If $f(u_2u_1) \neq f(u_0u_m)$ and $f(u_1u_0) \neq f(u_mu_{m-1})$, then without loss of generality, assume that $f(u_2u_1) = 1, f(u_1u_0) = 2, f(u_0u_m) = 3, f(u_mu_{m-1}) = 4$. Thus, f can be extended to a m-vdec of G by follows: Let

$f(u_0u_{m-1}) = 1$; $f(u_0u_i) = i+2, i = 2, 3, \cdots, m-2$; $f(u_iu_{i+1}) = i, i = 2, 3, \cdots, m-3$; $f(u_{m-2}u_{m-1}) = m-2$ if $m \neq 6$, otherwise, if $m = 6$, $f(u_{m-2}u_{m-1}) = 5$.

If $f(u_2u_1) = f(u_0u_m)$, we assume that $f(u_2u_1) = f(u_0u_m) = 1, f(u_1u_0) = 2, f(u_mu_{m-1}) = 3$. Then we extend f by follows: Let

$f(u_0u_i) = i+2, i = 2, 3, \cdots, m-1$; $f(u_iu_{i+1}) = i, i = 2, 3, \cdots, m-3$; $f(u_{m-2}u_{m-1}) = m-2$ if $m \neq 5$, otherwise, if $m = 5$, $f(u_{m-2}u_{m-1}) = 1$.

Under any cases it is easy to show that f is extended to a m-vdec of G. So the result holds. □

The proof of the Theorem. To prove this theorem we will construct the desired graphs with the property required in the theorem as follows.

Case 1. G is disconnected.

When $m = 3$, let G_3 be the union of C_3 and the paw, which is the union a C_3 and a K_2 with just one comment vertex (see Figure 1(a)). Obviously, $\Delta(G_3) = 3$ and it is an easy task to show $\chi'_{vd}(G_3) = 4$. Let G_3' is the subgraph of G_3 by deleting vertex u_1 and its incident edge u_1u_2. Then according to Lemma 2.2, $\chi'_{vd}(G_3') > 4$(in fact $\chi'_{vd}(G_3') = 5$). So G_3 is the desired graph.

When $m = 4$, let G_4 be the graph obtained from G_3 by adding a vertex u_5 and an edge u_2u_5, and obviously $\Delta(G_4) = d_{G_4}(u_2) = 4$ (see Figure 1(b)). Similarly, let G_4' is the subgraph of G_4 induced by $\{v_1, v_2, v_3, u_2, u_3, u_4\}$. According to Lemma 2.1 and Lemma 2.2, $\chi'_{vd}(G_4') > \chi'_{vd}(G_4)$.

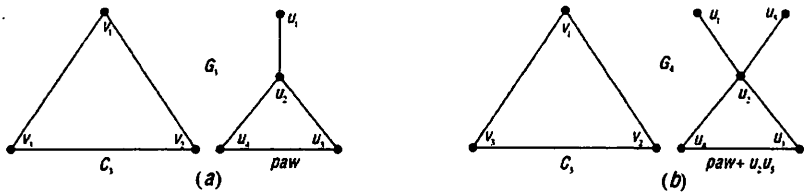

Figure 1: $G_3 = C_3 \cup$ paw and $G_4 = G_3 + u_2u_5$

When $m \geq 5$, for two positive integer m and $n = \lceil \frac{m^2-2m}{2} \rceil$, let C_n and F_{m+1} are n-vertex cycle and $(m+1)$-vertex fan respectively, and they are disjoint. Now we construct a disconnected graph G_m through the union of C_n and F_{m+1}, i.e. $G_m = C_n \cup F_{m+1}$ and $\Delta(G_m) = m$. Now we prove G_m is the desired graph.

First we prove $\chi'_{vd}(G_m) = m$. According to Lemma 2.1, it is easy to show $\chi'_{vd}(C_n) = m$. Let f be a m-vdec of C_n. Because of $n = \lceil \frac{m^2-2m}{2} \rceil$, there are at least two different elements of $\binom{[m]}{2}$ to be not included in $\bigcup_{i=1}^{n} C(u_i)$. Further, let $C_1 = \{c_1, c_2\}$ and $C_2 = \{c_3, c_4\}$ be different two 2-subsets of $[m]$ not in $\bigcup_{i=1}^{n} C(u_i)$. Obviously, $|\{c_1, c_2, c_3, c_4\}| \geq 3$. So there is an edge-4-coloring f of the subgraph of F_{m+1} induced by $\{u_2, u_1, u_0, u_m, u_{m-1}\}$ such that $C(u_1) \neq C(u_m)$, where the color-set is $\{c_1, c_2, c_3, c_4\}$. Thus, according to Lemma 2.3, f can be extendsed to a m-vdec of G_m, and $\chi'_{vd}(G_m) = m$.

