Enumeration of rooted loopless unicursal planar maps*

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Abstract

In this paper we investigate the number of rooted loopless unicursal planar maps and present some formulae for such maps with up to three parameters: the number of edges and the valencies of the two odd vertices.

Keywords: Loopless maps; Unicursal map; Enumerating function; Functional equation; Lagrangian inversion

MSC(2000): 05C45; 05C30

1. Introduction

Throughout this paper we consider the rooted maps on the plane. Definitions of terms not given here may be found in [22].

The concept of a rooted map was first introduced by W.T. Tutte. His series of census papers [29–31] laid the foundation for the theory. Since then, the theory has been developed by many scholars such as Arquès [1], Brown [7,8], Mullin et al. [26], Tutte [32], Bender et al. [2–4], Liskovets et al. [16–19], Bousquet-Mélou et al. [5,6], Walsh [33,34], Mednykh et al. [27,28], Gao [13,14], Liu [20–22] and Cai et al. [9–12]. In this article we consider one type of planar maps: loopless unicursal maps and discuss the enumerative problem of loopless unicursal planar maps with the valencies of the two odd vertices and the number of edges as three parameters.

A sum-free formula for the number of rooted unicursal planar maps with a given number of edges first appeared in [17]. In that paper, Liskovets and Walsh had also found a sum-free formula for the number of unicursal maps rooted in a vertex of odd valency and a formula for the number of rooted unicursal maps as a function of the odd vertex valencies. Several years later

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the enumeration of rooted unicursal planar maps with the valencies of the two odd vertices and the number of edges or the number of inner faces, the number of nonrooted vertices and the valencies of the two odd vertices of maps as parameters was investigated by Long and Cai [23,24]. Two summation-free were obtained. In 2013, Long and Cai [25] treated the enumeration of 4-regular unicursal planar maps with the number of nonrooted vertices and the valencies of the two odd vertices as three parameters and obtained two sum-free formulae.

Now, we define some basic concepts and terms. A *map* on an orientable surface is a connected graph cellularly embedded on the surface. A map M is said to be *rooted* if an edge with a direction along the edge, and a side of the edge is distinguished. We denote the root-edge of M by $e_r(M)$ and its tail vertex is chosen to be the *root-vertex* of this map; the face on the right-hand side of the root-edge is called the *root-face*. Without loss of generality, the root-face may be chosen as the infinite face.

A map is called *eulerian* if all the valencies of its vertices are even and a map is to be unicursal if it has exactly two vertices of odd valency. An *endpoint* is a vertex of valency 1; a unicursal map evidently can have at most two such vertices. A map is said to be *loopless* if it has no edge which is a loop. A *nearly loopless planar maps* is a planar map in which only the root-edge is a loop. A map with its root-face boundary consisting of only one edge (which is the loop) is called an *inner map*.

For any map M in a set of maps $\mathscr{M}$, let $M-e_r(M)$ and $M\bullet e_r(M)$ be the maps obtained by deleting $e_r(M)$, the root-edge, from M and contracting $e_r(M)$ into a vertex as the new root-vertex, respectively.

For the power series $f(x)$, $f(x,y)$ and $f(x,y,z)$, we employ the following notations:

$$\partial_x^m f(x), \quad \partial_{(x,y)}^{(m,n)} f(x,y) \quad \text{and} \quad \partial_{(x,y,z)}^{(m,n,s)} f(x,y,z)$$

to represent the coefficients of x^m in $f(x)$, $x^m y^n$ in $f(x,y)$ and $x^m y^n z^s$ in $f(x,y,z)$, respectively.

In what follows we will enumerate loopless unicursal planar maps rooted in a vertex of odd valency. Several explicit expressions of its enumerating functions will be derived.

2. Functional equations

In this section we will set up the functional equations satisfied by the enumerating functions for loopless unicursal planar maps rooted in a vertex of odd valency.

We first introduce some operations on the maps in $\mathscr{M}$.

For any two maps M_1 and M_2, whose respective roots are $r_1 = r(M_1)$ and $r_2 = r(M_2)$, let M be the map obtained by identifying the root-vertices v_{r_1} of M_1 and v_{r_2} of M_2 into a single vertex v_r, the only vertex of M belonging to both M_1 and M_2, where $e_r = e_{r_2}$ and M_1 is inside one of faces incident to v_{r_2}. The operation of getting M from M_1 and M_2 is called v_r-addition and the resulting map is written as $M = M_1 \dotplus M_2$. Further, for two sets of maps $\mathscr{M}_1$ and $\mathscr{M}_2$, the set of maps

$$\mathscr{M}_1 \odot \mathscr{M}_2 = \{M_1 \dot{+} M_2 \mid M_i \in \mathscr{M}_i, i = 1,2\} \tag{1}$$

is said to be v_r-production of $\mathscr{M}_1$ and $\mathscr{M}_2$.

