Association schemes based on the subspaces of type $(2,0,1)$ in singular symplectic space over finite fields

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Abstract The construction of association schemes based on the subspaces of type $(2,0,1)$ in singular symplectic space over finite fields is provided in this paper. Applying the matrix method and combinatorial design theory, all parameters of the association scheme are computed.

Keywords Association schemes, Singular symplectic space

§1 Introduction

In this section, we shall introduce the concepts of singular symplectic geometry over finite fields and association schemes, then we will give our main results. Notations and terminologies will be adopted from [1-2]. Now let us firstly introduce the concept of singular symplectic space[1] over $\mathbb{F}_q$.

Assume that $\mathbb{F}_q$ is a finite field with q elements, where q is a prime power. Let $\mathbb{F}_q^{(2\nu+l)}$ be the $(2\nu+l)$-dimensional row vector space over $\mathbb{F}_q$. Now let

$$K=\begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}, \qquad K_l=\begin{pmatrix} K & \\ & 0^{(l)} \end{pmatrix}.$$

The set of all $(2\nu+l)\times(2\nu+l)$ nonsingular matrices T over $\mathbb{F}_q$ satisfying $TK_lT'=K_l$ forms a group, called the *singular symplectic group* of degree $2\nu+l$ and index ν over $\mathbb{F}_q$ and denoted by $Sp_{2\nu+l,\nu}(\mathbb{F}_q)$. It can be readily verified that $Sp_{2\nu+l,\nu}(\mathbb{F}_q)$ consists of all $(2\nu+l)\times(2\nu+l)$ nonsingular matrices of form

$$T=\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where $T_{11}KT_{11}'=K$ and T_{22} is nonsingular.

Let $\mathbb{F}_q^{(2\nu+l)}$ be the $(2\nu+l)$-dimensional row vector space over $\mathbb{F}_q$. We have an action of $Sp_{2\nu+l,\nu}(\mathbb{F}_q)$ on $\mathbb{F}_q^{(2\nu+l)}$ defined as follows:

$$\mathbb{F}_q^{(2\nu+l)}\times Sp_{2\nu+l,\nu}(\mathbb{F}_q)\rightarrow\mathbb{F}_q^{(2\nu+l)}$$

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$$((x_1,x_2,\cdots,x_{2\nu+l}),T)\mapsto(x_1,x_2,\cdots,x_{2\nu+l})T.$$

The vector space $\mathbb{F}_q^{(2\nu+l)}$ together with the action of $Sp_{2\nu+l,\nu}(\mathbb{F}_q)$ is called the *singular symplectic space* over $\mathbb{F}_q$.

Let e_i $(1\le i\le 2\nu+l)$ be the row vector in $\mathbb{F}_q^{(2\nu+l)}$ whose i-th coordinate is 1 and all other coordinates are 0. Denote by E the l-dimensional subspace of $\mathbb{F}_q^{(2\nu+l)}$ generated by $e_{2\nu+1},e_{2\nu+2},\dots,e_{2\nu+l}$. An m-dimensional subspace P of $\mathbb{F}_q^{(2\nu+l)}$, is called a *subspace of type* (m,s,k) if

(i) PK_lP^t is cogredient to $M(m,s)$.

(ii) $\dim(P\cap E)=k$, where

$$M(m,s)=\begin{pmatrix} 0 & I^{(s)} & \\ -I^{(s)} & 0 & \\ & & 0^{(m-2s)}\end{pmatrix}.$$

Denote the set of all subspaces of type (m,s,k) in $\mathbb{F}_q^{(2\nu+l)}$ by $\mathcal{M}(m,s,k;2\nu+l,\nu)$. It can be verified that $\mathcal{M}(m,s,k;2\nu+l,\nu)$ is non-empty if and only if $0\le k\le l$, $2s\le m-k\le \nu+s$. Let $N(m,s,k;2\nu+l,\nu)=|\mathcal{M}(m,s,k;2\nu+l,\nu)|$, then

$$N(m,s,k;2\nu+l,\nu)=q^{2s(\nu+s-m+k)+(m-k)(l-k)}\frac{\prod\limits_{i=\nu+s-m+k+1}^{\nu}(q^{2i}-1)\prod\limits_{i=l-k+1}^{l}(q^i-1)}{\prod\limits_{i=1}^{s}(q^{2i}-1)\prod\limits_{i=1}^{m-k-2s}(q^i-1)\prod\limits_{i=1}^{k}(q^i-1)},$$

see the chapter 3 of reference [1].

Let P be an m-dimensional vector subspace of $\mathbb{F}_q^{(n)}$, then we write dim $P=m$. Let $v_1,v_2,\dots,v_m$ be a basis of P. We notice that $v_1,v_2,\dots,v_m$ are vector of $\mathbb{F}_q^{(n)}$. We usually use the $m\times n$ matrix $\begin{pmatrix} v_1\\ v_2\\ \vdots\\ v_m\end{pmatrix}$ to represent the vector subspace P, write $P=\begin{pmatrix} v_1\\ v_2\\ \vdots\\ v_m\end{pmatrix}$,i.e., we use the same letter P to denote a matrix which represents the vector subspace P, and call the matrix P a *matrix representation* of the vector subspace P.

Next we will introduce the concepts of *association schemes*[2].

Let X be a set of cardinality n and R_i $(i=0,1,\cdots,d)$ be nonempty subsets of $X\times X$ with the following properties that:

(1) $R_0=\{(x,x)|x\in X\}$;

(2) $X\times X=R_0\cup\ldots\cup R_d$ and $R_i\cap R_j=\emptyset$ for all $i\neq j$;

(3) For any R_i $(0 \leq i \leq d)$, define ${}^tR_i = \{(x,y)|(y,x) \in R_i\}$, then for any $i \in \{0,1,\cdots,d\}$, there is an $i' \in \{0,1,\cdots,d\}$ such that ${}^tR_i = R_{i'}$;

(4) For any $i,j,k \in \{0,1,\cdots,d\}$ and any pair $(x,y) \in R_k$, the number of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is constant. This constant is denoted by $p_{i,j}^k$, which is independent of the pair (x,y);

(5) $p_{i,j}^k = p_{j,i}^k$.

Such a configuration $\Gamma = \{X,\{R_i\},0 \leq i \leq d\}$ is called a commutative association scheme of class d on X. The non-negative integers $p_{i,j}^k$ are called the intersection numbers of Γ.

An association scheme with the additional property:

(6) $i = i'$ for all $i \in \{0,1,\cdots,d\}$ is called symmetric or Bose-Mesner type.

It can be readily verified that a symmetric association scheme is also a commutative one. There are some parameters in an association scheme $\Gamma = \{X,\{R_i\},0 \leq i \leq d\}$. i.e., d, v, n_i, $p_{i,j}^k$.

