A NOTE ON BI-NORMAL CAYLEY GRAPHS

JING JIAN LI, ZAI PING LU, AND GAIXIA WANG

Abstract. The aim of this paper is to answer a question proposed by Li [2] and prove that no connected bi-normal Cayley graph other than cycles of even length is 3-arc-transitive.

Keywords. Cayley graph, bi-Cayley graph, s-arc-transitive graph.

1. Introduction

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. We use $\mathsf{Aut}(\Gamma)$ to denote the automorphism group of Γ. The graph Γ is said to be *s-arc-transitive* if it has at least one s-arc and $\mathsf{Aut}(\Gamma)$ is transitive on both the vertices and the s-arcs of Γ, where an s-arc means a sequence $v_0, v_1, \cdots, v_s$ of $s+1$ vertices such that $\{v_{i-1}, v_i\} \in E(\Gamma)$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. A graph is said to be *s-transitive* if it is s-arc-transitive but not $(s+1)$-arc-transitive.

Let G be a finite group and S be a subset of G with $1 \notin S = S^{-1} := \{s^{-1} \mid s \in S\}$. The Cayley graph $\mathsf{Cay}(G, S)$ of G with respect to S is defined as the graph with vertex set G and edge set $\{\{x, y\} \mid yx^{-1} \in S\}$. Then $\mathsf{Cay}(G, S)$ admits a group $\hat{G} := \{\hat{g} : x \mapsto xg,\ x \in G \mid g \in G\}$ acting regularly on the vertices. The Cayley graph $\mathsf{Cay}(G, S)$ is said to be *normal* if $\hat{G}$ itself is normal in $\mathsf{Aut}(\mathsf{Cay}(G, S))$, or *bi-normal* if $\hat{G}$ has a subgroup of index 2 which is normal in $\mathsf{Aut}(\mathsf{Cay}(G, S))$.

The aim of this paper is to answer a question posed by Li [2]. For $s \geq 2$, Li [2] gave a characterization of s-transitive Cayley graphs. He proved that each connected s-transitive Cayley graph is normal with $s = 2$, or bi-normal (so bipartite) with $s \leq 3$, or a normal cover of one of finite number of graphs. Then the following interesting question was proposed:

Question 1.1. *Do there exist 3-transitive bi-normal Cayley graphs?*

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E-mail address: ljj@cfc.nankai.edu.cn(J.J. Li),
lu@nankai.edu.cn(Z.P. Lu),
wgx@cfc.nankai.edu.cn(G.X. Wang).

In this paper, we shall prove the following result.

Theorem 1.2. *Let Γ be a connected bi-normal Cayley graph of valency at least 3. Then Γ is not 3-arc-transitive.*

2. Bi-Cayley graphs

Note that a bi-normal Cayley graph is a bipartite graph admits a group acting regularly on both of the bipartition subsets. It is easily shown that such a graph is isomorphic to a bi-Cayley graph defined in the following.

Let G be a finite group and $S \subseteq G$ which possibly contains the identity element of G. The bi-Cayley graph, denoted by $\mathrm{BCay}(G, S)$, is defined to be the graph with vertex set $G \times \{l, r\}$ and edge set $\{\{(x, l), (y, r)\} \mid x, y \in G, yx^{-1} \in S\}$. Then $\mathrm{BCay}(G, S)$ is a well-defined bipartite graph with two bipartition subsets $G \times \{l\}$ and $G \times \{r\}$.

Let $\Gamma = \mathrm{BCay}(G, S)$ be a bi-Cayley graph. It is easily shown that $\mathrm{BCay}(G, S)$ is connected if and only if $G = \langle SS^{-1}\rangle$ (if and only if $G = \langle S^{-1}S\rangle$), see [1] for example. For each $g \in G$, we define

$$\hat{g} : G \times \{l, r\} \to G \times \{l, r\},\ (x, i) \mapsto (xg, i) \text{ for } i = l, r.$$

It is easy to see that $\hat{g}$ is an automorphism of $\mathrm{BCay}(G, S)$. Set $\hat{G} = \{\hat{g} \mid g \in G\}$. Then $g \mapsto \hat{g}$ gives an isomorphism from G to $\hat{G}$, and $\hat{G}$ acts regularly on both $G \times \{l\}$ and $G \times \{r\}$. In the following we shall consider the normalizer $N := \mathbf{N}_{\mathsf{Aut}(\Gamma)}(\hat{G})$ of $\hat{G}$ in $\mathsf{Aut}(\Gamma)$.

