On Metric Dimension of Generalized Petersen Graphs $P(n, 3)$*

Muhammad Imran[1], A.Q. Baig[2], M. K. Shafiq[2], Ioan Tomecu[3]

[1] Center for Advanced Mathematics and Physics (CAMP), National University of Science and Technology (NUST) Sector H-12, Islamabad, Pakistan
[2] Department of Mathematics, GC University Faisalabad, Pakistan
[3] Faculty of Mathematics and Computer Science, University of Bucharest Str. Academiei, 14, 010014 Bucharest, Romania
E-mail:{imrandhab, aqbaig1, kashif4v}@gmail.com, ioan@fmi.unibuc.ro

Abstract. In this paper we study the metric dimension of the generalized Petersen graphs $P(n, 3)$ by giving a partial answer to an open problem raised in [8]: Is $P(n, m)$ for $n \geq 7$ and $3 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, a family of graphs with constant metric dimension? We prove that the generalized Petersen graphs $P(n, 3)$ have metric dimension equal to 3 for $n \equiv 1 (\text{mod } 6)$, $n \geq 25$, and to 4 for $n \equiv 0 (\text{mod } 6)$, $n \geq 24$. For the remaining cases only 4 vertices appropriately chosen suffice to resolve all the vertices of $P(n, 3)$, thus implying that $dim(P(n, 3)) \leq 4$, except when $n \equiv 2 (\text{mod } 6)$, when $dim(P(n, 3)) \leq 5$.

Keywords: *Metric dimension, basis, resolving set, generalized Petersen graph*

1 Notation and auxiliary results

If G is a connected graph, the distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v|W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), d(v, w_3), \ldots, d(v, w_k))$. W is called a resolving set [4] or locating set [12] if

* This research is partially supported by NUST, Islamabad and Higher Education Commission of Pakistan.

every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a basis for G and this cardinality is the metric dimension of G, denoted by $dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [1, 4, 5, 7, 9, 11–13]).

For a given ordered set of vertices $W = \{w_1, w_2, \ldots, w_k\}$ of a graph G, the ith component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \backslash W$.

A useful property in finding $dim(G)$ is the following:

Lemma 1. *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap W \neq \emptyset$.*

Slater refereed to the metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set ([12], [13]). These concepts have also some applications in chemistry for representing chemical compounds ([4], [9]) or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [11].

By denoting $G + H$ the join of G and H a wheel W_n is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a fan is $F_n = K_1 + P_n$ for $n \geq 1$ and Jahangir graph $J_{2n}, (n \geq 2)$ (also known as gear graph) is obtained from the wheel W_{2n} by alternately deleting n spokes. Buczkowski *et al.* [1] determined the dimension of wheels W_n, Cáceres *et al.* [3] the dimension of fan F_n and Tomescu and Javaid [14] the dimension of Jahangir graph J_{2n}. The size of a smallest resolving set for the hypercube Q_n is known only for small n and it can take 'laborious computations' to find [3]. The metric dimension of all these graphs depends upon the number of vertices in the graph.

On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(G)$ is finite and does not depend upon the choice of G in $\mathcal{G}$. In [4] it was shown that a graph has metric dimension 1 if and only if it is a path, hence paths on n vertices constitute a family of graphs with constant metric dimension. Similarly, cycles with $n(n \geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon the number of vertices n.

In [8] Javaid *et al.* proved that some regular graphs namely generalized Petersen graphs $P(n, 2)$, antiprisms A_n and Harary graphs $H_{4,n}$ are families of graphs with constant metric dimension and raised an open problem.

Open Problem [8]: Is $P(n,m)$ for $n \geq 7$ and $3 \leq m \leq \lfloor\frac{n-1}{2}\rfloor$, a family of graphs with constant metric dimension?
Note that the problem of determining whether $dim(G) < k$ is an NP- complete problem [6].
In this paper we give a partial answer to this open problem and we show that the generalized Petersen graphs $P(n,3)$ constitute a family of regular graphs having bounded metric dimension and only 4 vertices appropriately chosen suffice to resolve all vertices of the generalized Petersen graphs $P(n,3)$ except $n \equiv 2 (\text{mod } 6)$, when this number equals 5. For $n \equiv 1 (\text{mod } 6)$ a minimal resolving set has cardinality equal to 3.
In what follows all indices i which do not satisfy inequalities $1 \leq i \leq n$ will be taken modulo n.

2 Upper bounds for metric dimension of generalized Petersen graphs $P(n,3)$

The generalized Petersen graph denoted by $P(n,m)$, where $n \geq 3$ and $1 \leq m \leq \lfloor\frac{n-1}{2}\rfloor$, is a cubic graph having vertex set

$$V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$$

and edge set

$$E = \{u_i u_{i+1}, u_i v_i, v_i v_{i+m} : 1 \leq i \leq n\}.$$

Generalized Petersen graphs were first defined by Watkins [15]. For $m = 1$ the generalized Petersen graph $P(n,1)$ is called prism, denoted by D_n. In [2] it was shown that

$$dim(P_m \times C_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Since the prism D_n is actually the cross product of P_2 with a cycle C_n, this implies that

$$dim(D_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

So, prisms constitute a family of 3-regular graphs with bounded metric dimension. In [8] it was proved that $dim(P(n,2)) = 3$ for every $n \geq 5$. Now we will find the metric dimension of the generalized Petersen graphs $P(n,3)$ when $n \equiv 0$ or $1 (\text{mod } 6)$ and an upper bound in the remaining cases.

When $m = 3$, $\{u_1, u_2, ..., u_n\}$ induces a cycle in $P(n, 3)$ with u_iu_{i+1} $(1 \leq i \leq n)$, as edges. If $n = 3l$ $(l \geq 3)$, then $\{v_1, v_2, \ldots, v_n\}$ induces 3 cycles of length l, otherwise it induces a cycle of length n with v_iv_{i+3} $(1 \leq i \leq n)$, as edges. For example, $P(8, 3)$ is the *Möbius-Kantor* graph [10].

Since generalized Petersen graphs $P(n, 3)$ form an important class of 3-regular graphs with $2n$ vertices and $3n$ edges, it is desirable to find their metric dimensions. For our purpose, we call the cycle induced by $\{u_1, u_2, \ldots, u_n\}$, outer cycle and cycle(s) induced by $\{v_1, v_2, \ldots, v_n\}$ inner cycle(s). Note that the choice of appropriate basis vertices (also referred to as landmarks) is core of the problem.