Furthermore, for any sets of maps $\mathscr{M}_1, \mathscr{M}_2, \cdots, \mathscr{M}_k$, we define a set of maps as

$$\mathscr{M}_1 \odot \cdots \odot \mathscr{M}_k = (\mathscr{M}_1 \odot \cdots \odot \mathscr{M}_{k-1}) \odot \mathscr{M}_k. \tag{2}$$

When $\mathscr{M}_1 = \mathscr{M}_2 = \cdots = \mathscr{M}_k = \mathscr{M}$, we have

$$\mathscr{M}_1 \odot \cdots \odot \mathscr{M}_k = \mathscr{M}^{\odot k}. \tag{3}$$

For any two maps M_1 and M_2, whose respective roots are $r_1 = r(M_1)$ and $r_2 = r(M_2)$, let M be the map obtained by identifying the root-edges e_{r_1} of M_1 and e_{r_2} of M_2 into a single edge e_r, the only edge of M belonging to both M_1 and M_2, where the root-face is as same as that of M_2 and M_1 is inside the inner face on the rooted edge of M_2. The operation of getting M from M_1 and M_2 is called e_r-addition and the resulting map is written as $M = M_1 \hat{+} M_2$. Further, for two sets of maps $\mathscr{M}_1$ and $\mathscr{M}_2$, the set of maps

$$\mathscr{M}_1 \oplus \mathscr{M}_2 = \{M_1 \hat{+} M_2 \mid M_i \in \mathscr{M}_i, i = 1,2\} \tag{4}$$

is said to be the e_r-production of $\mathscr{M}_1$ and $\mathscr{M}_2$.

Furthermore, for any sets of maps $\mathscr{M}_1, \mathscr{M}_2, \cdots, \mathscr{M}_k$, we define a set of maps as

$$\mathscr{M}_1 \oplus \cdots \oplus \mathscr{M}_k = (\mathscr{M}_1 \oplus \cdots \oplus \mathscr{M}_{k-1}) \oplus \mathscr{M}_k. \tag{5}$$

When $\mathscr{M}_1 = \mathscr{M}_2 = \cdots = \mathscr{M}_k = \mathscr{M}$, we have

$$\mathscr{M}_1 \oplus \cdots \oplus \mathscr{M}_k = \mathscr{M}^{\oplus k}. \tag{6}$$

Let

$$\begin{cases} \tilde{\nabla}\mathscr{M} = \sum\limits_{M \in \mathscr{M}} \{\nabla_i M \mid i = 1, 2, \cdots, val(M) - 1\}; \\ \nabla\mathscr{M} = \sum\limits_{M \in \mathscr{M}} \{\nabla_i M \mid i = 0, 1, 2, \cdots, val(M)\}, \end{cases} \tag{7}$$

where $\nabla_i M$ is the map obtained by splitting the root-vertex of M into two vertices v_r' and v_r'' with a new edge $< v_r', v_r'' >$ as the root-edge of the new map $\nabla_i M$ such that the valency of its root-vertex $val(\nabla_i M) = i + 1$.

Further, we define the following two subsets of a set $\mathscr{M}$:

$$\begin{cases} \mathscr{M}^e = \{M \in \mathscr{M} \mid val(M) \equiv 0 (mod\ 2)\}; \\ \mathscr{M}^o = \{M \in \mathscr{M} \mid val(M) \equiv 1 (mod\ 2)\}. \end{cases} \tag{8}$$

Let $\mathscr{E}$ and $\mathscr{U}$ be the sets of all rooted loopless Eulerian planar maps and loopless unicursal planar maps rooted in a vertex of odd valency, respectively. Suppose that their enumerating functions are

$$f_{\mathscr{E}}(x,y) = \sum_{U\in\mathscr{E}} x^{2m(U)}y^{n(U)}, \quad f_{\mathscr{U}}(x,y,z) = \sum_{M\in\mathscr{U}} x^{2m(M)+1}y^{n(M)}z^{2s(M)+1},$$

where $2m(U)$ and $n(U)$ are, respectively, the root-vertex valency and the number of edges of U, $2m(M)+1, n(M)$ and $2s(M)+1$ are, respectively, the root-vertex valency, the number of edges and the valency of the nonrooted odd-valent vertex of M.

Let $\mathscr{E}_{in}$ and $\mathscr{U}_{in}$ denote the sets of all inner Eulerian planar maps and inner unicursal planar maps rooted in a vertex of odd valency, respectively.

Lemma 2.1. Let $\mathscr{E}_{<in>} = \{M - e_r(M) \mid M \in \mathscr{E}_{in}\}$, $\mathscr{U}_{<in>} = \{M - e_r(M) \mid M \in \mathscr{U}_{in}\}$. Then,

$$\mathscr{E}_{<in>} = \mathscr{E}, \quad \mathscr{U}_{<in>} = \mathscr{U}. \tag{9}$$

Proof. The proof of the former formula is similar to that of the latter, so we only prove the latter formula.

For any $M \in \mathscr{U}_{<in>}$, since there is an $M' \in \mathscr{U}_{in}$ such that $M = M' - a'$, a' is the rooted edge of M' and M' has only one loop which is a', it follows that $M \in \mathscr{U}$. On the other hand, for any $M \in \mathscr{U}$, we may always construct a map M' by adding a loop a' as the outer-face boundary of M' from M. It is clear that $M = M' - a'$. We see that $M \in \mathscr{U}_{<in>}$. This proves the formula. □

The family $\mathscr{U}$ can be partitioned into two parts, i.e.,

$$\mathscr{U} = \mathscr{U}_1 + \mathscr{U}_2, \tag{10}$$

where $\mathscr{U}_1 = \{M \mid M \in \mathscr{U}$, the nonrooted end of $e_r(M)$ is an odd-valent vertex$\}$.

Further, we have

$$\mathscr{U}_i = \sum_{j\geq 1} \mathscr{U}_{ij}, \tag{11}$$

in which $\mathscr{U}_{ij} = \{M \mid M \in \mathscr{U}_i, e_r(M)$ is in a j-multi-edge$\}$.