Let $N^+ = N_{G\times\{l\}}$, the set-wise stabilizer of $G \times \{l\}$ in N. Then $N^+ = N_{G\times\{r\}}$, and either $N = N^+$ or N is transitive on $G \times \{l, r\}$. Noting that $\hat{G}$ is normal in N, it is easily shown that N^+ has index no more than 2 in N. Further, $N_{(1,i)} \le N^+$ for $i = l, r$, where $N_{(1,i)}$ is the stabilizer of $(1, i)$ in N.

Now we consider the point-wise stabilizers of $(1, l)$ and of $\{(1, l), (1, r)\}$ in N. For $\sigma \in \mathsf{Aut}(G)$ and $h \in G$, we define $\hat{\sigma}$, $\tilde{h}_l$ and $\tilde{h}_r$ as follows:

$$\begin{array}{llll} \hat{\sigma} : G \times \{l, r\} \to G \times \{l, r\}; & (x, l) \mapsto (x^\sigma, l), & (x, r) \mapsto (x^\sigma, r), \\ \tilde{h}_l : G \times \{l, r\} \to G \times \{l, r\}; & (x, l) \mapsto (h^{-1}x, l), & (x, r) \mapsto (x, r), \\ \tilde{h}_r : G \times \{l, r\} \to G \times \{l, r\}; & (x, l) \mapsto (x, l), & (x, r) \mapsto (h^{-1}x, r). \end{array}$$

Then $\hat{\sigma}$, $\tilde{h}_l$ and $\tilde{h}_r$ are well-defined permutations on $G\times\{l, r\}$ and fix $G\times\{l\}$ set-wise. Further, we have the following lemma (see also [3, 4]).

Lemma 2.1. *Let $g, h, k \in G$, $\sigma, \tau \in \mathsf{Aut}(G)$ and $\Gamma = \mathrm{BCay}(G, S)$. Then*

(1) $\widehat{\sigma\tau} = \hat{\sigma}\hat{\tau}$, $\hat{\sigma}^{-1}\hat{g}\hat{\sigma} = \widehat{g^\sigma}$, $\tilde{h}_l\hat{g} = \hat{g}\tilde{h}_l$, $\tilde{h}_r\hat{g} = \hat{g}\tilde{h}_r$ *and* $\tilde{h}_l\tilde{k}_r = \tilde{k}_r\tilde{h}_l$;

(2) $\hat{\sigma}\tilde{h}_l\tilde{k}_r$ *is an isomorphism from* $\mathrm{BCay}(G, S)$ *to* $\mathrm{BCay}(G, k^{-1}S^\sigma h)$;

(3) $\hat{\sigma}\tilde{h}_l\tilde{k}_r \in N^+$ *if and only if* $S = k^{-1}S^\sigma h$;

(4) $\hat{\sigma}\tilde{h}_l\tilde{k}_r \in N_{(1,l)}$ *if and only if* $h = 1$ *and* $S = k^{-1}S^\sigma$;

(5) $\hat{\sigma}\tilde{h}_l\tilde{k}_r \in N_{(1,r)}$ *if and only if* $k = 1$ *and* $S = S^\sigma h$;

(6) $\hat{\sigma}\tilde{h}_l\tilde{k}_r \in N_{(1,l)(1,r)}$ *if and only if* $h = k = 1$ *and* $S^\sigma = S$.

Proof. For any $x \in G$ and $i = l, r$, we have

$$(x,i)^{\widehat{\sigma\tau}} = ((x^\sigma)^\tau, i) = (x^\sigma, i)^{\hat{\tau}} = (x,i)^{\hat{\sigma}\hat{\tau}},$$
$$(x,i)^{\hat{g}\hat{\sigma}} = (xg,i)^{\hat{\sigma}} = ((xg)^\sigma, i) = (x,i)^{\hat{\sigma}\widehat{g^\sigma}}.$$

It follows that the first equation in (1) holds. One may easily check the other three equations in (1).

Set $\omega = \hat{\sigma}\tilde{h}_l\tilde{k}_r$. For $x, y \in G$, we have

$$\begin{aligned} & \{(x,l)^\omega, (y,r)^\omega\} = \{(h^{-1}x^\sigma, l), (k^{-1}y^\sigma, r)\} \in E(\mathrm{BCay}(G, k^{-1}S^\sigma h)) \\ \Leftrightarrow\ & k^{-1}y^\sigma(x^\sigma)^{-1}h \in k^{-1}S^\sigma h \Leftrightarrow (yx^{-1})^\sigma = y^\sigma(x^\sigma)^{-1} \in S^\sigma \Leftrightarrow yx^{-1} \in S \\ \Leftrightarrow\ & \{(x,l),(y,r)\} \in E(\Gamma). \end{aligned}$$

It implies that ω is an isomorphism from Γ to $\mathrm{BCay}(G, k^{-1}S^\sigma h)$.

