On Szeged polynomial of graphs with even number of vertices

Mehdi Eliasi[1] and Bijan Taeri[2]

Department of Mathematics, Faculty of Khansar, University of Isfahan
Isfahan 81746-73441, Iran
and
Department of Mathematical Sciences, Isfahan University of Technology
Isfahan 84156-83111, Iran

Abstract. The Szeged polynomial of a connected graph G, is defined as $Sz(G,x) = \sum_{e\in E(G)} x^{n_u(e)n_v(e)}$, where $n_u(e)$ is the number of vertices of G lying closer to u than to v, $n_v(e)$ is the number of vertices of G lying closer to v than to u and the summation goes over all edges $e = uv \in E(G)$ of G. Ashrafi et. al. (On Szeged polynomial of a graph, Bul. Iran. Math. Soc. **33** (2007) 37-46.) proved that if the number of the vertices of G is even, then $\deg(Sz(G,x)) \leq \frac{1}{4}|V(G)|^2$, where $V(G)$ is the set of vertices of G. In this paper we study the structure of graphs, with even number of vertices, for which the equality holds. Also we examine equality for the sum of graphs.

MSC 2000: Primary 05C12; Secondary 05A15, 05C05
Keywords: Szeged index, Szeged Polynomial

Introduction

Let G be a connected finite undirected graph without loops or multiple edges with vertex and edge sets $V(G)$ and $E(G)$, respectively. The distance between two vertices u and v of G, denoted by $d_G(u,v)$ (or simply $d(u,v)$ when G requires no explicit reference), is defined as the number of edges in a shortest path connecting u and v. For an edge $e = uv$ of G let $B_u(e)$ be the set of vertices closer to u than v,

$$B_u(e) = \{x \in V(G) \mid d(x,u) < d(x,v)\}.$$

Also let $n_u(e) = |B_u(e)|$. The set $B_u(e)$ is an important concept in metric graph theory (see for example [1, 2, 4, 5]). Ivan Gutman [3] defined the Szeged index, $Sz(G)$, of a graph G as

$$Sz(G) = \sum_{e\in E(G)} n_u(e)n_v(e).$$

[1]E-mail: eliasi@math.iut.ac.ir
[2]E-mail: b.taeri@cc.iut.ac.ir (Corresponding Author)

Recently Ashrafi et. al. [1] defined the Szeged polynomial, $Sz(G,x)$, of a connected graph G as

$$Sz(G,x) = \sum_{e\in E(G)} x^{n_u(e)n_v(e)}.$$

They proved that if $|V(G)|$ is an even number, then $\deg(Sz(G,x)) \leq \frac{1}{4}|V(G)|^2$. They characterized bipartite graphs with even number of vertices for which the upper bound is attained and posed the following question.

Question. Suppose that G is a graph with even number of vertices and $\deg(Sz(G,x)) = \frac{1}{4}|V(G)|^2$. What we can say about the structure of G?

For convenience we say that a connected graph G is an A-graph if $\deg(Sz(G,x)) = \frac{1}{4}|V(G)|^2$. In this paper we give an answer to the above question and characterize A-graphs with even number of vertices. Also we prove that the sum of two graphs with even number of vertices is an A-graph if and only if one of them is an A-graph. We define a class A^* of graphs and prove that the sum of two graphs is an A^*-graph if and only if one of them is an A^*-graph. Also present a sufficient condition so that the composition of two graphs be an A^*-graph.

Main results

Let G be a connected graph and $u \in V(G)$. The sum of distances between u and vertices of G is denoted by $d_G(u)$ (or simply $d(u)$ when G requires no explicit reference), that is

$$d_G(u) = \sum_{x\in V(G)} d(u,x).$$

First we state the following two results for the convenience of the reader.

Lemma 2.1. ([1, Theorem 4.4]) Suppose that G is a graph with even number of vertices. Then $\deg(Sz(G,x)) \leq \frac{1}{4}|V(G)|^2$.

Lemma 2.2. ([2, Lemma 2]) Let G be a connected graph and $e = uv \in E(G)$, with $u, v \in V(G)$. Then $n_u(e) - n_v(e) = d(v) - d(u)$.