Lemma 2.2. Let $\mathscr{U}_{(ij)} = \{M \bullet e_r(M) \mid M \in \mathscr{U}_{ij}\}, i = 1,2$ and $j \geq 1$. Then,

$$\begin{aligned}\mathscr{U}_{(1j)} =& \mathscr{E}_{in}^{\odot(j-1)} \odot \mathscr{E},\\ \mathscr{U}_{(2j)} =& \mathscr{E}_{in}^{\odot(j-1)} \odot \mathscr{U} + \left(\sum_{r+s=j-2} \mathscr{E}_{in}^{\odot r} \odot \mathscr{U}_{in} \odot \mathscr{E}_{in}^{\odot s}\right) \odot \mathscr{E},\end{aligned} \tag{12}$$

where $\mathscr{E}_{in}^{\odot(0)}$ is as a vertex map.

Proof. For any map $M \in \mathscr{U}_{1j}, e_r(M)$ contains the only two vertices of odd valency. Therefore the contracting of $e_r(M)$ will lead to a map $M \bullet e_r(M) \in \left(\mathscr{E}_{in}^{\odot(j-1)} \odot \mathscr{E}\right)$. On the other hand, by splitting the root-vertex into two, a map in $\mathscr{E}_{in}^{\odot(j-1)} \odot \mathscr{E}$ may produce any one of elements in $\mathscr{U}_{1j}$. This proves the first relation.

Let the set on the right hand side of the second part of (12) be denoted by $\mathcal{M}$ for convenience. For any map $M \in \mathscr{U}_{2j}$, the nonrooted end of $e_r(M)$ is an even-valent vertex. Therefore the contracting of $e_r(M)$ will lead to a map $M \bullet e_r(M) \in \mathcal{M}$. On the other hand, by splitting the root-vertex into two, a map in $\mathcal{M}$ may produce any one of elements in $\mathscr{U}_{2j}$. Thus, the second relation holds. □

Let

$$\begin{aligned}
\widetilde{\mathscr{M}_{1j}} &=(\tilde{\nabla}\mathscr{E}_{in})^{\oplus(j-1)} \oplus (\nabla\mathscr{E}),\\
\widetilde{\mathscr{M}_{2j}} &=(\tilde{\nabla}\mathscr{E}_{in})^{\oplus(j-1)} \oplus (\nabla\mathscr{U})\\
&\quad + \left[\sum_{r+s=j-2} (\tilde{\nabla}\mathscr{E}_{in})^{\oplus r} \oplus (\tilde{\nabla}\mathscr{U}_{in}) \oplus (\tilde{\nabla}\mathscr{E}_{in})^{\oplus s}\right] \oplus (\nabla\mathscr{E}),\\
\widetilde{\mathscr{M}_1} &=\sum_{j\geq 1} \widetilde{\mathscr{M}_{1j}}, \quad \widetilde{\mathscr{M}_2} = \sum_{j\geq 1} \widetilde{\mathscr{M}_{2j}}.
\end{aligned} \tag{13}$$

It is clear that

$$\widetilde{\mathscr{M}_{ij}} = \widetilde{\mathscr{M}_{ij}}^{e} + \widetilde{\mathscr{M}_{ij}}^{o} \tag{14}$$

for $i = 1, 2$ and $j \geq 1$.

Lemma 2.3. For $\mathscr{U}_{1j}$ and $\mathscr{U}_{2j}$, we have

$$\mathscr{U}_{1j} = \widetilde{\mathscr{M}_{1j}}^{o}, \quad \mathscr{U}_{2j} = \widetilde{\mathscr{M}_{2j}}^{o}. \tag{15}$$

Proof. From Lemma 2.2, by splitting the rooted vertex of maps in $\mathscr{U}_{(ij)}$, for $i = 1, 2$ and $j \geq 1$, the lemma follows. □

Theorem 2.1. The enumerating function $f = f_{\mathscr{U}}(x, y, z)$ satisfies the following equation:

$$\begin{aligned}
&\left\{1 + \frac{x^2y[(1+x^2yf_0)^2 + x^2(1+yh_0)^2 - 2x^2(1+yh_0)(1+x^2yf_0)]}{[(1+x^2yf_0)^2 - x^2(1+yh_0)^2]^2}\right\} f =\\
&\frac{xy[(1+x^2yf_0)^2 + x^2(1+yh_0)^2 - 2x^2(1+yh_0)(1+x^2yf_0)]h}{[(1+x^2yf_0)^2 - x^2(1+yh_0)^2]^2}\\
&+ \frac{xyz[z^2f_0(z,y) - x^2f_0(x,y)]}{z^2[1+x^2yf_0(x,y)]^2 - x^2[1+z^2yf_0(z,y)]^2},
\end{aligned} \tag{16}$$

where $f_0 = f_{\mathscr{E}}(x, y), h_0 = f_{\mathscr{E}}(1, y) = h_{\mathscr{E}}(y)$ and $h = f_{\mathscr{U}}(1, y, z) = h_{\mathscr{U}}(y, z)$.

Proof. Let $f_{\mathscr{U}_i}$ be the enumerating function of $\mathscr{U}_i$. We are now going to evaluate the contributions $f_{\mathscr{U}_i}$ of $\mathscr{U}_i$ to f $(i = 1, 2)$.