Note that ω fixes $G \times \{l\}$ set-wise and that ω normalizes $\hat{G}$ by (1). Then $\omega \in N^+$ if and only if $\omega \in \mathrm{Aut}(\Gamma)$. If $S = k^{-1}S^\sigma h$ then, by (2), ω is an automorphism of Γ. Assume $\omega \in \mathrm{Aut}(\Gamma)$. Then ω maps the neighborhood $S \times \{r\}$ of $(1,l)$ onto the neighborhood $Sh^{-1} \times \{r\}$ of $(1,l)^\omega = (h^{-1}, l)$. Noting $(S \times \{r\})^\omega = k^{-1}S^\sigma \times \{r\}$, we get $Sh^{-1} = k^{-1}S^\sigma$, and so $S = k^{-1}S^\sigma h$.

Note that $\hat{\sigma}$ fixes both $(1,l)$ and $(1,r)$, $\tilde{h}_l$ fixes $(1,r)$ and $\tilde{k}_r$ fixes $(1,l)$. Then (4), (5) and (6) hold. □

Remark 2.2. *By Lemma* 2.1 (2), *we get* $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G, k^{-1}S) \cong \mathrm{BCay}(G, Sh)$ *for any* $h, k \in G$. *In particular,* $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G, s^{-1}S)$ *for any* $s \in S$. *Thus, for a bi-Cayley graph* $\mathrm{BCay}(G,S)$, *one may assume that* S *contains the identity element of* G.

Theorem 2.3. (1) $N^+ = \{\hat{\sigma}\tilde{h}_l\tilde{k}_r \mid h,k \in G, \sigma \in \mathrm{Aut}(G),\ S = k^{-1}S^\sigma h\}$;

(2) $N_{(1,l)} = \{\hat{\sigma}\tilde{k}_r \mid \sigma \in \mathrm{Aut}(G),\ k \in G,\ S = k^{-1}S^\sigma\}$;

(3) $N_{(1,r)} = \{\hat{\sigma}\tilde{h}_l \mid \sigma \in \mathrm{Aut}(G),\ h \in G,\ S = S^\sigma h\}$;

(4) $N_{(1,l)(1,r)} = \{\hat{\sigma} \mid \sigma \in \mathrm{Aut}(G),\ S^\sigma = S\}$.

(5) *If* $\Gamma = \mathrm{BCay}(G,S)$ *is connected, then* $N_{(1,i)}$ *acts faithfully on the neighborhood of* $(1,i)$ *in* Γ, *where* $i = l, r$;

Proof. Let $\nu \in N$. Then ν normalizes $\hat{G}$ and so, for any $x \in G$, we have $\nu^{-1}\hat{x}\nu = \widehat{x'}$ for some $x' \in G$. Define $\sigma : G \to G$; $x \mapsto x'$. It is easily shown that σ is a well-defined bijection on G. For $x, y \in G$, we have

$$\begin{aligned} ((xy)^\sigma, l) &= ((xy)', l) = (1,l)^{\widehat{(xy)'}} = (1,l)^{\nu^{-1}\widehat{xy}\nu} \\ &= (1,l)^{\nu^{-1}\hat{x}\hat{y}\nu} = (1,l)^{\widehat{x'}\widehat{y'}} = (x'y', l) = (x^\sigma y^\sigma, l), \end{aligned}$$

and so $(xy)^\sigma = x^\sigma y^\sigma$. It implies $\sigma \in \mathrm{Aut}(G)$.

Assume $\nu \in N^+$. Then we may set $(1,l)^\nu = (h^{-1},l)$ and $(1,r)^\nu = (k^{-1},r)$ for some $h,k \in G$. Then $\nu = \hat{\sigma}\tilde{h}_l\tilde{k}_r$ follows from

$$(x,l)^{\hat{\sigma}\tilde{h}_l\tilde{k}_r} = (h^{-1}x^\sigma,l) = (h^{-1},l)^{\hat{x}'} = (h^{-1},l)^{\nu^{-1}\hat{x}\nu} = (1,l)^{\hat{x}\nu} = (x,l)^\nu,$$
$$(x,r)^{\hat{\sigma}\tilde{h}_l\tilde{k}_r} = (k^{-1}x^\sigma,r) = (k^{-1},r)^{\hat{x}'} = (k^{-1},r)^{\nu^{-1}\hat{x}\nu} = (1,r)^{\hat{x}\nu} = (x,r)^\nu.$$

By Lemma 2.1 (3), we have $S = k^{-1}S^\sigma h$, and so (1) of the present theorem holds. Recall that $N_{(1,i)} \leq N^+$ for $i = l,r$. Then (2), (3) and (4) of the present theorem follow from (1) and Lemma 2.1.