Next, consider the subgraph G_m' of G_m by deleting edges u_0u_i for $i = 2, 3, \cdots, m-1$, i.e. G_m' is the union of two disjoint cycle C_{m+1} and C_n. According to Lemma 2.2, it has $\chi'_{vd}(G_m') > m$ for $m+1+n = \frac{m^2+2}{2} > \binom{m}{2}$. So $\chi'_{vd}(G_m') > \chi'_{vd}(G_m)$.

Case 2. G is connected.

When $\Delta(G) = 3$ or $\Delta(G) = 4$, let G_3 and G_4 be the graphs obtained from a 5-vertex cycle C_5 by adding a chord v_1v_3 and two chords v_1v_3, v_1v_4 respectively, i.e. $G_3 = C_5 + \{v_1v_3\}$ and $G_4 = C_5 + \{v_1v_3, v_1v_4\}$. Obviously, $\Delta(G_3) = 3$ and $\Delta(G_4) = 4$, and it is easy to show $\chi'_{vd}(G_3) = \chi'_{vd}(G_4) = 4$. In addition, C_5 is a subgraph of G_3 and G_4, and $\chi'_{vd}(C_5) = 5$ by Lemma

2.1, so G_3, G_4 are the desired graphs.

When $\Delta(G) \geq 5$.

For integer $m \geq 5$ and $n = \lceil \frac{m^2-2m}{2} \rceil$, we construct a connected graph G with maximum degree m in the following way. Suppose $C_n = v_1v_2\cdots v_nv_1$ is a n-vertex cycle, and F_{m+1} is a $(m+1)$-vertex fan defined above, and they are disjoint. Let G_m be the graph obtained from $C_n \cup F_{m+1}$ by deleting edge v_1v_n and adding edges v_1u_1 and v_nu_m, i.e. $G_m = C_n \cup F_{m+1} - v_1v_n + \{v_1u_1, v_nu_m\}$. It is clear that $\Delta(G) = m$.

According to Lemma 2.1, $\chi'_{vd}(C_n) = m$. Suppose that f is a m-vdec of C_n. Without loss of generality, assume that $f(v_1v_n) = 2$. Then f can be extended to a m-vdec of G_m by coloring edges of F_{m+1} as follows. Let $f(v_1u_1) = 2$, $f(v_nu_m) = 2$; $f(u_0u_i) = i, i = 1, 2, \cdots, m$; $f(u_iu_{i+1}) = i+2, i = 1, 2, \cdots, m-2$; and $f(u_{m-1}u_m) = 1$.

Because $d_{G_m}(v_i) = 2$ for $i = 1, 2, \cdots, n$, $d_{G_m}(u_i) = 3$ for $i = 1, 2, \cdots, m$ and $d_{G_m}(u_0) = m$, we only need to show that $C(v_i) \neq C(v_j)$ for any $i \neq j \in [1, n]$ and $C(u_i) \neq C(u_j)$ for $i \neq j \in [1, m]$. The former holds immediately by the selection that f is a m-vdec of C_n. Further, according to the method of edge-coloring of F_{m+1}, we have $C(u_1) = \{1, 2, 3\}; C(u_m) = \{1, 2, m\}; C(u_{m-1}) = \{1, m-1, m\}; C(u_i) = \{i, i+2, i+1\}$ for $i = 2, 3, \cdots, m-2$. So $C(u_i) \neq C(u_j)$ for $i \neq j \in [1, m]$, and f is a m-vdec of G_m.

Now, let G'_m be the subgraph of G_m by deleting vertex u_0 and all of its incident edges, i.e. $G'_m = C_{m+n}$. Because of $m + n = \frac{m^2}{2} > \binom{m}{2}$, it follows that $\chi'_{vd}(G'_m) > m = \chi'_{vd}(G_m)$. So G_m is the desired graph.

Considered the two cases discussed above, the proof of the theorem is finished. □

Note that there are in fact many kinds of graphs G (differ from the G_m constructed in the proof of the theorem) with the property described in the theorem. Here, we will present an interesting example.

Let $C'_n = v'_1v'_2\cdots v'_{11}v'_1$ be a copy of a 11-vertex cycle $C_{11} = v_1v_2\cdots v_{11}v_1$. First, construct a 5-regular graph H from C_n, C'_n by adding edges v_iu_i, v_iu_{i+1}, v_iu_{i+2} for $i = 1, 2, \cdots, 11$, where the subscript addition is taken modulo 11. Next, add a new vertex w and seven edges wu_2 and wu_{2i-1} for $i = 1, 2, \cdots, 6$ to H, and then denote the resulting graph by G (see Figure2).