By Lemma 2.1, we have

$$f_{\mathscr{E}_{in}} = x^2 y f_0, \quad f_{\mathscr{U}_{in}} = x^2 y f. \tag{17}$$

By (11), (13), (14), (15) and (17), we have

$$\begin{aligned}
f_{\widetilde{\mathscr{M}_1}} =& xyz \sum_{j\geq 1}\Bigg(\sum_{M\in\mathscr{E}_{in}} \sum_{k=1}^{2m(M)-1} x^k y^{n(M)} z^{2m(M)-k}\Bigg)^{j-1} \\
&\times \Bigg(\sum_{M\in\mathscr{E}} \sum_{k=0}^{2m(M)} x^k y^{n(M)} z^{2m(M)-k}\Bigg) \\
=& \frac{xyz[zf_0(z,y) - xf_0(x,y)]}{z[1+x^2yf_0(x,y)] - x[1+z^2yf_0(z,y)]} \\
=& \frac{xyz[z^2f_0(z,y) - x^2f_0(x,y)]}{z^2[1+x^2yf_0(x,y)]^2 - x^2[1+z^2yf_0(z,y)]^2} + f_{\widetilde{\mathscr{M}_1}^c}, \\
f_{\widetilde{\mathscr{M}_2}} =& xy \sum_{j\geq 1}\Bigg(\sum_{M\in\mathscr{E}_{in}} \sum_{k=1}^{2m(M)-1} x^k y^{n(M)}\Bigg)^{j-1} \\
&\times \Bigg(\sum_{M\in\mathscr{U}} \sum_{k=0}^{2m(M)+1} x^k y^{n(M)} z^{2s(M)+1}\Bigg) \\
&+ xy\Bigg(\sum_{M\in\mathscr{U}_{in}} \sum_{k=1}^{2m(M)} x^k y^{n(M)} z^{2s(M)+1}\Bigg) \\
&\times \Bigg[\sum_{j\geq 1} j\Bigg(\sum_{M\in\mathscr{E}_{in}} \sum_{k=1}^{2m(M)-1} x^k y^{n(M)}\Bigg)^{j-1}\Bigg]\Bigg(\sum_{M\in\mathscr{E}} \sum_{k=0}^{2m(M)} x^k y^{n(M)}\Bigg) \\
=& \frac{xy(1-x)(h-xf)}{[1-x-xy(h_0-xf_0)]^2} \\
=& \frac{xy(h-xf)}{[(1+x^2yf_0)^2 - x^2(1+yh_0)^2]^2} \\
&\times [(1+x^2yf_0)^2 + x^2(1+yh_0)^2 - 2x^2(1+yh_0)(1+x^2yf_0)] + f_{\widetilde{\mathscr{M}_2}^c}, \\
f_{\mathscr{U}_1} =& \frac{xyz[z^2f_0(z,y) - x^2f_0(x,y)]}{z^2[1+x^2yf_0(x,y)]^2 - x^2[1+z^2yf_0(z,y)]^2}, \\
f_{\mathscr{U}_2} =& \frac{xy(h-xf)[(1+x^2yf_0)^2 + x^2(1+yh_0)^2 - 2x^2(1+yh_0)(1+x^2yf_0)]}{[(1+x^2yf_0)^2 - x^2(1+yh_0)^2]^2}.
\end{aligned}$$

Since $f = f_{\mathscr{U}_1} + f_{\mathscr{U}_2}$, the theorem follows. □

Let $z = 1$ in (16). Then we have

Theorem 2.2. The enumerating function $g = g_{\mathscr{U}}(x,y) = f_{\mathscr{U}}(x,y,1)$ satisfies the equation as follows:

$$\left\{1 + \frac{x^2y[(1+x^2yf_0)^2 + x^2(1+yh_0)^2 - 2x^2(1+yh_0)(1+x^2yf_0)]}{[(1+x^2yf_0)^2 - x^2(1+yh_0)^2]^2}\right\}g =$$
$$\frac{xy[(1+x^2yf_0)^2 + x^2(1+yh_0)^2 - 2x^2(1+yh_0)(1+x^2yf_0)]H}{[(1+x^2yf_0)^2 - x^2(1+yh_0)^2]^2}$$
$$+ \frac{xy(h_0 - x^2f_0)}{(1+x^2yf_0)^2 - x^2(1+yh_0)^2}, \tag{18}$$

where $f_0 = f_{\mathscr{E}}(x,y), h_0 = f_{\mathscr{E}}(1,y) = h_{\mathscr{E}}(y)$ and $H = g_{\mathscr{U}}(1,y) = h_{\mathscr{U}}(y,1) = f_{\mathscr{U}}(1,y,1) = H_{\mathscr{U}}(y)$.

3. Enumeration

In this section we present the explicit formulae for enumerating functions $f = f_{\mathscr{U}}(x,y,z)$, $g = g_{\mathscr{U}}(x,y), h = h_{\mathscr{U}}(y,z)$ and $H = H_{\mathscr{U}}(y)$ by using Lagrangian inversion. Before stating our main results we will use the following well-known results coming from (1), (5) and (7) in [11] by taking $f_0 = f_{\mathscr{E}}(x,y) = h_{\mathscr{E}_{nl}}(x,y)$:

The enumerating function $f_0 = f_{\mathscr{E}}(x,y)$ satisfies the following quadratic equation:

$$x^4y^2f_0^3 + 2x^2yf_0^2 + [1 - x^2y - x^2(1+yh_0)^2]f_0$$
$$+ x^2(1+yh_0) - 1 = 0, \tag{19}$$

where $h_0 = f_{\mathscr{E}}(1,y) = h_{\mathscr{E}}(y)$ and

$$y = \eta(1-\eta)(1-\eta^2)^2, \quad yh_0 = \eta(1-\eta-\eta^2), \tag{20}$$

$$\begin{cases} x^2y = \xi(1-\eta)(1-\xi\eta)^2, \quad y = \eta(1-\eta)(1-\eta^2)^2; \\ x^2yf_0(x,y) = \xi(1-\eta-\xi\eta). \end{cases} \tag{21}$$