(a) d is called the class of the association scheme.

(b) v is called the size of X.

(c) For a fixed $x \in X$, n_i is the number of y satisfying $(x,y) \in R_i$, that is to say, $n_i = |\{y|x \in X,(x,y) \in R_i\}|$.

(d) The non-negative integers $p_{i,j}^k$ are called the intersection numbers of Γ.

Furthermore, we will give some parameter relational expressions in an commutative association scheme:

$$\sum_{j=0}^{d} p_{i,j}^k = n_i, \qquad n_i p_{j,k}^i = n_j p_{i,k}^j;$$

$$v = n_1 + n_2 + \cdots + n_d + 1, \qquad p_{j,k}^i = p_{k,j}^i.$$

where i, j, $k = 1, 2,\cdots, d$.

Dual polar schemes are well-known as association schemes. Applying the matrix method, Wan and Dai[3] computed all parameters of dual polar schemes. As a generalization of dual polar schemes, Rieck[4] constructed association schemes by the subspaces of a given dimension in a finite classical polar space; Guo, Wang and Li[5–6] constructed association schemes in singular classical spaces and singular pseudo-symplectic spaces. As a generalization of bilinear forms schemes, Wang, Guo and Li[7–9] constructed association schemes in attenuated spaces and singular general linear space. As a generalization of symmetric schemes, Gao and He[10] constructed association schemes based on singular symplectic geometry over finite fields.

In this paper, we provide a new symmetric association scheme based on singular symplectic geometry over finite fields. This paper is organized as follows. In Section 2, an association scheme is constructed by the subspaces of type $(2,0,1)$ in singular symplectic space. In Section 3, all preliminaries of the association schemes are computed.

§2 The construction of association schemes

Assume that $\nu \geq 2$, $l \geq 3$. Suppose that X be the set of all subspaces of type $(2,0,1)$ in $(2\nu+l)$-dimensional singular symplectic space $\mathbb{F}_q^{(2\nu+l)}$. For the elements X_1, X_2 of X, consider the type of the subspace X_1+X_2.

Define $R_0 = \{(X_1,X_2)|X_1 = X_2\}$.

If $X_1 \neq X_2$, since both X_1 and X_2 are the set of all subspaces of type $(2,0,1)$ in $(2\nu+l)$-dimensional singular symplectic space $\mathbb{F}_q^{(2\nu+l)}$. Suppose that

$$X_1 = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & 0 & x_{23} \end{pmatrix}_{\nu\ \ \nu\ \ l}, \qquad X_2 = \begin{pmatrix} x'_{11} & x'_{12} & x'_{13} \\ 0 & 0 & x'_{23} \end{pmatrix}_{\nu\ \ \nu\ \ l}.$$

where both $(x_{11}\ x_{12})$ and $(x'_{11}\ x'_{12})$ are 1-dimensional subspaces in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, both x_{23} and x'_{23} are 1-dimensional subspaces in the l-dimensional row vector space $\mathbb{F}_q^{(l)}$.

(i) If dim $(X_1+X_2) = 3$, then we discuss the following two cases:

(1) If dim $(X_1 \cap X_2 \cap E) = 1$, then there are the following three subcases to be considered:

(a) If rank $\begin{pmatrix} x_{11} & x_{12} \\ x'_{11} & x'_{12} \end{pmatrix} = 1$, then X_1+X_2 is a subspace of type $(3,0,2)$. In this case, the 1-st association relationship is defined by $(X_1,X_2) \in R_1$.

(b) If rank $\begin{pmatrix} x_{11} & x_{12} \\ x'_{11} & x'_{12} \end{pmatrix} = 2$ and $\begin{pmatrix} x_{11} & x_{12} \\ x'_{11} & x'_{12} \end{pmatrix}$ is a subspace of type $(2,0)$ in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, then X_1+X_2 is a subspace of type $(3,0,1)$. In this case, the 2-nd association relationship is defined by $(X_1,X_2) \in R_2$.

(c) If rank $\begin{pmatrix} x_{11} & x_{12} \\ x'_{11} & x'_{12} \end{pmatrix} = 2$ and $\begin{pmatrix} x_{11} & x_{12} \\ x'_{11} & x'_{12} \end{pmatrix}$ is a subspace of type $(2,1)$ in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, then X_1+X_2 is a subspace of type $(3,1,1)$. In this case, the 3-rd association relationship is defined by $(X_1,X_2) \in R_3$.

(2) If dim $(X_1 \cap X_2 \cap E) = 0$, then there exists the unique case:

(d) That is to say, X_1+X_2 is a subspace of type $(3,0,2)$. In this case, the 4-th association relationship is defined by $(X_1,X_2) \in R_4$.

(ii) If dim $(X_1+X_2) = 4$, then there are also the following three cases to be considered:

(e) If rank $\begin{pmatrix} x_{11} & x_{12} \\ x'_{11} & x'_{12} \end{pmatrix} = 1$, then X_1+X_2 is a subspace of type $(4,0,3)$. In this case, the 5-th association relationship is defined by $(X_1,X_2) \in R_5$.

(f) If rank $\begin{pmatrix} x_{11} & x_{12} \\ x'_{11} & x'_{12} \end{pmatrix} = 2$ and $\begin{pmatrix} x_{11} & x_{12} \\ x'_{11} & x'_{12} \end{pmatrix}$ is a subspace of type $(2,0)$ in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, then X_1+X_2 is a subspace of type $(4,0,2)$. In this case, the 6-th association relationship is defined by $(X_1,X_2) \in R_6$.

(g) If rank $\begin{pmatrix} x_{11} & x_{12} \\ x'_{11} & x'_{12} \end{pmatrix} = 2$ and $\begin{pmatrix} x_{11} & x_{12} \\ x'_{11} & x'_{12} \end{pmatrix}$ is a subspace of type $(2,1)$ in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, then X_1+X_2 is a subspace of type $(4,1,2)$. In this case, the 7-th association relationship is defined by $(X_1,X_2) \in R_7$.

To sum up, $\Gamma = (X, \{R_i\}_{0\le i\le 7})$ is an association classification of class 7 on X.

Next, we will show that $\Gamma = (X, \{R_i\}_{0\le i\le 7})$ is a symmetric association scheme of class 7 on X.

Theorem 2.1 Suppose that X be the set of all subspaces of type $(2,0,1)$ in $(2\nu+l)$-dimensional singular symplectic space $\mathbb{F}_q^{(2\nu+l)}$, where $\nu \ge 2$, $l \ge 3$. For the elements X_1, X_2 of X, if X_1 and X_2 have the above 7 association relationships, then $\Gamma = (X, \{R_i\}_{0\le i\le 7})$ is a symmetrical association scheme of class 7 on X.