Theorem 1. *For the generalized Petersen graph $P(n, 3)$ we have*
(a) $dim(P(n, 3)) \leq 3$ for $n \equiv 1(mod\ 6)$ and $n \geq 13$;
(b) $dim(P(n, 3)) \leq 4$ for $n \equiv 0, 3, 4, 5(mod\ 6)$ and $n \geq 17$.
(c) $dim(P(n, 3)) \leq 5$ for $n \equiv 2(mod\ 6)$ and $n \geq 8$.

Proof. We denote $W = \{v_1, v_2, v_3, u_4\}$ for $n \equiv 0, 3, 4, 5(\text{mod } 6)$ and $W = \{v_1, v_{3k-1}, v_{6k}\}$ for $n \equiv 1(\text{mod } 6)$, $n = 6k + 1$. We show that the set W distinguishes the vertices of $P(n, 3)$ for $n \not\equiv 2(\text{mod } 6)$. For this purpose, we give the representation of $V(P(n, 3))$ in these cases.

Case (i) $n = 6k, k \in \mathbf{Z}^+, k \geq 2$. For every $n \geq 12$, the representations of vertices on the outer cycle are

$r(u_1|W) = (1, 2, 3, 3)$, $r(u_2|W) = (2, 1, 2, 2)$, $r(u_3|W) = (3, 2, 1, 1)$.

$$r(u_{2+3i}|W) = \begin{cases} (3, 2, 3, 1), & i = 1; \\ (i + 2, i + 1, i + 2, i + 2), & 2 \leq i \leq k; \\ (2k - i + 2, 2k - i + 1, 2k - i + 2, 2k - i + 4), & k + 1 \leq i \leq 2k - 1. \end{cases}$$

$$r(u_{3+3i}|W) = \begin{cases} (4, 3, 2, 2), & i = 1; \\ (i + 3, i + 2, i + 1, i + 3), & 2 \leq i \leq k - 1; \\ (2k - i + 1, 2k - i + 2, 2k - i + 1, 2k - i + 3), & k \leq i \leq 2k - 1. \end{cases}$$

$$r(u_{4+3i}|W) = \begin{cases} (i + 2, i + 3, i + 2, i + 2), & 1 \leq i \leq k - 1; \\ (2k - i, 2k - i + 1, 2k - i + 2, 2k - i + 2), & k \leq i \leq 2k - 2. \end{cases}$$

Representations of vertices with respect to W on the inner cycles are

$$r(v_{1+3i}|W) = \begin{cases} (i, i + 3, i + 2, i), & 1 \leq i \leq k; \\ (2k - i, 2k - i + 3, 2k - i + 4, 2k - i + 2), & k + 1 \leq i \leq 2k - 1. \end{cases}$$

$r(v_{2+3i}|W) =$

$$\begin{cases} (i+3, i, i+3, i+1), & 1 \le i \le k; \\ (2k-i+3, 2k-i, 2k-i+3, 2k-i+3), & k+1 \le i \le 2k-1. \end{cases}$$

$r(v_{3+3i}|W) =$

$$\begin{cases} (i+4, i+3, i, i+2), & 1 \le i \le k-1; \\ (2k-i+2, 2k-i+3, 2k-i, 2k-i+2), & k \le i \le 2k-1. \end{cases}$$

Case (ii) $n = 6k+1, k \in \mathbf{Z}^+$, $k \ge 2$. For every $n \ge 13$, the representations of vertices with respect to W are the following:
Representations of vertices on the outer cycle are

$$r(u_{3i}|W) = \begin{cases} (i+2, k-i+2, i+2), & 1 \le i \le k-1; \\ (k+2, 2, k+1), & i = k; \\ (2k-i+3, i-k+2, 2k-i+1), & k+1 \le i \le 2k-1; \\ (3, k+1, 1), & i = 2k. \end{cases}$$

$$r(u_{3i-1}|W) = \begin{cases} (i+1, k-i+1, i+1), & 1 \le i \le k; \\ (2k-i+2, i-k+1, 2k-i+2), & k+1 \le i \le 2k. \end{cases}$$

$$r(u_{3i-2}|W) = \begin{cases} (i, k-i+2, i+2), & 1 \le i \le k; \\ (k+1, 3, k+2), & i = k+1; \\ (2k-i+3, i-k+2, 2k-i+3), & k+2 \le i \le 2k; \\ (2, k+2, 2), & i = 2k+1. \end{cases}$$

Representations of vertices with respect to W on the inner cycle are

$$r(v_{3i-1}|W) = \begin{cases} (i+2, k-i, i), & 1 \le i \le k-1; \\ (k+1, 0, k), & i = k; \\ (k, 1, k+1), & i = k+1; \\ (2k-i+1, i-k, 2k-i+3), & k+2 \le i \le 2k. \end{cases}$$

$$r(v_{3i}|W) = \begin{cases} (i+3, k-i+3, i+3), & 1 \le i \le k-2; \\ (i+3, k-i+3, 2k-i), & k-1 \le i \le k; \\ (2k-i+4, i-k+3, 2k-i), & k+1 \le i \le 2k-2; \\ (5, k+1, 1), & i = 2k-1; \\ (4, k, 0), & i = 2k. \end{cases}$$

$$r(v_{3i+1}|W) = \begin{cases} (0, k+1, 4), & i = 0; \\ (i, k-i+2, i+4), & 1 \le i \le k-1; \\ (k, 4, k+3), & i = k; \\ (k+1, 5, k+2), & i = k+1; \\ (2k-i+3, i-k+4, 2k-i+3), & k+2 \le i \le 2k-1; \\ (3, k+3, 3), & i = 2k. \end{cases}$$

Case (iii) $n = 6k+3, k \in \mathbf{Z}^+$. For $P(9,3)$, the representations of the vertices are $r(u_1|W) = (1,2,3,3)$, $r(u_2|W) = (2,1,2,2)$, $r(u_3|W) = (3,2,1,1)$,

$r(u_5|W) = (3,2,3,1)$, $r(u_6|W) = (3,3,2,2)$, $r(u_7|W) = (2,3,3,3)$, $r(u_8|W) = (3,2,3,4)$, $r(u_9|W) = (2,3,2,4)$, $r(v_4|W) = (1,4,3,1)$, $r(v_5|W) = (4,1,4,2)$, $r(v_6|W) = (4,4,1,3)$, $r(v_7|W) = (1,4,4,2)$, $r(v_8|W) = (4,1,4,3)$, $r(v_9|W) = (3,4,1,3)$. For every $n \geq 15$, representations of vertices with respect to W are the following:
Representations of vertices on the outer cycle are
$r(u_1|W) = (1,2,3,3)$, $r(u_2|W) = (2,1,2,2)$, $r(u_3|W) = (3,2,1,1)$.