Let θ be the root of characteristic equation of (18). Then we get

$$1 + \frac{\theta^2y}{\{[1+\theta^2yf_0(\theta,y)]^2 - \theta^2(1+yh_0)^2\}^2}$$
$$\times \{[1+\theta^2yf_0(\theta,y)]^2 + \theta^2(1+yh_0)^2 - 2\theta^2(1+yh_0)[1+\theta^2yf_0(\theta,y)]\} = 0,$$
$$\frac{\theta y\{[1+\theta^2yf_0(\theta,y)]^2 + \theta^2(1+yh_0)^2 - 2\theta^2(1+yh_0)[1+\theta^2yf_0(\theta,y)]\}H}{\{[1+\theta^2yf_0(\theta,y)]^2 - \theta^2(1+yh_0)^2\}^2}$$
$$+ \frac{\theta y[h_0 - \theta^2f_0(\theta,y)]}{[1+\theta^2yf_0(\theta,y)]^2 - \theta^2(1+yh_0)^2} = 0. \tag{22}$$

By (22) we have

$$H = \frac{\theta^2 y[h_0 - \theta^2 f_0(\theta, y)]}{[1 + \theta^2 y f_0(\theta, y)]^2 - \theta^2(1 + yh_0)^2}. \tag{23}$$

By (20), (21), the first part of (22) and (23), one may find the parametric expression of $H_{\mathscr{U}} = H_{\mathscr{U}}(y)$ as follows:

$$y = \eta(1-\eta)(1-\eta^2)^2, \quad H = \frac{\eta}{(1-\eta^2)^2}. \tag{24}$$

Theorem 3.1. The enumerating function $H = H_{\mathscr{U}}(y)$ has the following explicit expression:

$$H_{\mathscr{U}}(y) = y + \sum_{n\geq 2} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(2n+j+2)!(2n-2j-4)!R_n(j)}{n!(2n+2)!j!(n-2j-1)!} y^n, \tag{25}$$

where $R_n(j) = (2n-2j-2)(2n-2j-3) + 3(n-2j-1)(n-2j-2)$.
Proof. Applying Lagrangian inversion with one parameter [35] to (24), we obtain

$$\begin{aligned} H_{\mathscr{U}}(y) &= \sum_{n\geq 1} \frac{y^n}{n!} \frac{d^{n-1}}{d\eta^{n-1}} \left\{ \frac{1+3\eta^2}{(1-\eta)^n(1-\eta^2)^{2n+3}} \right\}\Bigg|_{\eta=0} \\ &= \sum_{n\geq 1} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{n} \left[\binom{2n-2j-2}{n-2j-1} + 3\binom{2n-2j-4}{n-2j-3} \right] \binom{2n+j+2}{j} y^n \\ &= y + \sum_{n\geq 2} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(2n+j+2)!(2n-2j-4)!R_n(j)}{n!(2n+2)!j!(n-2j-1)!} y^n, \end{aligned}$$

where $R_n(j) = (2n-2j-2)(2n-2j-3) + 3(n-2j-1)(n-2j-2)$.

This completes the proof of Theorem 3.1. □

Accordingly, one may count loopless unicursal planar maps rooted in a vertex of odd valency by edges. For instance, there are 6 such maps with 3 edges as shown in Fig. 1.

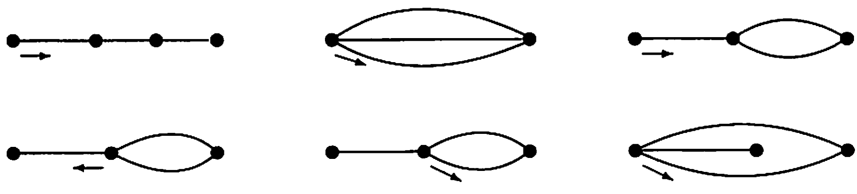

Fig. 1

By substituting (20), (21) and (24) into Eq. (18) and regrouping the terms, one may find the following parametric expression of the function $g_{\mathscr{U}} = g_{\mathscr{U}}(x, y)$:

$$x^2 y = \xi(1-\eta)(1-\xi\eta)^2, \quad y = \eta(1-\eta)(1-\eta^2)^2,$$
$$x^{-1} g = \frac{\frac{\eta(1-\eta-\eta^2-\xi\eta)}{(1-\xi\eta)^3} - \frac{\eta^2[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]}{(1-\frac{\xi}{\eta})(1-\xi\eta)^4}}{1 - \frac{\xi[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]}{(1-\frac{\xi}{\eta})(1-\xi\eta)^2}}. \tag{26}$$

According to (26), we obtain

$$\Delta_{(\xi,\eta)} = \begin{vmatrix} \frac{1-3\xi\eta}{1-\xi\eta} & * \\ 0 & \frac{1-\eta-6\eta^2}{1-\eta^2} \end{vmatrix} = \frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\xi\eta)(1-\eta^2)}. \tag{27}$$