Proof: Obviously, the association relations R_i $(0 \le i \le 7)$ is symmetric. We only need to show that each R_i $(i = 0,1,\cdots,7)$ is an orbit of $Sp_{2\nu+l,\nu}(\mathbb{F}_q)$ in a natural action on $X \times X$.

For $i = 0$, i.e., $X_1 = X_2$. Since $Sp_{2\nu+l,\nu}(\mathbb{F}_q)$ acts transitively on each set of subspaces of the same type in $(2\nu+l)$-dimensional singular symplectic space $\mathbb{F}_q^{(2\nu+l)}$, $Sp_{2\nu+l,\nu}(\mathbb{F}_q)$ acts transitively on R_0.

For $i = 1$, i.e., X_1+X_2 is a subspace of type $(3,0,2)$ defined by the 1-st association relationship. If $(X_1,X_2) \in R_1$, $(Y_1,Y_2) \in R_1$. Pick four elements X_1,X_2,Y_1,Y_2 of X. Without loss of generality, we can take

$$X_1 = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & 0 & x_{23} \end{pmatrix}_{\substack{\\ \nu \quad \nu \quad l}}, \qquad X_2 = \begin{pmatrix} x_{11} & x_{12} & x'_{13} \\ 0 & 0 & x_{23} \end{pmatrix}_{\substack{\\ \nu \quad \nu \quad l}},$$

$$Y_1 = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ 0 & 0 & y_{23} \end{pmatrix}_{\substack{\\ \nu \quad \nu \quad l}}, \qquad Y_2 = \begin{pmatrix} y_{11} & y_{12} & y'_{13} \\ 0 & 0 & y_{23} \end{pmatrix}_{\substack{\\ \nu \quad \nu \quad l}}$$

to be matrixes representation of the subspaces X_1,X_2,Y_1 and Y_2 respectively. Then the subspaces X_1+X_2, Y_1+Y_2 can be represented by $(2\nu+l)\times(2\nu+l)$ matrices of the form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & -x_{13}+x'_{13} \end{pmatrix}, \qquad \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ 0 & 0 & y_{23} \\ 0 & 0 & -y_{13}+y'_{13} \end{pmatrix}$$

respectively. Since both $(x_{11}\ x_{12})$ and $(y_{11}\ y_{12})$ are subspaces of type $(1,0)$ in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, there exits a $T_{11} \in Sp_{2\nu}(\mathbb{F}_q)$ such that

$$(x_{11}\ x_{12}) = (y_{11}\ y_{12})T_{11}. \tag{2.1}$$

Additionally, because both $\begin{pmatrix} x_{23} \\ -x_{13}+x'_{13} \end{pmatrix}$ and $\begin{pmatrix} y_{23} \\ -y_{13}+y'_{13} \end{pmatrix}$ are $2 \times l$ matrixes with rank 2, there exists a $T_{22} \in GL_l(\mathbb{F}_q)$ such that

$$\begin{pmatrix} x_{23} \\ -x_{13}+x'_{13} \end{pmatrix} = \begin{pmatrix} y_{23} \\ -y_{13}+y'_{13} \end{pmatrix} T_{22}. \tag{2.2}$$

We can find a $2v \times l$ matrix X such that $(y_{11}\ y_{12})X = x_{13} - y_{13}T_{22}$. Suppose that

$$T = \begin{pmatrix} T_{11} & X \\ 0 & T_{22} \end{pmatrix} \in Sp_{2v+l,v}(\mathbb{F}_q),$$

then

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & -x_{13}+x'_{13} \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ 0 & 0 & y_{23} \\ 0 & 0 & -y_{13}+y'_{13} \end{pmatrix} T. \tag{2.3}$$

From (2.1), (2.2) and (2.3), it can be readily verified that

$$X_1 = Y_1 T,\ X_2 = Y_2 T.$$

Hence, $Sp_{2v+l,v}(\mathbb{F}_q)$ acts transitively on R_1. Similarly, $Sp_{2v+l,v}(\mathbb{F}_q)$ acts transitively on $R_2, R_3, \cdots, R_7$.

According to the above statements, $\Gamma = (X, \{R_i\}_{0 \le i \le 7})$ is a symmetrical association scheme of class 7 on X.

§3 The computation of the preliminaries

Theorem 3.1 The preliminaries $d, v, n_i (1 \le i \le 7)$ of the association scheme $\Gamma = (X, \{R_i\}_{0 \le i \le 7})$ are

$d = 7,$ $\quad v = q^{l-1}\frac{(q^{2v}-1)(q^l-1)}{(q-1)^2},$ $\quad n_1 = q^{l-1} - 1,$

$n_2 = \frac{q^l(q^{2(v-1)}-1)}{q-1},$ $\quad n_3 = q^{2v+l-2},$ $\quad n_4 = \frac{q^2(q^{l-1}-1)}{q-1},$

$n_5 = \frac{q^2(q^{l-2}-1)(q^{l-1}-1)}{(q-1)(q^2-1)},$ $\quad n_6 = \frac{q^{l+2}(q^{2(v-1)}-1)(q^{l-1}-1)}{(q-1)^2},$ $\quad n_7 = \frac{q^{2v+l}(q^{l-1}-1)}{q-1},$

respectively.

Proof: From the construction of section 2, we have known $d = 7$. In order to prove Theorem 3.1, we also need to prove Propositions 3.1-3.8.

Proposition 3.1 The size of X is

$$\upsilon = q^{l-1}\frac{(q^{2\nu}-1)(q^l-1)}{(q-1)^2}.$$

Proof: Since X is the set of all subspaces of type $(2,0,1)$ in $(2\nu+l)$-dimensional singular symplectic space $\mathbb{F}_q^{(2\nu+l)}$, we have

$$\upsilon = |X| = N(2,0,1;2\nu+l,\nu) = q^{l-1}\frac{(q^{2\nu}-1)(q^l-1)}{(q-1)^2}.$$

Proposition 3.2 For a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V,V_1) \in R_1$ is

$$n_1 = q^{l-1} - 1.$$

Proof: Let V be a fixed subspace of type $(2,0,1)$ in $\mathbb{F}_q^{(2\nu+l)}$. From dim $(V \cap V_1 \cap E) = 0$, we can assume that the subspaces V and V_1 have the matrix representation of the forms

$$V = \begin{pmatrix} e_1 \\ e_{2\nu+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.1}$$
$$\qquad\qquad\qquad\; 1 \quad \nu-1 \quad 1 \quad \nu-1 \quad 1 \quad l-1$$

$$V_1 = \begin{pmatrix} e_{2\nu+1} \\ x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & 0 & x_{16} \end{pmatrix}, \tag{3.2}$$
$$\qquad\qquad\qquad\; 1 \quad \nu-1 \quad 1 \quad \nu-1 \quad 1 \quad l-1$$

respectively. Then the subspace $V + V_1$ can be represented by the matrix of the form

$$\begin{pmatrix} e_1 \\ e_{2\nu+1} \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & 0 & x_{16} \end{pmatrix}. \tag{3.3}$$

Since

$$\text{rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix} = 1,$$

we can assume that $(x_{11}\, x_{12}\, x_{13}\, x_{14}) = (1\,0\,0\,0)$. Additionally, we can deduce that x_{16} is nonzero from dim $(V + V_1) = 3$. Thus, the number of x_{16} is $q^{l-1} - 1$.