$$r(u_{2+3i}|W) = \begin{cases} (3,2,3,1), & i = 1; \\ (i+2, i+1, i+2, i+2), & 2 \leq i \leq k; \\ (k+2, k+1, k+2, k+3), & i = k+1; \\ (2k-i+3, 2k-i+2, 2k-i+3, 2k-i+5), & k+2 \leq i \leq 2k. \end{cases}$$

$$r(u_{3+3i}|W) = \begin{cases} (4,3,2,2), & i = 1; \\ (i+3, i+2, i+1, i+3), & 2 \leq i \leq k-1; \\ (k+2, k+2, k+1, k+3), & i = k; \\ (2k-i+2, 2k-i+3, 2k-i+2, 2k-i+4), & k+1 \leq i \leq 2k. \end{cases}$$

$$r(u_{4+3i}|W) = \begin{cases} (i+2, i+3, i+2, i+2), & 1 \leq i \leq k-1; \\ (k+1, k+2, k+2, k+2), & i = k; \\ (2k-i+1, 2k-i+2, 2k-i+3, 2k-i+3), & k+1 \leq i \leq 2k-1. \end{cases}$$

Representations of the vertices with respect to W on the inner cycles are

$$r(v_{1+3i}|W) = \begin{cases} (i, i+3, i+2, i), & 1 \leq i \leq k; \\ (k, k+3, k+3, k+1), & i = k+1; \\ (2k-i+1, 2k-i+4, 2k-i+5, 2k-i+3), & k+2 \leq i \leq 2k. \end{cases}$$

$$r(v_{2+3i}|W) = \begin{cases} (i+3, i, i+3, i+1), & 1 \leq i \leq k; \\ (k+3, k, k+3, k+2), & i = k+1; \\ (2k-i+4, 2k-i+1, 2k-i+4, 2k-i+4), & k+2 \leq i \leq 2k. \end{cases}$$

$r(v_{3+3i}|W) =$

$$\begin{cases} (i+4, i+3, i, i+2), & 1 \leq i \leq k-1; \\ (k+3, k+3, k, k+2), & i = k; \\ (2k-i+3, 2k-i+4, 2k-i+1, 2k-i+3), & k+1 \leq i \leq 2k. \end{cases}$$

Case (iv) $n = 6k+4, k \in \mathbf{Z}^+$. For every $n \geq 10$, the representations of the vertices on the outer cycle are
$r(u_1|W) = (1,2,3,3)$, $r(u_2|W) = (2,1,2,2)$, $r(u_3|W) = (3,2,1,1)$.

$r(u_{2+3i}|W) =$

$$\begin{cases} (3,2,3,1), & i = 1; \\ (i+2, i+1, i+2, i+2), & 2 \leq i \leq k; \\ (2k-i+2, 2k-i+3, 2k-i+4, 2k-i+4), & k+1 \leq i \leq 2k. \end{cases}$$

$r(u_{3+3i}|W) =$

$$\begin{cases} (4,3,2,2), & i = 1; \\ (i+3, i+2, i+1, i+3), & 2 \leq i \leq k; \\ (2k-i+3, 2k-i+2, 2k-i+3, 2k-i+5), & k+1 \leq i \leq 2k. \end{cases}$$

$r(u_{4+3i}|W) =$

$$\begin{cases} (i+2, i+3, i+2, i+2), & 1 \leq i \leq k; \\ (2k-i+2, 2k-i+3, 2k-i+2, 2k-i+4), & k+1 \leq i \leq 2k. \end{cases}$$

Representations of vertices on inner cycle are

$r(v_{1+3i}|W) =$

$$\begin{cases} (i, i+3, i+2, i), & 1 \leq i \leq k; \\ (k+1, k+4, k+1, k+1), & i = k+1; \\ (2k-i+4, 2k-i+5, 2k-i+2, 2k-i+4), & k+2 \leq i \leq 2k+1. \end{cases}$$

$r(v_{2+3i}|W) =$

$$\begin{cases} (i+3, i, i+3, i+1), & 1 \leq i \leq k-1; \\ (k+1, k, k+3, k+1), & i = k; \\ (k, k+1, k+4, k+2), & i = k+1; \\ (2k-i+1, 2k-i+4, 2k-i+5, 2k-i+3), & k+2 \leq i \leq 2k. \end{cases}$$

$r(v_{3+3i}|W) =$

$$\begin{cases} (i+4, i+3, i, i+2), & 1 \leq i \leq k-1; \\ (k+4, k+1, k, k+2), & i = k; \\ (k+3, k, k+1, k+3), & i = k+1; \\ (2k-i+4, 2k-i+1, 2k-i+4, 2k-i+4), & k+2 \leq i \leq 2k. \end{cases}$$

Case (v) $n = 6k+5, k \in \mathbf{Z}^+$. For every $n \geq 17$, the representations of vertices on the outer cycle are
$r(u_1|W) = (1,2,3,3)$, $r(u_2|W) = (2,1,2,2)$, $r(u_3|W) = (3,2,1,1)$.

$r(u_{2+3i}|W) =$

$$\begin{cases} (3,2,3,1), & i = 1; \\ (i+2, i+1, i+2, i+2), & 2 \leq i \leq k; \\ (k+2, k+2, k+2, k+3), & i = k+1; \\ (2k-i+3, 2k-i+4, 2k-i+3, 2k-i+5), & k+2 \leq i \leq 2k+1. \end{cases}$$

$r(u_{3+3i}|W) =$

$$\begin{cases} (4,3,2,2), & i = 1; \\ (i+3, i+2, i+1, i+3), & 2 \leq i \leq k-1; \\ (k+2, k+2, k+1, k+3), & i = k; \\ (k+1, k+2, k+2, k+3), & i = k+1; \\ (2k-i+2, 2k-i+3, 2k-i+4, 2k-i+4), & k+2 \leq i \leq 2k. \end{cases}$$

$r(u_{4+3i}|W) =$

$$\begin{cases} (i+2, i+3, i+2, i+2), & 1 \leq i \leq k-1; \\ (k+2, k+2, k+2, k+2), & i = k; \\ (k+2, k+1, k+2, k+3), & i = k+1; \\ (2k-i+3, 2k-i+2, 2k-i+3, 2k-i+5), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of vertices with respect to W on inner cycle are