Because the maximum degree of G is 7, it naturally has $\chi'_{vd}(G) \geq 7$. Now, we give an edge 7-coloring f of G by coloring the edges of G through (1) to (6) as follows:

(1) the edges $wu_1, wu_2, wu_3, wu_5, wu_7, wu_9, wu_{11}$ are colored by 2, 3, 4, 5, 6, 7, 1, respectively;

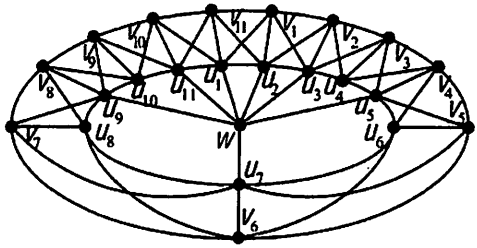

Figure 2: a graph G

(2) the edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_{10}, v_{10}v_{11}, v_{11}v_1$ are colored by 7, 1, 6, 1, 4, 1, 5, 2, 3, 6, 5, respectively;

(3) the edges $u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6, u_6u_7, u_7u_8, u_8u_9, u_9u_{10}, u_{10}u_{11}, u_{11}u_1$ are colored by 5, 1, 7, 3, 1, 2, 1, 6, 2, 3, 6, respectively;

(4) the edges $v_1u_1, v_2u_2, v_3u_3, v_4u_4, v_5u_5, v_6u_6, v_7u_7, v_8u_8, v_9u_9, v_{10}u_{10}, v_{11}u_{11}$ are colored by 3, 4, 5, 4, 6, 7, 7, 4, 1, 1, 2, respectively;

(5) the edges $v_1u_2, v_2u_3, v_3u_4, v_4u_5, v_5u_6, v_6u_7, v_7u_8, v_8u_9, v_9u_{10}, v_{10}u_{11}, v_{11}u_1$ are colored by 2, 2, 2, 7, 5, 5, 3, 3, 5, 5, 4, respectively; and

(6) the edges $v_1u_3, v_2u_4, v_3u_5, v_4u_6, v_5u_7, v_6u_8, v_7u_9, v_8u_{10}, v_9u_{11}, v_{10}u_1, v_{11}u_2$ are colored by 6, 6, 4, 3, 3, 2, 4, 6, 4, 7, 7, respectively.

It is easy to show f is a 7-vdec of G, but $\chi'_{vd}(H) > 7$ for $|V(H)| = 22 > \binom{7}{5}$. Therefore, $\chi'_{vd}(G) < \chi'_{vd}(H)$.

From the above discussions, we have an observation as follow: For a n-vertex graph G, if there are $m(< n)$ vertices, $v_1, v_2, \cdots, v_m$, of G such that $G - \{v_1, v_2, \cdots, v_m\} \triangleq H$ is a $k(\geq 2)$-regular graph, and $\binom{\chi'_{vd}(G)}{k} < n - m$, then G has the property that $\chi'_{vd}(G) < \chi'_{vd}(H)$. However, if G itself is a regular graph, then G may not have this property. So we propose the following conjecture.

Conjecture 2. For any k-regular graph G with $k \geq 2$, there exists no proper vdec-subgraphs H of G such that $\chi'_{vd}(H) > \chi'_{vd}(G)$.

References

[1] J. A. Bondy and U. S. R. Murty. Graph Theory, *Springer*, 1976.

[2] M. Aigner, E. triesch, and Z. Tuza. Irregular assignments and vertex-distinguishing edge-colorings of graphs. *Combinatorics* 90 (A. Barlotti et al., eds.), Elsevier Science Pub., New York (1992), 1-9.

[3] A. C. Burris and R. H. Schilp. Vertex-distinguishing proper edge-colorings. *J. Graph Theory* 26 (2) (1997), 70-82.

[4] J. Černý, M. Horňák, and R. Soták. Observability of a graph. *Math. Slovaca* 46 (1996), 21-31.

[5] F. Harary and J. S. Msybee (eds.). Graphs and Applications. *Wiley-Interscience*, New York (1985), 147-162.

[6] M. Horňák and R. Soták. Observability of complete multipartite graphs with equipotent parts. *Ars Comb.* 41 (1995), 289-301.

[7] M. Horňák and R. Soták. Asymptotic behaviour of the observability of Q_n. *Discrete Math.* 176 (1997), 139-148.

[8] C. Bazgan, A. Harkat-Benhamdine, Hao Li, and M. Woźniak. On the vertex-distinguishing proper edge-coloring of graphs. *J. Combinatorial Theory B* 75 (1999), 288-301.

[9] Janka Rudašová, Roman Sotká. Vertex-distinguishing proper edge colourings of some regular graphs. *Discrete Math.* 308 (2008) 795 C 802.

[10] Bin Liu and Guizhen Liu. Vertex-Distinguishing Edge Colorings of Graphs with Degree Sum Conditions. *Graphs and Combinatorics* (2010) 26:781C791.