Hence, for a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V,V_1) \in R_1$ is $n_1 = q^{l-1} - 1$.

Proposition 3.3 For a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V,V_1) \in R_2$ is

$$n_2 = \frac{q^l(q^{2(\nu-1)}-1)}{q-1}.$$

Proof: Suppose that the choices of subspaces V, V_1, $V+V_1$ are as same as (3.1), (3.2), (3.3), respectively. Since rank $\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix} = 2$ and $\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix}$ is a subspace of type (2,0) in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, then x_{11} is arbitrary, $x_{13}=0$ and $(x_{12}\, x_{14})$ are a 1-dimensional subspace in $2(\nu-1)$-dimensional symplectic space $\mathbb{F}_q^{2(\nu-1)}$. Additionally, we can deduce that x_{16} is arbitrary from dim $(V+V_1)=3$. Therefore, the number of subspaces V_1 is $\frac{q^l(q^{2(\nu-1)}-1)}{q-1}$.

Hence, for a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V, V_1) \in R_2$ is

$$n_2 = \frac{q^l(q^{2(\nu-1)}-1)}{q-1}.$$

Proposition 3.4 For a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V, V_1) \in R_3$ is

$$n_3 = q^{2\nu+l-2}.$$

Proof: Suppose that the choices of subspaces V, V_1, $V+V_1$ are as same as (3.1), (3.2), (3.3), respectively. Since rank $\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix} = 2$ and $\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix}$ is a subspace of type $(2,1)$ in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, then $x_{13} \neq 0$, x_{11}, x_{12} and x_{14} are arbitrary. Additionally, we can deduce that x_{16} is arbitrary from dim $(V+V_1)=3$. So the number of subspaces V_1 is $q^{l-1}\frac{(q-1)q^{2(\nu-1)+1}}{q-1} = q^{2\nu+l-2}$.

Hence, for a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V, V_1) \in R_3$ is

$$n_3 = q^{2\nu+l-2}.$$

Proposition 3.5 For a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V, V_1) \in R_4$ is

$$n_4 = \frac{q^2(q^{l-1}-1)}{q-1}.$$

Proof: Suppose that the choice of subspace V is as same as (3.1) and the subspace V_1 has the matrix representation of the form

$$V_1 = \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & y_{15} & y_{16} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \end{pmatrix}, \quad (3.4)$$
$$\begin{matrix} \quad 1 & \nu-1 & 1 & \nu-1 & 1 & l-1 \end{matrix}$$

then the subspace $V+V_1$ has the matrix representation of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & y_{15} & y_{16} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \end{pmatrix}. \qquad (3.5)$$

From dim $(V \cap V_1 \cap E) = 0$ and dim $(V + V_1) = 3$. We can deduce that $x_{11} \neq 0$, $x_{13} = 0$, x_{12}, x_{14} and x_{16} are zero vectors, x_{15}, y_{15} are arbitrary and y_{16} is a 1-dimensional subspace in $(l-1)$-dimensional row vector space $\mathbb{F}_q^{(l-1)}$. So the number of subspaces V_1 is $\frac{q^2(q-1)(q^{l-1}-1)}{(q-1)^2} = \frac{q^2(q^{l-1}-1)}{q-1}$.

Hence, for a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V, V_1) \in R_4$ is

$$n_4 = \frac{q^2(q^{l-1}-1)}{q-1}.$$

Proposition 3.6 For a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V, V_1) \in R_5$ is

$$n_5 = \frac{q^2(q^{l-2}-1)(q^{l-1}-1)}{(q-1)(q^2-1)}.$$

Proof: Suppose that the choices of subspaces V, V_1, $V + V_1$ are the same as (3.1), (3.4), (3.5), respectively. Since rank $\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix} = 1$, we can assume that $(x_{11}\ x_{12}\ x_{13}\ x_{14}) = (1\ 0\ 0\ 0)$. As a result of dim $(V + V_1) = 4$, we can deduce that both x_{15} and y_{15} are arbitrary and $\begin{pmatrix} y_{16} \\ x_{16} \end{pmatrix}$ is a 2-dimensional subspace in $(l-1)$-dimensional row vector space $\mathbb{F}_q^{(l-1)}$. Thus, the number of subspaces V_1 is

$$q^2 \begin{bmatrix} l-1 \\ 2 \end{bmatrix}_q = \frac{q^2(q^{l-2}-1)(q^{l-1}-1)}{(q-1)(q^2-1)}.$$

Hence, for a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V, V_1) \in R_5$ is

$$n_5 = \frac{q^2(q^{l-2}-1)(q^{l-1}-1)}{(q-1)(q^2-1)}.$$

Proposition 3.7 For a fixed subspace $V \in X$, the number of subspaces $V_1 \in X$ satisfying $(V, V_1) \in R_6$ is

$$n_6 = \frac{q^{l+2}(q^{2(\nu-1)}-1)(q^{l-1}-1)}{(q-1)^2}.$$

Proof: Suppose that the choices of subspaces V, V_1, $V + V_1$ are as same as (3.1), (3.4), (3.5), respectively. Since rank $\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix} = 2$ and

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix}$ is a subspace of type (2,0) in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, we can deduce that x_{11} is arbitrary, $x_{13}=0$ and $(x_{12}\ x_{14})$ is a 1-dimensional subspace in $2(\nu-1)$-dimensional symplectic space $\mathbb{F}_q^{2(\nu-1)}$. Additionally, from dim $(V+V_1)=4$, we can deduce that x_{15}, x_{16} and y_{15} are arbitrary, y_{16} is a 1-dimensional subspace in $(l-1)$-dimensional row vector space $\mathbb{F}_q^{(l-1)}$. So the number of subspaces V_1 is $q^{2+(l-1)}\frac{q(q^{2(\nu-1)}-1)}{q-1}\frac{(q^{l-1}-1)}{q-1}=\frac{q^{l+2}(q^{2(\nu-1)}-1)(q^{l-1}-1)}{(q-1)^2}$.