$r(v_{1+3i}|W) =$

$$\begin{cases} (i, i+3, i+2, i), & 1 \leq i \leq k-1; \\ (k, k+2, k+2, k), & i = k; \\ (k+1, k+1, k+3, k+1), & i = k+1; \\ (k+2, k, k+3, k+2), & i = k+2; \\ (2k-i+5, 2k-i+2, 2k-i+5, 2k-i+5), & k+3 \leq i \leq 2k+1. \end{cases}$$

$r(v_{2+3i}|W) =$

$$\begin{cases} (i+3, i, i+3, i+1), & 1 \leq i \leq k-1; \\ (k+3, k, k+2, k+1), & i = k; \\ (k+3, k+1, k+1, k+2), & i = k+1; \\ (k+2, k+2, k, k+2), & i = k+2; \\ (2k-i+4, 2k-i+5, 2k-i+2, 2k-i+4), & k+3 \leq i \leq 2k+1. \end{cases}$$

$r(v_{3+3i}|W) =$

$$\begin{cases} (i+4, i+3, i, i+2), & 1 \leq i \leq k-2; \\ (k+2, k+2, k-1, k+1), & i = k-1; \\ (k+1, k+3, k, k+2), & i = k; \\ (k, k+3, k+1, k+2), & i = k+1; \\ (k-1, k+2, k+2, k+1), & i = k+2; \\ (2k-i+1, 2k-i+4, 2k-i+5, 2k-i+3), & k+3 \leq i \leq 2k. \end{cases}$$

We note that there are no two vertices in the inner cycle(s) with same representations. Also, there are no two vertices in the inner cycle(s) and outer cycle having the same representations and no two vertices on outer cycle having the same representations. This implies that $W = \{v_1, v_2, v_3, u_4\}$ is a resolving set for $V(P(n,3))$ when $n \equiv 0$, 3, 4, 5(mod 6) implying that in these cases $dim(P(n,3)) \leq 4$. Also $W = \{v_1, v_{3k-1}, v_{6k}\}$ is a resolving set for $n = 6k+1$, when $dim(P(n,3)) \leq 3$.

Case (vi) $n = 6k+2, k \in \mathbf{Z}^+$. It is straightforward to verify that $W_1 = \{v_1, v_2, v_3, u_4\}$ and $W_2 = \{v_1, v_2, u_7, u_{10}\}$ are resolving sets for $P(8,3)$ and $P(14,3)$, respectively. For every $n \geq 20$, we show that $W = \{v_1, v_2, v_3, u_4, u_{3k+5}\}$ is a resolving set. For this, we first give representations of vertices with respect to $W' = \{v_1, v_2, v_3, u_4\}$. The representations of the vertices on the outer cycle are
$r(u_1|W') = (1,2,3,3)$, $r(u_2|W') = (2,1,2,2)$, $r(u_3|W') = (3,2,1,1)$.
$r(u_{2+3i}|W') =$

$$\begin{cases} (3,2,3,1), & i = 1; \\ (i+2, i+1, i+2, i+2), & 2 \leq i \leq k; \\ (2k-i+2, 2k-i+3, 2k-i+2, 2k-i+4), & k+1 \leq i \leq 2k. \end{cases}$$

$r(u_{3+3i}|W') =$

$$\begin{cases} (4,3,2,2), & i=1; \\ (i+3,i+2,i+1,i+3), & 2 \le i \le k-1; \\ (k+1,k+2,k+1,k+3), & i=k; \\ (2k-i+1,2k-i+2,2k-i+3,2k-i+3), & k+1 \le i \le 2k-1. \end{cases}$$

$r(u_{4+3i}|W') =$

$$\begin{cases} (i+2,i+3,i+2,i+2), & 1 \le i \le k-1; \\ (k+2,k+1,k+2,k+2), & i=k; \\ (2k-i+2,2k-i+1,2k-i+2,2k-i+4), & k+1 \le i \le 2k-1. \end{cases}$$

Representations of vertices with respect to W' on inner cycle are

$r(v_{1+3i}|W') =$

$$\begin{cases} (i,i+3,i+2,i), & 1 \le i \le k-1; \\ (k,k+1,k+2,k), & i=k; \\ (k+1,k,k+3,k+1), & i=k+1; \\ (2k-i+4,2k-i+1,2k-i+4,2k-i+4), & k+2 \le i \le 2k. \end{cases}$$

$r(v_{2+3i}|W') =$

$$\begin{cases} (i+3,i,i+3,i+1), & 1 \le i \le k-1; \\ (k+3,k,k+1,k+1), & i=k; \\ (k+2,k+1,k,k+2), & i=k+1; \\ (2k-i+3,2k-i+4,2k-i+1,2k-i+3), & k+2 \le i \le 2k. \end{cases}$$

$r(v_{3+3i}|W') =$

$$\begin{cases} (i+4,i+3,i,i+2), & 1 \le i \le k-2; \\ (k+1,k+2,k-1,k+1), & i=k-1; \\ (k,k+3,k,k+2), & i=k; \\ (k-1,k+2,k+1,k+1), & i=k+1; \\ (2k-i,2k-i+3,2k-i+4,2k-i+2), & k+2 \le i \le 2k-1. \end{cases}$$

Consequently, $r(u_{2+3k}|W') = r(u_{4+3k}|W') = (k+2,k+1,k+2,k+2)$; $r(u_{5+3k}|W') = r(u_{3+3k}|W') = (k+1,k+2,k+1,k+3)$; $r(u_{3k}|W') = r(v_{5+3k}|W') = (k+2,k+1,k,k+2)$; $r(v_{8+3k}|W') = r(v_{3k}|W') = (k+1,k+2,k-1,k+1)$.

The vertex u_{3k+5} distinguishes these pairs of vertices with same representations as $d(u_{3k+5}, u_{3k+2}) = 3$, $d(u_{3k+5}, u_{3k+4}) = 1$, $d(u_{3k+5}, u_{3k}) = 5$, $d(u_{3k+5}, v_{3k+5}) = 1$, $d(u_{3k+5}, v_{3k+8}) = 2$ and $d(u_{3k+5}, v_{3k}) = 4$. This suggests that $W = \{v_1, v_2, v_3, u_4, u_{3k+5}\}$ is a resolving set for $V(P(n,3))$ in this case implying that $dim(P(n,3)) \leq 5$.