Theorem 3.2. The enumerating function $g = g_{\mathscr{U}}(x, y)$ has the following explicit expression:

$$g_{\mathscr{U}}(x, y) = xy + \sum_{n\geq 2}\sum_{m=0}^{n-1} g_{m,n} x^{2m+1} y^n, \tag{28}$$

where

$$g_{m,n} = \sum_{l=0}^{\min\{\lfloor\frac{m+j}{2}\rfloor, \lfloor\frac{n+t+r+j-m-2}{2}\rfloor\}} \sum_{r=0}^{l+1}\sum_{j=0}^{r+1}\sum_{t=0}^{m+j-2l}\sum_{i=0}^{m+j-2l-t}\sum_{k=0}^{\lfloor\frac{q-p+1}{2}\rfloor} (-1)^{j+1} 2^i$$
$$\times \binom{r}{i}\binom{2n-2m+k}{k} Q_{m,n}(l,r,t,i,j,k), \tag{29}$$

in which

$$Q_{m,n}(l,r,t,i,j,k) =$$
$$\frac{2m+2l-2p+3}{2m+2l+3}\binom{l+t-1}{t}\binom{l}{r}\binom{r+1}{j}\binom{2m+2l+p+2}{p}$$
$$\times \left[\binom{n+q-j-p-2k-1}{q-p-2k+1} - 6\binom{n+q-j-p-2k-2}{q-p-2k-1}\right]$$
$$+ \left[\frac{2m+2l-2p+1}{2m+2l+3}\binom{l+t-1}{t}\binom{l}{r}\binom{2m+2l+p+3}{p+1}\right.$$
$$\left. + \frac{2m+2l-2p+4}{2m+2l+4}\binom{l+t}{t}\binom{l+1}{r}\binom{2m+2l+p+3}{p}\right]\binom{r}{j}$$
$$\times \left[\binom{n+q-j-p-2k-3}{q-p-2k-1} - 6\binom{n+q-j-p-2k-4}{q-p-2k-3}\right], \tag{30}$$

where $p = m + j - 2l - t - i - 1, q = n + t + r + j - m - 2l - 3$.

Proof. By employing Lagrangian theorem with two variables [15], from (26) and (27) one may find that

$$g_{\mathscr{U}}(x,y) = \sum_{m,n\geq 0} \partial^{(m,n)}_{(\xi,\eta)} \frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\eta)^{m+n}(1-\xi\eta)^{2m+1}(1-\eta^2)^{2n+1}}$$

$$\times \frac{\frac{\eta(1-\eta-\eta^2-\xi\eta)}{(1-\xi\eta)^3} - \frac{\eta^2[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]}{(1-\frac{\xi}{\eta})(1-\xi\eta)^4}}{1-\frac{\xi[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]}{(1-\frac{\xi}{\eta})(1-\xi\eta)^2}} x^{2m+1}y^{m+n}$$

$$= \sum_{n\geq 0}\sum_{m=0}^{n} \partial^{(m,n-m)}_{(\xi,\eta)} \frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\eta)^{n}(1-\xi\eta)^{2m+1}(1-\eta^2)^{2n-2m+1}}$$

$$\times \frac{\frac{\eta(1-\eta-\eta^2-\xi\eta)}{(1-\xi\eta)^3} - \frac{\eta^2[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]}{(1-\frac{\xi}{\eta})(1-\xi\eta)^4}}{1-\frac{\xi[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]}{(1-\frac{\xi}{\eta})(1-\xi\eta)^2}} x^{2m+1}y^{n}$$

$$= xy + \sum_{n\geq 2}\sum_{m=0}^{n-1}\sum_{l=0}^{m} \Bigg[\partial^{(m-l,n-m-1)}_{(\xi,\eta)} \frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\eta)^{n}(1-\xi\eta)^{2m+2l+4}}$$

$$\times \frac{(1-\eta-\eta^2-\xi\eta)[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]^l}{(1-\eta^2)^{2n-2m+1}(1-\frac{\xi}{\eta})^l}$$

$$- \partial^{(m-l,n-m-2)}_{(\xi,\eta)} \frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\eta)^{n}(1-\xi\eta)^{2m+2l+5}}$$

$$\times \frac{[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]^{l+1}}{(1-\eta^2)^{2n-2m+1}(1-\frac{\xi}{\eta})^{l+1}} \Bigg] x^{2m+1}y^n$$

$$= xy + \sum_{n\geq 2}\sum_{m=0}^{n-1} g_{m,n} x^{2m+1} y^n,$$

where

$$g_{m,n} = \sum_{l=0}^{m} \Bigg[\partial^{(m-l,n-m-1)}_{(\xi,\eta)} \frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\eta)^{n}(1-\xi\eta)^{2m+2l+4}}$$

$$\times \frac{(1-\eta-\eta^2-\xi\eta)[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]^l}{(1-\eta^2)^{2n-2m+1}(1-\frac{\xi}{\eta})^l}$$

$$- \partial^{(m-l,n-m-2)}_{(\xi,\eta)} \frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\eta)^{n}(1-\xi\eta)^{2m+2l+5}}$$

$$\times \frac{[\xi\eta^2 + (1+2\xi)(\xi\eta+\eta-1)]^{l+1}}{(1-\eta^2)^{2n-2m+1}(1-\frac{\xi}{\eta})^{l+1}}\Bigg]$$

$$= \sum_{l=0}^{m}\sum_{t=0}^{m-l}\Bigg[\binom{l+t-1}{t}\partial_{(\xi,\eta)}^{(m-l-t,n-m+t-1)}\frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\eta)^n(1-\xi\eta)^{2m+2l+4}}$$