Hence, for a fixed subspace $V\in X$, the number of subspaces $V_1\in X$ satisfying $(V,V_1)\in R_6$ is

$$n_6=\frac{q^{l+2}(q^{2(\nu-1)}-1)(q^{l-1}-1)}{(q-1)^2}.$$

Proposition 3.8 For a fixed subspace $V\in X$, the number of subspaces $V_1\in X$ satisfying $(V,V_1)\in R_7$ is

$$n_7=\frac{q^{2\nu+l}(q^{l-1}-1)}{q-1}.$$

Proof: Suppose that the choices of subspaces V, V_1, $V+V_1$ are as same as (3.1), (3.4), (3.5), respectively. Since rank $\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix}=2$ and $\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix}$ is a subspace of type $(2,1)$ in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, we can deduce that $x_{13}\neq 0$, x_{11}, x_{12} and x_{14} are arbitrary. Additionally, from $\dim(V+V_1)=4$, we can deduce that x_{15}, x_{16} and y_{15} are arbitrary, y_{16} is a 1-dimensional subspace in $(l-1)$-dimensional row vector space $\mathbb{F}_q^{(l-1)}$. So the number of subspaces V_1 is

$$\frac{q(q-1)q^{2\nu+l-1}}{q-1}\frac{(q^{l-1}-1)}{q-1}=\frac{q^{2\nu+l}(q^{l-1}-1)}{q-1}.$$

Hence, for a fixed subspace $V\in X$, the number of subspaces $V_1\in X$ satisfying $(V,V_1)\in R_7$ is

$$n_7=\frac{q^{2\nu+l}(q^{l-1}-1)}{q-1}.$$

Theorem 3.2 The intersection numbers $p^i_{j,k}(1\le i,j,k\le 7)$ of the association scheme $\Gamma=(X,\{R_i\}_{0\le i\le 7})$ are listed as follows in the Table 1:

Table 1: The Intersection Numbers $p^i_{j,k}(1\le i,j,k\le 7)$

$p_{11}^1 = q(q^{l-2}-1)$	$p_{12}^1 = 0$	$p_{13}^1 = p_{14}^1 = p_{15}^1 = p_{16}^1 = p_{17}^1 = 0$
$p_{22}^1 = \frac{(q^{2(v-1)}-1)q^l}{q-1}$		$p_{23}^1 = p_{24}^1 = p_{25}^1 = p_{26}^1 = p_{27}^1 = 0$
$p_{33}^1 = q^{2v+l-2}$		$p_{34}^1 = p_{35}^1 = p_{36}^1 = p_{37}^1 = 0$
$p_{44}^1 = q^2$	$p_{45}^1 = \frac{q^3(q^{l-2}-1)}{q-1}$	$p_{46}^1 = p_{47}^1 = 0$
$p_{55}^1 = \frac{q^4(q^{l-2}-1)}{q-1}$		$p_{56}^1 = p_{57}^1 = 0$
$p_{66}^1 = \frac{q^{l+3}(q^{2(v-1)}-1)(q^{l-2}-1)}{(q-1)^2}$		$p_{67}^1 = 0$
$p_{77}^1 = \frac{q^{2v+l-1}(q^{l-2}-1)}{q-1}$		
$p_{22}^2 = \frac{q^{l+1}(q^{2(v-2)}-1)}{q-1}$	$p_{23}^2 = q^{2v+l-3}$	$p_{24}^2 = p_{25}^2 = p_{26}^2 = p_{27}^2 = 0.$
$p_{33}^2 = (q-1)q^{2v+l-3}$		$p_{34}^2 = p_{35}^2 = p_{36}^2 = p_{37}^2 = 0$
$p_{44}^2 = p_{45}^2 = p_{47}^2 = 0$		$p_{46}^2 = \frac{q^2(q^{l-1}-1)}{q-1}$
$p_{55}^2 = p_{57}^2 = 0$		$p_{56}^2 = \frac{q^2(q^{l-1}-1)(q^{l-2}-1)}{(q-1)(q^2-1)}$
$p_{66}^2 = \frac{q^{l+3}(q^{2(v-2)}-1)(q^{l-1}-1)}{(q-1)^2}$		$p_{67}^2 = \frac{q^{2v+l-1}(q^{l-1}-1)}{q-1}$
$p_{77}^2 = q^{2v+l-1}(q^{l-1}-1)$		
$p_{33}^3 = (q-1)q^{2v+l-3}$		$p_{34}^3 = p_{35}^3 = p_{36}^3 = p_{37}^3 = 0$
$p_{44}^3 = p_{45}^3 = p_{46}^3 = 0$		$p_{47}^3 = \frac{q(q^{l-1}-1)}{q-1}$
$p_{55}^3 = p_{56}^3 = 0$		$p_{57}^3 = \frac{q^2(q^{l-1}-1)(q^{l-2}-1)}{q^2-1}$
$p_{66}^3 = \frac{q^{l+1}(q^{2(v-1)}-1)(q^{l-1}-1)}{(q-1)^2}$		$p_{67}^3 = \frac{q^{l+1}(q^{2(v-1)}-1)(q^{l-1}-1)}{q-1}$
$p_{77}^3 = q^{2v+l-1}(q^{l-1}-1)$		
$p_{44}^4 = \frac{q^2(q^{l-2}-1)}{q-1}$	$p_{45}^4 = q(q^{l-1}-1)$	$p_{46}^4 = p_{47}^4 = 0$
$p_{55}^4 = \frac{q^4(q^{l-3}-1)(q^{l-2}-1)}{q^2-1}$		$p_{56}^5 = p_{57}^5 = 0$
$p_{66}^4 = \frac{q^{l+3}(q^{2(v-1)}-1)(q^{l-2}-1)}{(q-1)^2}$		$p_{67}^4 = 0$
$p_{77}^4 = \frac{q^{2v+l+1}(q^{l-2}-1)}{q-1}$		
$p_{55}^5 = \frac{q^6(q^{l-3}-1)(q^{l-4}-1)}{(q-1)(q^2-1)}$		$p_{56}^5 = p_{57}^5 = 0$
$p_{66}^5 = \frac{q^{l+5}(q^{2(v-1)}-1)(q^{l-3}-1)}{(q-1)^2}$		$p_{67}^5 = 0$
$p_{77}^5 = \frac{q^{2v+l+2}(q^{l-3}-1)}{q-1}$		
$p_{66}^6 = \frac{q^{l+4}(q^{l-2}-1)(q^{2(v-2)}-1)}{(q-1)^2}$	$p_{67}^6 = \frac{q^{2v+l}(q^{l-2}-1)}{q-1}$	$p_{77}^6 = q^{2v+l}(q^{l-2}-1)$
$p_{77}^7 = q^{2v+l}(q^{l-2}-1)$		

To prove Theorem 3.2, we have to prove the following propositions 3.9-3.15 in the following part. In this part, we only enumerate several classical intersection numbers $p_{j,k}^i (1 \le i,j,k \le 7)$.