□

3 Metric dimension of $P(n,3)$ for $n \equiv 0, 1 (\text{mod } 6)$

In this section we will prove that $dim(P(n,3)) \geq 3$ for $n \equiv 1(\text{mod } 6)$ and $n \geq 25$ and $dim(P(n,3)) \geq 4$ for $n \equiv 0(\text{mod } 6)$ and $n \geq 24$, yielding exact values of $dim(P(n,3))$ in these cases by Theorem 2.1. For this purpose we need some more notations and definitions. Without loss of generality we can suppose that the vertices of the outer cycle are $u_1, u_2, \ldots, u_n$ in the clockwise direction. For two vertices u_i and u_j $(i \neq j)$ we shall define the "clockwise distance" from u_i to u_j, denoted by $d^*(u_i, u_j)$ the distance, measured in clockwise direction, from u_i to u_j, in the subgraph induced by the outer cycle. For example, $d^*(u_1, u_n) = n - 1$ and $d^*(u_n, u_1) = 1$; in general we have $d^*(u_i, u_j) + d^*(u_j, u_i) = n$. This definition can be extended to any two vertices of $P(n,3)$ for $i \neq j$ by: $d^*(u_i, v_j) = d^*(v_i, u_j) = d^*(v_i, v_j) = d^*(u_i, u_j)$.
Consider a vertex on the outer cycle, say u_1. A vertex u_i is called a good vertex for u_1 if u_i and u_{i+2} have equal distances to u_1, i. e., $d(u_1, u_i) = d(u_1, u_{i+2})$; otherwise u_i is called a bad vertex for u_1. This definition can be extended to vertices of the inner cycle: u_i is a good vertex for v_1 if $d(v_1, u_i) = d(v_1, u_{i+2})$ and bad otherwise.
In figure 1 we have represented by black dots all good vertices for u_1 when $n = 6k + 1 \geq 25$.
It is important to note that the set of good vertices for v_1 is deduced from the set of good vertices for u_1 by adding 4 new vertices, namely u_2, u_3, u_{6k-1} and u_{6k-2}. Similarly, a vertex u_j is said to be good for the pair $\{u_1, u_i\}$ if $d(u_1, u_j) = d(u_1, u_{j+2})$ and $d(u_i, u_j) = d(u_i, u_{j+2})$. If u_k is a good vertex for the pairs $\{u_1, u_i\}$ and $\{u_1, u_j\}$ then u_k is also a good vertex for the triplet $\{u_1, u_i, u_j\}$, i. e., $d(u_1, u_k) = d(u_1, u_{k+2}), d(u_i, u_k) = d(u_i, u_{k+2})$ and $d(u_j, u_k) = d(u_j, u_{k+2})$.
Due to the rotational symmetry of $P(n,3)$ we deduce

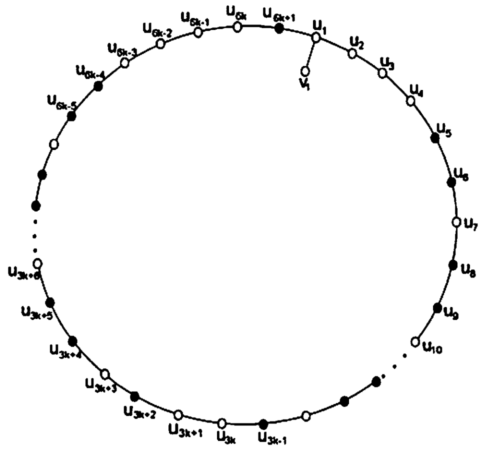

Figure 1: Good vertices for $u_1 (n = 6k + 1)$.

Lemma 2. *For any two vertices u_i and u_j on the outer cycle of $P(n, 3)$ we have $d(u_i, u_j) = d(u_{i+r}, u_{j+r})$ for any $1 \le r \le n - 1$.*

In order to find good vertices for pairs of vertices belonging to the outer cycle the following lemma will be useful.

Lemma 3. *Let $1 \le i \le n - 2$. If u_j is good for u_1 and u_{j-i} is also good for u_1, then u_j is good for the pair $\{u_1, u_{i+1}\}$.*

Proof: By hypothesis we can write $d(u_1, u_j) = d(u_1, u_{j+2})$ and $d(u_1, u_{j-i}) = d(u_1, u_{j-i+2})$. By Lemma 3.1 the last equality is equivalent to $d(u_{i+1}, u_j) = d(u_{i+1}, u_{j+2})$. □

Theorem 2. *$dim(P(n, 3)) = 3$ if $n = 6k + 1$ and $n \ge 25$.*

Proof: We shall prove that $dim(P(n, 3)) \ge 3$ in this case, by showing that there is no resolving set of $V(P(n, 3))$ consisting of two vertices, X and Y. If both vertices X and Y belong to the outer cycle, we can suppose that $X = u_1$. We distinguish three cases:

1)$d^*(u_1, Y) \equiv 0 (\text{mod } 3)$. We choose vertex u_{6k-4}. Since vertices $u_{6k-7}, u_{6k-10}, \ldots, u_{3k+5}, u_{3k+2}, u_{3k-1}, u_{3k-4}, \ldots, u_8, u_5$ are good vertices for u_1, but u_2, u_{6k} and u_{6k-3} are bad vertices for u_1 (see fig. 1), applying Lemma 3.2 we find that u_{6k-4} is a good vertex for any pair $\{u_1, Y\}$ such that $Y \notin \{u_{6k-5}, u_{6k-2}, u_{6k+1}\}$.
But in this case u_5 is a good vertex for the pairs $\{u_1, u_{6k-2}\}$ and $\{u_1, u_{6k+1}\}$ if $k \ge 4$ and for $\{u_1, u_{6k-5}\}$ if $k \ge 5$; for $k = 4$ we have that $u_{3k} = u_{12}$ is not a good vertex for u_1. It remains to consider the pair $\{u_1, u_{6k-5}\} = \{u_1, u_{19}\}$

when $k = 4$. In this case u_9 is a good vertex for this pair. It follows that any pair $\{u_1, Y\}$ cannot be a resolving set having two vertices.