$$\times \frac{(1-\eta-\eta^2-\xi\eta)[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]^l}{(1-\eta^2)^{2n-2m+1}}$$

$$- \binom{l+t}{t}\partial_{(\xi,\eta)}^{(m-l-t,n-m+t-2)}\frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\eta)^n(1-\xi\eta)^{2m+2l+5}}$$

$$\times \frac{[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]^{l+1}}{(1-\eta^2)^{2n-2m+1}}\Bigg]$$

$$= \sum_{l=0}^{\min\{\lfloor\frac{m+j}{2}\rfloor,\lfloor\frac{n+t+r+j-m-2}{2}\rfloor\}}\sum_{r=0}^{l+1}\sum_{j=0}^{r+1}\sum_{t=0}^{m+j-2l}\sum_{i=0}^{m+j-2l-t}(-1)^{j+1}2^i\binom{r}{i}$$

$$\times \Bigg[\binom{l+t-1}{t}\binom{l}{r}\binom{r+1}{j}\partial_{(\xi,\eta)}^{(p,q+1)}\frac{(1-3\xi\eta)}{(1-\eta)^{n-j}(1-\xi\eta)^{2m+2l+4}}$$

$$\times \frac{(1-\eta-6\eta^2)}{(1-\eta^2)^{2n-2m+1}} + \binom{l+t-1}{t}\binom{l}{r}\binom{r}{j}$$

$$\times \partial_{(\xi,\eta)}^{(p+1,q)}\frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\eta)^{n-j}(1-\xi\eta)^{2m+2l+4}(1-\eta^2)^{2n-2m+1}} + \binom{l+t}{t}\binom{l+1}{r}$$

$$\times \binom{r}{j}\partial_{(\xi,\eta)}^{(p,q-1)}\frac{(1-3\xi\eta)(1-\eta-6\eta^2)}{(1-\eta)^{n-j}(1-\xi\eta)^{2m+2l+5}(1-\eta^2)^{2n-2m+1}}\Bigg],$$

in which $p = m+j-2l-t-i-1, q = n+t+r+j-m-2l-3$. Thus,

$$g_{m,n} = \sum_{l=0}^{\min\{\lfloor\frac{m+j}{2}\rfloor,\lfloor\frac{n+t+r+j-m-2}{2}\rfloor\}}\sum_{r=0}^{l+1}\sum_{j=0}^{r+1}\sum_{t=0}^{m+j-2l}\sum_{i=0}^{m+j-2l-t}(-1)^{j+1}2^i\binom{r}{i}$$

$$\times \left\{ \binom{l+t-1}{t} \binom{l}{r} \binom{r+1}{j} \frac{(2m+2l+p+2)!(2m+2l-2p+3)}{p!(2m+2l+3)!} \right.$$

$$\times\, \partial_\eta^{q-p+1} \frac{1-\eta-6\eta^2}{(1-\eta)^{n-j}(1-\eta^2)^{2n-2m+1}}$$

$$+ \left[\binom{l+t-1}{t} \binom{l}{r} \binom{r}{j} \frac{(2m+2l+p+3)!(2m+2l-2p+1)}{(p+1)!(2m+2l+3)!} \right.$$

$$\left. + \binom{l+t}{t} \binom{l+1}{r} \binom{r}{j} \frac{(2m+2l+p+3)!(2m+2l-2p+4)}{p!(2m+2l+4)!} \right]$$

$$\left. \times\, \partial_\eta^{q-p-1} \frac{1-\eta-6\eta^2}{(1-\eta)^{n-j}(1-\eta^2)^{2n-2m+1}} \right\}$$

$$= \sum_{l=0}^{\min\{\lfloor \frac{m+j}{2} \rfloor, \lfloor \frac{n+t+r+j-m-2}{2} \rfloor\}} \sum_{r=0}^{l+1} \sum_{j=0}^{r+1} \sum_{t=0}^{m+j-2l} \sum_{i=0}^{m+j-2l-t} \sum_{k=0}^{\lfloor \frac{q-p+1}{2} \rfloor} (-1)^{j+1} 2^i \binom{r}{i}$$

$$\binom{2n-2m+k}{k} \left\{ \frac{2m+2l-2p+3}{2m+2l+3} \binom{l+t-1}{t} \binom{l}{r} \binom{r+1}{j} \right.$$

$$\times \binom{2m+2l+p+2}{p} \left[\binom{n+q-j-p-2k-1}{q-p-2k+1} \right.$$

$$\left. - 6 \binom{n+q-j-p-2k-2}{q-p-2k-1} \right]$$

$$+ \left[\frac{2m+2l-2p+1}{2m+2l+3} \binom{l+t-1}{t} \binom{l}{r} \binom{2m+2l+p+3}{p+1} \right.$$

$$\left. + \frac{2m+2l-2p+4}{2m+2l+4} \binom{l+t}{t} \binom{l+1}{r} \binom{2m+2l+p+3}{p} \right] \binom{r}{j}$$

$$\left. \times \left[\binom{n+q-j-p-2k-3}{q-p-2k-1} - 6 \binom{n+q-j-p-2k-4}{q-p-2k-3} \right] \right\},$$

which is equivalent to the theorem.

By (21) we have

$$\begin{cases} z^2 y = \lambda(1-\eta)(1-\lambda\eta)^2, \quad y = \eta(1-\eta)(1-\eta^2)^2; \\ z^2 y f_0(z,y) = \lambda(1-\eta-\lambda\eta), \end{cases} \tag{31}$$

where λ is a new parameter corresponding to z.