Proposition 3.9 Suppose that V and W are two fixed subspaces of X with $(V,W) \in R_1$. Then the number of subspaces $U \in X$ satisfying $(V,U) \in R_1$ and $(U,W) \in R_1$ is

$$p_{11}^1 = q(q^{l-2}-1).$$

Proof: Suppose that the matrix representation form of the subspace V is as same as (3.1). Assume that W is a fixed subspace of type $(2,0,1)$ in $\mathbb{F}_q^{(2v+l)}$ with the matrix representation of the form

$$W=\begin{pmatrix} e_{2v+1} \\ e_1+e_{2v+2} \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.6)$$
$$\begin{matrix} 1 & v-1 & 1 & v-1 & 1 & 1 & l-2 \end{matrix}$$

then $(V,W) \in R_1$.

Suppose that the subspace U has the matrix representation of the form

$$U=\begin{pmatrix} e_{2v+1} \\ x \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & 0 & x_{16} & x_{17} \end{pmatrix}. \quad (3.7)$$
$$\begin{matrix} 1 & v-1 & 1 & v-1 & 1 & 1 & l-2 \end{matrix}$$

Since

$$\operatorname{rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix}=1,$$

we can assume that $(x_{11}\, x_{12}\, x_{13}\, x_{14})=(1\,0\,0\,0)$. Additionally, we can deduce that $(x_{16}\, x_{17})$ a 1-dimensional subspace in $(l-1)$-dimensional row vector space $\mathbb{F}_q^{(l-1)}$ from dim $(V+U)=3$. Then the subspace $U+W$ has the matrix representation of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & x_{16} & x_{17} \end{pmatrix}.$$
$$\begin{matrix} 1 & v-1 & 1 & v-1 & 1 & 1 & l-2 \end{matrix}$$

Furthermore, from $\dim(U+W)=3$, we can deduce that x_{16} is arbitrary and x_{17} is nonzero. So the number of subspaces U is

$$q(q^{l-2}-1).$$

Hence, for $(V,W) \in R_1$, the number of subspaces $U \in X$ satisfying $(V,U) \in R_1$ and $(U,W) \in R_1$ is

$$p_{11}^1=q(q^{l-2}-1).$$

Similarly, suppose that the matrix representation forms of the subspaces V, W, U are as same as (3.1), (3.6), (3.7), respectively, it can be readily verified that $p_{12}^1=p_{13}^1=0$.

Furthermore, suppose that the matrix representation forms of the subspaces V, W are as same as (3.1), (3.6), respectively. Suppose that the subspace U has the matrix representation of the form

$$U=\begin{pmatrix} y \\ x \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & y_{15} & y_{16} & y_{17} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} \end{pmatrix}. \quad (3.8)$$
$$\begin{matrix} 1 & v-1 & 1 & v-1 & 1 & 1 & l-2 \end{matrix}$$

It can be readily verified that $p_{14}^1 = p_{15}^1 = p_{16}^1 = p_{17}^1 = 0$ and the value of $p_{j,k}^1 (2 \leq j \leq k \leq 7)$ as listed in the table 1.

Proposition 3.10 Suppose that V and W are two fixed subspaces of X with $(V,W) \in R_2$. Then the number of subspaces $U \in X$ satisfying $(V,U) \in R_2$ and $(U,W) \in R_2$ is

$$p_{22}^2 = \frac{q^{l+1}(q^{2(\nu-2)} - 1)}{q-1}.$$

Proof: Suppose that the matrix representation form of the subspace V is as same as (3.1). Assume that W is a fixed subspace of type $(2,0,1)$ in $\mathbb{F}_q^{(2\nu+l)}$ with the matrix representation of the form

$$W = \begin{pmatrix} e_2 \\ e_{2\nu+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.9)$$
$$\qquad\qquad\qquad 1 \quad 1 \quad \nu-2 \quad 1 \quad \nu-1 \quad 1 \quad l-1$$

then $(V,W) \in R_2$.

Suppose that the subspace U has the matrix representation of the form

$$U = \begin{pmatrix} e_{2\nu+1} \\ x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & 0 & x_{18} \end{pmatrix}. \quad (3.10)$$
$$\qquad\qquad\qquad 1 \quad 1 \quad \nu-2 \quad 1 \quad 1 \quad \nu-2 \quad 1 \quad l-1$$

Then the subspaces $V+U$, $U+W$ can be represented by the following matrixes of the forms

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & 0 & x_{18} \end{pmatrix}, \quad (3.11)$$
$$\qquad 1 \quad 1 \quad \nu-2 \quad 1 \quad 1 \quad \nu-2 \quad 1 \quad l-1$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & 0 & x_{18} \end{pmatrix}, \quad (3.12)$$
$$\qquad 1 \quad 1 \quad \nu-2 \quad 1 \quad 1 \quad \nu-2 \quad 1 \quad l-1$$

respectively.

We can assume that the subspace $U \in X$ satisfying $(V,U) \in R_2$ and $(U,W) \in R_2$ must be like that

$$\begin{cases} x_{14} = x_{15} = 0 \\ x_{11},\ x_{12},\ x_{18} \text{ are arbitrary} \\ (x_{13}\ x_{16}) \text{ is a 1-dimensional subspace in } \mathbb{F}_q^{2(\nu-2)} \end{cases}.$$

Therefore, the number of subspaces U is $\frac{q^{l+1}(q^{2(\nu-2)}-1)}{q-1}$.

Hence, for $(V,W)\in R_2$, the number of subspaces $U\in X$ satisfying $(V,U)\in R_2$ and $(U,W)\in R_2$ is

$$p_{22}^2=\frac{q^{l+1}(q^{2(\nu-2)}-1)}{q-1}.$$

Similarly, suppose that the matrix representation forms of the subspaces V, W, U are as same as (3.1), (3.9), (3.10), respectively, it can be readily verified that $p_{23}^2=q^{2\nu+l-3}$.