Now, by Remark 2.2, we may assume that S contains the identity element of G. Thus $(1,j)$ belongs to the neighborhood of $(1,i)$, where $\{i,j\} = \{l,r\}$. Since $1 \in S$ and Γ is connected, we have $G = \langle S^{-1}S\rangle = \langle S\rangle$. Noting that $S \times \{r\}$ is the neighborhood of $(1,l)$ and $S^{-1} \times \{l\}$ is the neighborhood of $(1,r)$, it follows from (4) that the stabilizer $N_{(1,i)(1,j)}$ of $(1,j)$ in $N_{(1,i)}$ acts faithfully on $S \times \{r\}$ for $i = l$ and on $S^{-1} \times \{l\}$ for $i = r$. Thus $N_{(1,i)}$ is faithful on the neighborhood of $(1,i)$. □

Note that $(s,r)^{\hat{\sigma}} = (s,r)$ implies $s^\sigma = s$ and $(s^{-1})^\sigma = s^{-1}$ for $s \in S$ and $\sigma \in \mathsf{Aut}(G)$ with $S^\sigma = S$. We have the following corollary.

Corollary 2.4. *Assume that* $|S \cup \{1\}| \geq 3|$. *If* $1 \neq s \in S$, *then* $N_{(1,r)(1,l)(s,r)}$ *is not transitive on* $S^{-1} \times \{l\} \setminus \{(1,l)\}$. *In particular, if* $1 \in S$, *then* $N_{(1,r)(1,l)}$ *is not transitive on the 3-arcs in* $\mathrm{BCay}(G,S)$ *which contains the arc* $((1,r),(1,l))$.

3. Proof of Theorem 1.2

Let Γ be a connected bi-normal Cayley graph. Then $\mathsf{Aut}(\Gamma)$ has a normal subgroup, say G, which is semiregular and has exactly two orbits on $V(\Gamma)$. Note that these two G-orbits give an $\mathsf{Aut}(\Gamma)$-invariant partition of $V(\Gamma)$. It follows that either Γ is not arc-transitive, or Γ is a bipartite graph and those two G-orbits are the bipartition subsets of Γ. Then the following argument completes the proof of Theorem 1.2.

Now let Γ be a connected bipartite graph with $G \leq \mathsf{Aut}(\Gamma)$ acting regularly on both two bipartition subsets U_l and U_r of Γ. Then it is easily shown that Γ is a regular graph. Let $\{u_l,u_r\} \in E(\Gamma)$ with $u_l \in U_l$ and $u_r \in U_r$. Then each vertex in U_i can be written uniquely as u_i^x for some $x \in G$, where $i = l,r$. Define

$$\phi : V(\Gamma) \to G \times \{l,r\};\ u_i^x \mapsto (x,i),\ i = l,r.$$

Then ϕ is a bijection. Set $S = \{s \in G \mid \{u_l,u_r^s\} \in E(\Gamma)\}$. Then $1 \in S$ and, for $x,y \in G$, we have

$$\{u_l^x,u_r^y\} \in E(\Gamma) \Leftrightarrow \{u_l,u_r^{yx^{-1}}\} \in E(\Gamma) \Leftrightarrow yx^{-1} \in S.$$

Thus ϕ is an isomorphism from Γ to the bi-Cayley graph $\mathrm{BCay}(G, S)$. Further, $\phi^{-1}g\phi = \hat{g} \in \hat{G}$ for all $g \in G$, and $X \leq \mathrm{N}_{\mathsf{Aut}(\Gamma)}(G)$ implies $\phi^{-1}X\phi \leq \mathrm{N}_A(\hat{G})$, where $A = \mathsf{Aut}(\mathrm{BCay}(G, S))$. Assume further that Γ is not a cycle. Then $|S| = |S \cup \{1\}| \geq 3$. It follows from Corollary 2.4 that X is intransitive on the 3-arcs of Γ for $X \leq \mathsf{Aut}(\Gamma)$ (with $G \leq X \leq \mathrm{N}_{\mathsf{Aut}(\Gamma)}(G)$). In particular, Γ is not 3-arc-transitive if it is a connected bi-normal Cayley graph. This completes the proof of Theorem 1.2.

The following result is a consequence of the above argument.

Theorem 3.1. *Let Γ be a connected bipartite graph with $G \leq \mathsf{Aut}(\Gamma)$ acting regularly on both two bipartition subsets of Γ. If $X \leq \mathrm{N}_{\mathsf{Aut}(\Gamma)}(G)$, then X is intransitive on the 3-arcs of Γ except that Γ is a cycle of even length.*

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Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China

E-mail address: ljj@cfc.nankai.edu.cn

Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China

E-mail address: lu@nankai.edu.cn

Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China

E-mail address: wgx@cfc.nankai.edu.cn