2)$d^*(u_1, Y) \equiv 1 (\text{mod } 3)$. We consider vertex u_5. By starting from u_5 and going in the counter-clockwise direction at distances 1,4,7,... the only bad vertices encountered are u_4, u_1 and u_{6k-1}. By Lemma 3.2 we deduce that u_5 is a good vertex for any pair $\{u_1, Y\}$ such that $Y \notin \{u_2, u_5, u_8\}$. In a similar manner we get that u_{6k-4} is a good vertex for the pairs $\{u_1, u_2\}$ and $\{u_1, u_5\}$ if $k \geq 4$ and for $\{u_1, u_8\}$ if $k \geq 5$. For $k = 4$ the vertex u_{16} is a good vertex for $\{u_1, u_8\}$.

3)$d^*(u_1, Y) \equiv 2 (\text{mod } 3)$. In this case in order to minimize the number of bad vertices for the pairs $\{u_1, Y\}$ we choose vertex u_6. This vertex is a good vertex for any pair $\{u_1, Y\}$ such that $Y \notin \{u_3, u_6, u_9\}$. Vertex u_{6k-5} is a good vertex for all pairs $\{u_1, u_3\}$, $\{u_1, u_6\}$ and $\{u_1, u_9\}$ for any $k \geq 4$. If both X and Y belong to the inner cycle we can consider that $X = v_1$ and $Y = v_i$ $(i > 1)$; this case can be reduced to the case when $X = u_1$ and $Y = u_i$ since the set of good vertices for v_p includes the set of good vertices for u_p for every $1 \leq p \leq n$.
If $X = u_i$ and $Y = v_i$ then any good vertex for u_i is also a good vertex for v_i, hence for the pair $\{X, Y\}$. The remaining case when $X = u_i, Y = v_j$ and $i \neq j$ can also be reduced to the case $X = u_i, Y = u_j$. It follows that there is no resolving set containing two vertices, which concludes the proof. □

Theorem 3. *If $n = 6k$ and $n \geq 24$ then $dim(P(n, 3)) = 4$.*

Proof: By Theorem 2.1 it is necessary only to show that $dim(P(n, 3)) \geq 4$, or that there is no resolving set of $V(P(n, 3))$ consisting of three vertices, X, Y and Z. By the same reasoning as in the proof of Theorem 3.1 it is sufficient to consider only the case when X, Y, Z belong to the outer cycle since the set of good vertices for v_1 can be deduced from the set of good vertices for u_1 (represented in figure 2) by adding vertices u_2, u_3, u_{6k-2} and u_{6k-3}. As in the case $n = 6k + 1$ we shall see that for any three vertices X, Y, Z such that $d^*(X, Y) < d^*(X, Z)$ it is possible to find a pair of vertices at distance 2 on the outer cycle, $\{u_i, u_{i+2}\}$ having equal distances to X, Y and Z, respectively. If $n = 6k$ and X, Y, Z are on the outer cycle, we can suppose that $X = u_1$. By denoting $(x, y) \equiv (a, b) (\text{mod } 3)$ if $x \equiv a (\text{mod } 3)$ and $y \equiv b (\text{mod } 3)$, the following 9 cases can occur: $(d^*(u_1, Y), d^*(u_1, Z))$ is congruent modulo 3 to:
1)(0,0); 2)(1,1); 3)(2,2); 4)(0,1); 5)(0,2); 6)(1,0); 7)(1,2); 8)(2,0); 9)(2,1).

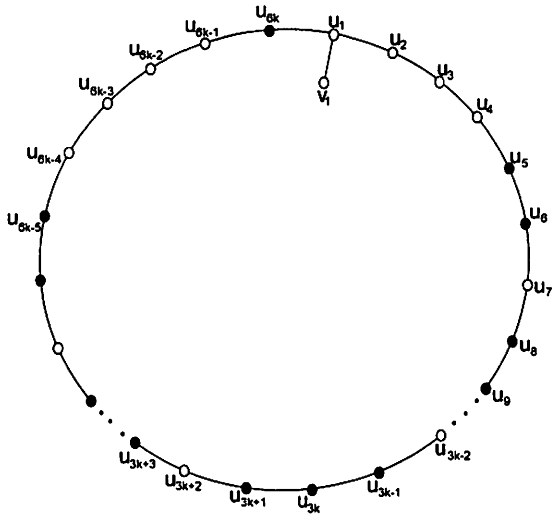

Figure 2: Good vertices for u_1 $(n = 6k)$.

Some of these cases can be reduced to another cases. For example, from case 2 by permutation $X \to Y, Y \to Z, Z \to X$ we obtain case 5 and by permutation $X \to Z, Y \to X, Z \to Y$ we get case 8.
The graph of reducibility between cases is illustrated in figure 3.

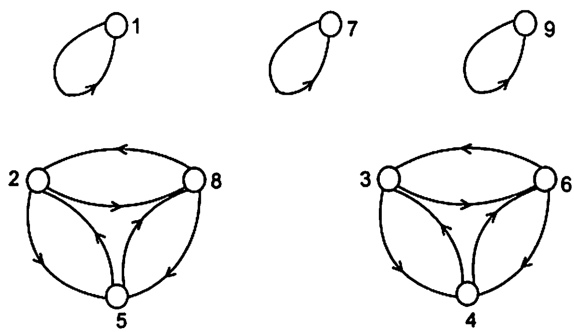

Figure 3: Reducibility between cases.

It follows that it is sufficient to consider only cases 1,2,3,7,9.

Case 1. If we choose good vertex u_6 and we go in the counter-clockwise direction, reaching vertices $u_3, u_{6k}, u_{6k-6}, \ldots, u_9$ we encounter only two bad vertices, u_3 and u_{6k-3} (see figure 2). By Lemma 3.2 it follows that u_6 is a good vertex for all pairs $\{u_1, Y\}$ where $d^*(u_1, Y) \equiv 0 (\text{mod } 3)$

and $Y \notin \{u_4, u_{10}\}$. This implies that u_6 is a good vertex for all triplets $\{u_1, Y, Z\}$, unless $Y = u_4$ and $Z \in \{u_7, u_{10}, u_{13}, \ldots, u_{6k-2}\}$; $Y = u_{10}$ and $Z \in \{u_7, u_{13}, u_{16}, \ldots, u_{6k-2}\}$. For these triplets we must find other good vertices on the outer cycle. Similarly, u_{3k+6} is a good vertex for u_1 since $6k-6 \geq 3k+6$ and for all pairs $\{u_1, Y\}$ where $d^*(u_1, Y) \equiv 0 (\text{mod } 3)$ and $Y \notin \{u_{3k+4}, u_{3k+10}\}$. Consequently, we have found a good vertex (u_6 or u_{3k+6}) for all triplets $\{u_1, Y, Z\}$ such that $\{Y, Z\} \neq \{u_4, u_{3k+4}\}, \{u_4, u_{3k+10}\}, \{u_{10}, u_{3k+4}\}, \{u_{10}, u_{3k+10}\}$. Finally, u_9 is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 0 (\text{mod } 3)$ and $Y \notin \{u_7, u_{13}\}$. Since $k \geq 4$ we have $3k+4 > 13$ and u_9 is a good vertex for the remaining triplets $\{u_1, Y, Z\}$, where $Y \in \{u_4, u_{10}\}$ and $Z \in \{u_{3k+4}, u_{3k+10}\}$.