Furthermore, suppose that the matrix representation forms of the subspaces V, W are as same as (3.1), (3.9) respectively. Suppose that the subspace U has the matrix representation of the form

$$U=\begin{pmatrix} y \\ x \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & y_{17} & y_{18} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \end{pmatrix}. \qquad (3.13)$$
$$\begin{matrix} 1 & 1 & \nu-2 & 1 & 1 & \nu-2 & 1 & l-1 \end{matrix}$$

It can be readily verified that $p_{24}^2=p_{25}^2=p_{26}^2=p_{27}^2=0$ and the value of $p_{j,k}^2(3\le j\le k\le 7)$ as listed in the table 1.

Proposition 3.11 Suppose that V and W are two fixed subspaces of X with $(V,W)\in R_3$. Then the number of subspaces $U\in X$ satisfying $(V,U)\in R_3$ and $(U,W)\in R_3$ is

$$p_{33}^3=(q-1)q^{2\nu+l-3}.$$

Proof: Suppose that the matrix representation form of the subspace V is as same as (3.1). Assume that W is a fixed subspace of type $(2,0,1)$ in $\mathbb{F}_q^{(2\nu+l)}$ with the matrix representation of the form

$$W=\begin{pmatrix} e_{\nu+1} \\ e_{2\nu+1} \end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad (3.14)$$
$$\begin{matrix} 1 & \nu-1 & 1 & \nu-1 & 1 & l-1 \end{matrix}$$

then $(V,W)\in R_3$.

Suppose that U has the matrix representation of the form

$$U=\begin{pmatrix} e_{2\nu+1} \\ x \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & 0 & x_{16} \end{pmatrix}. \qquad (3.15)$$
$$\begin{matrix} 1 & \nu-1 & 1 & \nu-1 & 1 & l-1 \end{matrix}$$

Since rank $\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix}=2$ and $\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix}$ is a subspace of type $(2,1)$ in 2ν-dimensional symplectic space $\mathbb{F}_q^{(2\nu)}$, it follows that $x_{13}\ne 0$, x_{11}, x_{12} and x_{14} are arbitrary. Additionally, to make sure that $(U,W)\in R_3$, we also need $x_{11}\ne 0$. Therefore, the number of subspaces U is

$$\frac{(q-1)^2q^{2\nu+l-3}}{q-1}=(q-1)q^{2\nu+l-3}.$$

Hence, for $(V,W)\in R_3$, the number of subspaces $U\in X$ satisfying $(V,U)\in R_3$ and $(U,W)\in R_3$ is

$$p_{33}^3=(q-1)q^{2\nu+l-3}.$$

Similarly, suppose that the matrix representation forms of the subspaces V, W, U are as same as (3.1), (3.14), (3.15), respectively, it can be readily verified that $p_{34}^3=0$.

Furthermore, suppose that the matrix representation forms of the subspaces of V, W are as same as (3.1), (3.14), respectively. Suppose that the subspace U has the matrix representation of the form

$$U=\begin{pmatrix} y \\ x \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & y_{15} & y_{16} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \end{pmatrix}. \qquad (3.16)$$
$$\begin{matrix} 1 & \nu-1 & 1 & \nu-1 & 1 & l-1 \end{matrix}$$

It can be readily verified that $p_{35}^3=p_{36}^3=p_{37}^3=0$ and the value of $p_{j,k}^3(4\le j\le k\le 7)$ as listed in the table 1.

Proposition 3.12 Suppose that V and W are two fixed subspaces of X with $(V,W)\in R_4$. Then the number of subspaces $U\in X$ satisfying $(V,U)\in R_4$ and $(U,W)\in R_4$ is

$$p_{44}^4=\frac{q^2(q^{l-2}-1)}{q-1}.$$

Proof: Suppose that the matrix representation form of the subspace V is as same as (3.1). Assume that W is a fixed subspace of type $(2,0,1)$ in $\mathbb{F}_q^{(2\nu+l)}$ with the matrix representation of the form

$$W=\begin{pmatrix} e_1 \\ e_{2\nu+2} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad (3.17)$$
$$\begin{matrix} 1 & \nu-1 & 1 & \nu-1 & 1 & 1 & l-2 \end{matrix}$$

then $(V,W)\in R_4$.

Suppose that U has the matrix representation of the form

$$U=\begin{pmatrix} y \\ e_1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & y_{15} & y_{16} & y_{17} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (3.18)$$
$$\begin{matrix} 1 & \nu-1 & 1 & \nu-1 & 1 & 1 & l-2 \end{matrix}$$

Since $\dim(V\cap U\cap E)=0$ and $\dim(V+U)=3$, we can deduce that y_{15} is arbitrary and $(y_{16}\ y_{17})$ is a 1-dimensional subspace in $(l-1)$-dimensional row vector space $\mathbb{F}_q^{(l-1)}$. Additionally, to make sure that $(U,W)\in R_4$, we also need $y_{16}\neq 0$ and y_{17} is a 1-dimensional subspace in $(l-2)$-dimensional row vector space $\mathbb{F}_q^{(l-2)}$. Therefore, the number of subspaces U is

$$\frac{q^2(q^{l-2}-1)}{q-1}.$$

Hence, for $(V,W)\in R_4$, the number of subspaces $U\in X$ satisfying $(V,U)\in R_4$ and $(U,W)\in R_4$ is

$$p^4_{44}=\frac{q^2(q^{l-2}-1)}{q-1}.$$

Similarly, suppose that the matrix representation forms of the subspaces V, W are as same as (3.1), (3.17), respectively. Suppose that the subspace U has the matrix representation of the form

$$U=\begin{pmatrix} y \\ x \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & y_{15} & y_{16} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \end{pmatrix}. \tag{3.19}$$
$$\quad\quad\quad\quad 1 \quad v-1 \quad 1 \quad v-1 \quad 1 \quad l-1$$

It can be readily verified $p^4_{45}=q(q^{l-1}-1)$, $p^4_{46}=p^4_{47}=0$ and the value of $p^4_{j,k}(5\le j\le k\le 7)$ as listed in the table 1.

Proposition 3.13 Suppose that V and W are two fixed subspaces of X with $(V,W)\in R_5$. Then the number of subspaces $U\in X$ satisfying $(V,U)\in R_5$ and $(U,W)\in R_5$ is

$$p^5_{55}=\frac{q^6(q^{l-3}-1)(q^{l-4}-1)}{(q-1)(q^2-1)}.$$

Proof: Suppose that the matrix representation form of the subspace V is as same as (3.1). Assume that W is a fixed subspace of type $(2,0,1)$ in $\mathbb{F}_q^{(2v+l)}$ with the matrix representation of the form

$$W=\begin{pmatrix} e_1+e_{2v+2} \\ e_{2v+3} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{3.20}$$
$$\quad\quad\quad\quad 1 \quad v-1 \quad 1 \quad v-1 \quad 1 \quad 1 \quad 1 \quad l-3$$

then $(V,W)\in R_5$.