Similarly, by (16), (20), (21) and (31), one may find the parametric expression of $h = h_{\mathscr{M}}(y,z)$ as follows:

$$y = \eta(1-\eta)(1-\eta^2)^2, \quad z^2 y = \lambda(1-\eta)(1-\lambda\eta)^2,$$
$$z^{-1}h = \frac{\eta}{(1-\lambda\eta)^2}, \tag{32}$$

from which we get

$$\Delta_{(\eta,\lambda)} = \begin{vmatrix} \frac{1-\eta-6\eta^2}{1-\eta^2} & 0 \\ * & \frac{1-3\lambda\eta}{1-\lambda\eta} \end{vmatrix} = \frac{(1-\eta-6\eta^2)(1-3\lambda\eta)}{(1-\eta^2)(1-\lambda\eta)}. \tag{33}$$

Theorem 3.3. The enumerating function $h = h_{\mathscr{U}}(y,z)$ has the following explicit expression:

$$h_{\mathscr{U}}(y,z) = yz + \sum_{n\geq 2} \sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{n-2s-1}{2} \rfloor} \frac{2(3s+1)!(2n-2s+i)!}{s!(2s+2)!(2n-2s)!(n-1)!}$$
$$\times \frac{(2n-2s-2i-4)!J_{n,s}(i)}{i!(n-2s-2i-1)!} y^n z^{2s+1}, \tag{34}$$

where $J_{n,s}(i) = (n-1)^2 + (n-2s-2i-2)(12s-5n+12i+5)$.

Proof. By using Lagrangian theorem with two variables [15], from (32) and (33) one may find that

$$h_{\mathscr{U}}(y,z) = \sum_{n\geq 1} \sum_{s\geq 0} \partial_{(\eta,\lambda)}^{(n-1,s)} \frac{(1-\eta-6\eta^2)(1-3\lambda\eta)}{(1-\eta)^{n+s}(1-\eta^2)^{2n+1}(1-\lambda\eta)^{2s+3}} y^{n+s} z^{2s+1}$$

$$= yz + \sum_{n\geq 2} \sum_{s=0}^{n-1} \partial_{(\eta,\lambda)}^{(n-s-1,s)} \frac{(1-\eta-6\eta^2)(1-3\lambda\eta) y^n z^{2s+1}}{(1-\eta)^n (1-\eta^2)^{2n-2s+1}(1-\lambda\eta)^{2s+3}}$$

$$= yz + \sum_{n\geq 2} \sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2(3s+1)!}{s!(2s+2)!} \partial_\eta^{n-2s-1} \frac{(1-\eta-6\eta^2) y^n z^{2s+1}}{(1-\eta)^n (1-\eta^2)^{2n-2s+1}}$$

$$= yz + \sum_{n\geq 2} \sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{n-2s-1}{2} \rfloor} \frac{2(3s+1)!}{s!(2s+2)!} \binom{2n-2s+i}{i}$$
$$\times \partial_\eta^{n-2s-2i-1} \frac{1-\eta-6\eta^2}{(1-\eta)^n} y^n z^{2s+1}$$

$$= yz + \sum_{n\geq 2} \sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{n-2s-1}{2} \rfloor} \frac{2(3s+1)!(2n-2s+i)!}{s!(2s+2)!(2n-2s)!(n-1)!}$$
$$\times \frac{(2n-2s-2i-4)!J_{n,s}(i)}{i!(n-2s-2i-1)!} y^n z^{2s+1},$$

where $J_{n,s}(i) = (n-1)^2 + (n-2s-2i-2)(12s-5n+12i+5)$.

This completes the proof of Theorem 3.3. □

By substituting (20), (21), (31) and (32) into Equ. (16), one may find the following parametric expression of the function $f = f_{\mathscr{U}}(x,y,z)$:

$$x^2y = \xi(1-\eta)(1-\xi\eta)^2, \quad y = \eta(1-\eta)(1-\eta^2)^2,$$
$$z^2y = \lambda(1-\eta)(1-\lambda\eta)^2,$$
$$x^{-1}y^{-1}z^{-1}f = \frac{\frac{1-\eta-\eta\lambda-\xi\eta}{(1-\eta)(1-\lambda\eta)^2(1-\xi\eta)^2(1-\xi\lambda)} - \frac{\eta[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]}{(1-\eta)(1-\frac{\xi}{\eta})(1-\xi\eta)^4(1-\lambda\eta)^2}}{1-\frac{\xi[\xi\eta^2+(1+2\xi)(\xi\eta+\eta-1)]}{(1-\frac{\xi}{\eta})(1-\xi\eta)^2}}. \quad (35)$$

According to (35), we get

$$\Delta_{(\xi,\eta,\lambda)} = \begin{vmatrix} \frac{1-3\xi\eta}{1-\xi\eta} & * & 0 \\ 0 & \frac{1-\eta-6\eta^2}{1-\eta^2} & 0 \\ 0 & * & \frac{1-3\lambda\eta}{1-\lambda\eta} \end{vmatrix} = \frac{(1-3\xi\eta)(1-\eta-6\eta^2)(1-3\lambda\eta)}{(1-\xi\eta)(1-\eta^2)(1-\lambda\eta)}. \quad (36)$$

Of course, the parametric expressions given by (35) and (36) allow us to employ Lagrangian inversion with three variables for finding an explicit expression of the enumerating function $f_{\mathscr{U}}(x,y,z)$. But that would take up too much space in this article; so we leave it for a forthcoming article.

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