Case 2. In a similar way we get that u_{6k-5} is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 1 (\text{mod } 3)$ and $Y \notin \{u_{6k-7}, u_{6k-1}\}$, therefore u_{6k-5} is a good vertex for all triplets $\{u_1, Y, Z\}$, unless $Y = u_{6k-7}$ and $Z \in \{u_2, u_5, \ldots, u_{6k-10}, u_{6k-4}, u_{6k-1}\}$; $Y = u_{6k-1}$ and $Z \in \{u_2, u_5, \ldots, u_{6k-10}, u_{6k-4}\}$. Since $6k-8 > 3k-2$ it follows that u_{6k-8} is a good vertex for u_1 and for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 1 (\text{mod } 3)$ and $Y \notin \{u_{6k-10}, u_{6k-4}\}$. We have found a good vertex (u_{6k-5} or u_{6k-8}) for all triplets $\{u_1, Y, Z\}$ such that $\{Y, Z\} \neq \{u_{6k-7}, u_{6k-10}\}, \{u_{6k-7}, u_{6k-4}\}, \{u_{6k-1}, u_{6k-10}\}, \{u_{6k-1}, u_{6k-4}\}$.
Since $k \geq 4$ we find for triplets $\{u_1, u_{6k-7}, u_{6k-10}\}, \{u_1, u_{6k-7}, u_{6k-4}\}, \{u_1, u_{6k-1}, u_{6k-10}\}$ and $\{u_1, u_{6k-1}, u_{6k-4}\}$ good vertices u_{3k-4}, u_{3k-4}, u_{6k-11} and u_{3k-1}, respectively (e. g., using Lemma 3.2).

Case 3. We deduce that u_5 is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 2 (\text{mod } 3)$ and $Y \notin \{u_3, u_9\}$. It follows that u_5 is a good vertex for all triplets $\{u_1, Y, Z\}$, unless $Y = u_3$ and $Z \in \{u_6, u_9, \ldots, u_{6k}\}$; $Y = u_9$ and $Z \in \{u_6, u_{12}, u_{15}, \ldots, u_{6k}\}$. Also u_{3k-1} is a good vertex for u_1 and for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 2 (\text{mod } 3)$ and $Y \notin \{u_{3k-3}, u_{3k+3}\}$. It follows that there exists a good vertex (u_5 or u_{3k-1}) for all triplets $\{u_1, Y, Z\}$ such that $\{Y, Z\} \neq \{u_3, u_{3k-3}\}, \{u_3, u_{3k+3}\}, \{u_9, u_{3k-3}\}, \{u_9, u_{3k+3}\}$ (note that for $k = 4$ the third triplet must be eliminated from the list).
Now u_8 is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 2 (\text{mod } 3)$ and $Y \notin \{u_6, u_{12}\}$. It follows that u_8 is a good vertex for all four remaining triplets, except $\{u_1, u_3, u_{12}\}$ for $k = 5$ and $n = 30$. For this last triplet $u_{3k+1} = u_{16}$ is a good vertex.

Cases 7 and 9 can be reduced to the case 7 without imposing any inequality between the distances $d^*(X, Y)$ and $d^*(X, Z)$.
Let $A = \{u_2, u_5, u_8, \ldots, u_{6k-1}\}$ and $B = \{u_3, u_6, u_9, \ldots, u_{6k}\}$. It is necessary to prove that for any triplet $\{u_1, Y, Z\}$, where $Y \in A$ and $Z \in B$ there is a good vertex on the outer cycle. From the previous case we have seen that u_5 is a bad vertex for pairs $\{u_1, Z\}$, where $Z \in B$ if and only if $Z \in \{u_3, u_9\}$ and a bad vertex for pairs $\{u_1, Y\}$, where $Y \in A$ if and only

if $Y \in \{u_2, u_5, u_8, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-1}\}$. It follows that u_5 is a good vertex for all triplets $\{u_1, Y, Z\}$, where $Y \in A$ and $Z \in B$, unless: $Y \in A$ and $Z \in \{u_3, u_9\}$; $Y \in \{u_2, u_5, u_8, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-1}\}$ and $Z \in B$ (these sets of pairs $\{Y, Z\}$ will be denoted by α and β, respectively). For the remaining triplets $\{u_1, Y, Z\}$, where $\{Y, Z\} \in \alpha \cup \beta$ we must find other good vertices on the outer cycle. Consider now vertex u_{3k+1}. This vertex is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in A \backslash \{u_{3k-1}, u_{3k+5}\}$. Since $3k - 7 \geq 5$ it follows that u_{3k+1} is also a good vertex for all pairs $\{u_1, Z\}$, where $Z \in \{u_3, u_9\}$. Therefore the set α is reduced to the set α_1 of pairs $\{Y, Z\}$ such that $Y \in \{u_{3k-1}, u_{3k+5}\}$ and $Z \in \{u_3, u_9\}$. Now u_8 is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in \{u_{14}, u_{17}, \ldots, u_{3k+8}\}$ and all pairs $\{u_1, Z\}$, where $Z \in \{u_3, u_9, u_{15}, u_{18}, \ldots, u_{6k}\}$. Since $3k - 1 \geq 14$ for $k \geq 5$ it follows that u_8 is a good vertex for all pairs in α_1 if $k \geq 5$ and for $k = 4$ we must consider only the pairs $\{u_{11}, u_3\}$ and $\{u_{11}, u_9\}$. From figure 2 we deduce that $u_{3k+3} = u_{15}$ is a good vertex for $\{u_1, u_{11}, u_3\}$ and $u_{3k+4} = u_{16}$ is a good vertex for $\{u_1, u_{11}, u_9\}$.