Suppose that the subspace U has the matrix representation of the form

$$U=\begin{pmatrix} y \\ x \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & y_{15} & y_{16} & y_{17} & y_{18} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \end{pmatrix}. \tag{3.21}$$
$$\quad\quad\quad\quad 1 \quad v-1 \quad 1 \quad v-1 \quad 1 \quad 1 \quad 1 \quad l-3$$

To make sure that $(V,U)\in R_5$, $(U,W)\in R_5$, we can assume that

$$\begin{cases} x_{15},\ x_{16},\ x_{17},\ y_{15},\ y_{16},\ y_{17} \text{ are arbitrary} \\ (x_{11}\ x_{12}\ x_{13}\ x_{14})=(1\ 0\ 0\ 0) \\ \begin{pmatrix} y_{18} \\ x_{18} \end{pmatrix} \text{ is a 2-dimensional subspace in } \mathbb{F}_q^{(l-3)} \end{cases}.$$

Therefore, the number of subspaces U is

$$q^6\begin{bmatrix} l-3 \\ 2 \end{bmatrix}_q=\frac{q^6(q^{l-3}-1)(q^{l-4}-1)}{(q-1)(q^2-1)}.$$

Hence, for $(V,W) \in R_5$, the number of subspaces $U \in X$ satisfying $(V,U) \in R_5$ and $(U,W) \in R_5$ is

$$p_{55}^{5} = \frac{q^6(q^{l-3}-1)(q^{l-4}-1)}{(q-1)(q^2-1)}.$$

Similarly, suppose that the matrix representation forms of the subspaces V, W, U are as same as (3.1), (3.20), (3.21), respectively. It can be readily verified $p_{56}^{5} = p_{57}^{5} = 0$ and the value of $p_{j,k}^{5} (6 \le j \le k \le 7)$ as listed in the table 1.

Proposition 3.14 Suppose that V and W are two fixed subspaces of X with $(V,W) \in R_6$. Then the number of subspaces $U \in X$ satisfying $(V,U) \in R_6$ and $(U,W) \in R_6$ is

$$p_{66}^{6} = \frac{q^{l+4}(q^{l-2}-1)(q^{2(\nu-2)}-1)}{(q-1)^2}.$$

Proof: Suppose that the matrix representation form of the subspace V is as same as (3.1). Assume that W is a fixed subspace of type $(2,0,1)$ in $\mathbb{F}_q^{(2\nu+l)}$ with the matrix representation of the form

$$W = \begin{pmatrix} e_2 \\ e_{2\nu+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.22)$$
$$\begin{matrix} 1 & 1 & \nu-2 & 1 & \nu-1 & 1 & 1 & l-2 \end{matrix}$$

then $(V,W) \in R_6$.

Suppose that the subspace U has the matrix representation of the form

$$U = \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & y_{17} & y_{18} & y_{19} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \end{pmatrix}. \quad (3.23)$$
$$\begin{matrix} 1 & 1 & \nu-2 & 1 & 1 & \nu-2 & 1 & 1 & l-2 \end{matrix}$$

To make sure that $(V,U) \in R_6$, $(U,W) \in R_6$, we can assume that

$$\begin{cases} x_{11}, x_{12}, x_{17}, x_{18}, x_{19}, y_{17}, y_{18} \text{ are arbitrary} \\ x_{14} = x_{15} = 0 \\ (x_{13}\ x_{16}) \text{ is a 1-dimensional subspace in } \mathbb{F}_q^{2(\nu-2)} \\ y_{19} \text{ is a 1-dimensional subspace in } \mathbb{F}_q^{(l-2)} \end{cases}.$$

Therefore, the number of subspaces U is

$$q^{l+4}\frac{q^{2(\nu-2)}-1}{q-1}\frac{q^{l-2}-1}{q-1} = \frac{q^{l+4}(q^{l-2}-1)(q^{2(\nu-2)}-1)}{(q-1)^2}.$$

Hence, for $(V,W) \in R_6$, the number of subspaces $U \in X$ satisfying $(V,U) \in R_6$ and $(U,W) \in R_6$ is

$$p_{66}^{6} = \frac{q^{l+4}(q^{l-2}-1)(q^{2(\nu-2)}-1)}{(q-1)^2}.$$

Similarly, suppose that the matrix representation forms of the subspaces V, W, U are as same as (3.1), (3.22), (3.23), respectively. We can get

$$p_{67}^{6}=\frac{q^{2\nu+l}(q^{l-2}-1)}{q-1}, \qquad p_{77}^{6}=q^{2\nu+l}(q^{l-2}-1).$$

Proposition 3.15 Suppose that V and W are two fixed subspaces of X with $(V,W)\in R_7$. Then the number of subspaces $U\in X$ satisfying $(V,U)\in R_7$ and $(U,W)\in R_7$ is

$$p_{77}^{7}=q^{2\nu+l}(q^{l-2}-1).$$

Proof: Suppose that the matrix representation form of the subspace V is as same as (3.1). Assume that W is a fixed subspace of type $(2,0,1)$ in $\mathbb{F}_q^{(2\nu+l)}$ with the matrix representation of the form

$$W=\begin{pmatrix} e_{\nu+1} \\ e_{2\nu+2} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tag{3.24}$$
$$\begin{matrix} 1 & \nu-1 & 1 & \nu-1 & 1 & 1 & l-2 \end{matrix}$$

then $(V,W)\in R_7$.

Suppose that the subspace U has the matrix representation of the form

$$U=\begin{pmatrix} y \\ x \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 & y_{15} & y_{16} & y_{17} \\ x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} \end{pmatrix}. \quad \tag{3.25}$$
$$\begin{matrix} 1 & \nu-1 & 1 & \nu-1 & 1 & 1 & l-2 \end{matrix}$$

To make sure that $(V,U)\in R_7$, $(U,W)\in R_7$, we can assume that

$$\begin{cases} x_{11}\neq 0,\ x_{13}\neq 0 \\ x_{12},\ x_{14},\ x_{15},\ x_{16},\ y_{15},\ y_{16} \text{ are arbitrary} \\ y_{17} \text{ is a 1-dimensional subspace in } \mathbb{F}_q^{(l-2)} \end{cases}.$$

Therefore, the number of subspaces U is $\frac{(q-1)^2q^{2\nu+l}(q^{l-2}-1)}{(q-1)^2}=q^{2\nu+l}(q^{l-2}-1)$.

Hence, for $(V,W)\in R_7$, the number of subspaces $U\in X$ satisfying $(V,U)\in R_7$ and $(U,W)\in R_7$ is

$$p_{77}^{7}=q^{2\nu+l}(q^{l-2}-1).$$

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