It remains to find good vertices for pairs in β. Since u_{6k-5} is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in A \backslash \{u_{6k-7}, u_{6k-1}\}$ and a good vertex for all pairs $\{u_1, Z\}$, where $Z \in \{u_{3k-3}, u_{3k}, \ldots, u_{6k-9}\}$, β is reduced to $\gamma \cup \delta$, where γ consists of all pairs $\{Y, Z\}$ with $Y \in \{u_{6k-7}, u_{6k-1}\}$ and $Z \in B$ and δ of $\{Y, Z\}$ with $Y \in \{u_2, u_5, u_8, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-10}, u_{6k-4}\}$ and $Z \in \{u_3, u_6, \ldots, u_{3k-6}, u_{6k-6}, u_{6k-3}, u_{6k}\}$.

u_{3k-1} is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in \{u_{3k+5}, u_{3k+8}, \ldots, u_{6k-1}\}$ and all pairs $\{u_1, Z\}$, where $Z \in B \backslash \{u_{3k-3}, u_{3k+3}\}$. Therefore γ is reduced to ε, consisting of all pairs $\{Y, Z\}$ such that $Y \in \{u_{6k-7}, u_{6k-1}\}$ and $Z \in \{u_{3k-3}, u_{3k+3}\}$ and δ is reduced to μ, which consists of $\{Y, Z\}$ with $Y \in \{u_2, u_5, u_8\}$ and $Z \in \{u_3, u_6, u_{3k-6}, u_{6k-6}, u_{6k-3}, u_{6k}\}$.

u_6 is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in \{u_2, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-1}\}$ and all pairs $\{u_1, Z\}$, where $Z \in \{u_{12}, u_{15}, \ldots, u_{3k+6}\}$.

If $k \geq 5$ then $6k - 7 \geq 3k + 8$ and $3k - 3 \geq 12$; therefore u_6 is a good vertex for all triples $\{u_1, Y, Z\}$, where $\{Y, Z\} \in \varepsilon$. If $k = 4$ the pairs in ε are $\{u_{17}, u_9\}, \{u_{17}, u_{15}\}, \{u_{23}, u_9\}$ and $\{u_{23}, u_{15}\}$; good vertices for these pairs are vertices u_{24}, u_5, u_{16} and u_9, respectively, which also are good vertices for u_1.

It remains to study the pairs from μ. u_{3k} is a good vertex for all pairs $\{u_1, Y\}$, where $Y \in \{u_2, u_5, \ldots, u_{3k-4}\}$ and all pairs $\{u_1, Z\}$, where $Z \in \{u_{3k+6}, u_{3k+9}, \ldots, u_{6k}\}$. Since $3k - 4 \geq 8$ and $6k - 6 \geq 3k + 6$ it follows that μ is reduced to the set ν of pairs $\{Y, Z\}$, where $Y \in \{u_2, u_5, u_8\}$ and $Z \in \{u_3, u_6, u_{3k-6}\}$.

Finally, u_{3k+1} is a good vertex for all pairs $\{u_1, Y\}$, where
$Y \in A \backslash \{u_{3k-1}, u_{3k+5}\}$ and for all pairs $\{u_1, Z\}$, where
$Z \in \{u_3, u_6, \ldots, u_{3k-3}\}$.

Consequently, u_{3k+1} is a good vertex for u_1 and all pairs from ν. □

4 Concluding remarks

In this paper we have studied the metric dimension of the generalized Petersen graphs $P(n,3)$ by giving a partial answer to an open problem raised in [8]. We proved that the metric dimension of the generalized Petersen graphs $P(n,3)$ is bounded and determined exact value of the metric dimension when $n \equiv 0$ or $1 (\text{mod } 6)$. In the remaining cases we showed that $dim(P(n,3)) \leq 4$ when $n \equiv 3, 4, 5 (\text{mod } 6)$, $n \geq 9$ and $dim(P(n,3)) \leq 5$ when $n \equiv 2 (\text{mod } 6)$ and $n \geq 26$. In the same way as for the case $n \equiv 1 (\text{mod } 6)$ it is not difficult to show that $dim(P(n,3)) \geq 3$ when $n \equiv 2, 3, 4, 5 (\text{mod } 6)$. We close this section by raising a question as an open problem.
Open Problem: Find the exact value of the metric dimension of $P(n,3)$ when $n \equiv 2, 3, 4, 5 (\text{mod } 6)$.

Acknowledgement

The authors are very grateful to the referee for his/her careful reading with corrections and useful comments.

References

1. P. S. Buczkowski, G. Chartrand, C. Poisson, P. Zhang, On k-dimensional graphs and their bases, *Periodica Math. Hung.*, 46(1)(2003), 9 – 15.
2. J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of some families of graphs, *Electronic Notes in Discrete Math.*, 22(2005), 129 – 133.
3. J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of Cartesian products of graphs, *SIAM J. Discrete Math.*, 2(21), (2007), 423 – 441.
4. G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, 105(1 – 3)(2000), 99 – 113.
5. G. Chartrand, P. Zhang, The theory and applications of resolvability in graphs, *Congressus Numerantium*, 160(2003), 47 – 68.
6. M. R. Garey, D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, New York, 1979.
7. F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combin.*, 2(1976), 191 – 195.
8. I. Javaid, M. T. Rahim, K. Ali, Families of regular graphs with constant metric dimension, *Utilitas Math.*, 75(2008), 21 – 33.
9. M. A. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *J. Biopharm. Statist.*, 3(1993), 203 – 236.
10. E. Lijnen, A. Celemans, Oriented 2-cell embeddings of a graph and their symmetry classification: Generating algorithms and case study of the Möbius-Kantor graph, *J. Chem. Inf. Comput. Sci.*, (2004), 44(5), 1552-1564.

11. R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Computer Vision, Graphics, and Image Processing,* 25(1984), 113 – 121.
12. P. J. Slater, Leaves of trees, *Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory, and Computing* (Florida Atlantic Univ., Boca Raton, Fl. 1975), *Congressus Numerantium*, XIV, Winnipeg, Man., 1975, *Utilitas Math.*, 549–559.
13. P. J. Slater, Dominating and reference sets in graphs, *J. Math. Phys. Sci.*, 22(1998), 445 – 455.
14. I. Tomescu, I. Javaid, On the metric dimension of the Jahangir graph, *Bull. Math. Soc. Sci. Math. Roum.*, 50(98), 4(2007), 371 – 376.
15. I. Tomescu, M. Imran, On metric and partition dimensions of some infinite regular graphs, *Bull. Math. Soc. Sci. Math. Roum.*, 52(100), 4(2009), 461 – 472.
16. M. E. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs, *J. Combin. Theory* 6(1969), 152 – 164.