By using the above Lemmas we can prove the main result of the paper which improves the inequality in Lemma 2.1.

Theorem 2.3. Let G be a connected graph with n vertices. Then

$$\deg(Sz(G,x)) \leq \frac{1}{4} \max_{e=uv\in E(G)} \{n^2 - (n_u(e) - n_v(e))^2\}.$$

Proof. Suppose that $e = uv$ is an edge of G. Then

$$n \geq n_u(e) + n_v(e). \tag{1}$$

Also by Lemma 2.2 we have

$$d(v) - d(u) = n_u(e) - n_v(e). \tag{2}$$

Adding both sides of (1) and (2) we have

$$n + d(v) - d(u) \geq 2n_u(e)$$

and subtracting (2) from(1) we have

$$n + d(u) - d(v) \geq 2n_v(e).$$

Therefore

$$\frac{1}{2}(n + d(v) - d(u)) \geq n_u(e) \geq 0$$

and

$$\frac{1}{2}(n + d(u) - d(v)) \geq n_v(e) \geq 0.$$

Multiplying above inequities we obtain that

$$\frac{1}{4}\left(n^2 - (d(u) - d(v))^2\right) \geq n_v(e)n_u(e) \geq 0. \tag{3}$$

But for computing $Sz(G, x)$ we must calculate the sum of terms $x^{n_v(e)n_u(e)}$, where the summation goes over $e = uv \in E(G)$. Thus from inequity (3) we have

$$\deg(Sz(G,x)) \leq \frac{1}{4} \max_{uv \in E(G)} \{n^2 - (d(u) - d(v))^2\}$$

and so by Lemma 2.2

$$\deg(Sz(G,x)) \leq \frac{1}{4} \max_{uv \in E(G)} \{n^2 - (n_u(e) - n_v(e))^2\},$$

which completes the proof. ∎

Now we can determine the structure of A-graphs with even number of vertices

Corollary 2.4. Let G be a connected graph with n vertices and n is even. Then G is an A-graph if and only if there exists an edge $e = uv$ such that $n_u(e) = n_v(e) = \frac{1}{2}n$.

Proof. Suppose that G ia an A-graph. Then there exists an edge $e = uv$ such that $n_u(e)n_v(e) = \frac{1}{4}n^2$. Also by (3) in Theorem 2.3 and Lemma 2.2 we have $\frac{1}{4}\left(n^2 - (n_u(e) - n_v(e))^2\right) \geq n_u(e)n_v(e)$. Therefore

$$\frac{1}{4}n^2 \geq \frac{1}{4}\left(n^2 - (n_u(e) - n_v(e))^2\right) \geq n_v(e)n_u(e) = \frac{1}{4}n^2.$$

Thus $n_u(e) = n_v(e) = \frac{1}{2}n$.

Conversely suppose that $n_u(e) = n_v(e) = \frac{1}{2}n$, for some $e = uv \in E(G)$. Then $n_u(e)n_v(e) = \frac{1}{4}n^2$ and so by Theorem 2.3, $\deg(Sz(G,x)) = \frac{1}{4}n^2$. Thus G is an A-graph. ■

Now we investigate bipartite A-graphs

Recall that the sum $G_1 + G_2$, of two connected graphs G_1 and G_2, has the vertex set $V(G_1+G_2) = V_1 \times V_2$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 + G_2$ are adjacent if and only if $[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2$ and $u_1v_1 \in E(G_1)]$.

Corollary 2.5. Let G and H be two connected graphs with even number of vertices. Then $G + H$ is A-graph if and only if G or H is A-graph.

Proof. Let $X = G + H$. Assume X is A-graph and let $e = uv$ be an edge of X that $n_u(e) = n_v(e) = \frac{1}{2}|X|$. We may assume, without loss of generality, that $u = (u_1, u_2)$, $v = (v_1, u_2)$, where $u_1v_1 \in E(G)$. According to the proof of Theorem 1 in [6] for each $(u_1, u_2), (x, y) \in V(X)$ we have

$$d_X((x,y),(u_1,u_2)) = d_G(x,u_1) + d_H(y,u_2).$$

Hence

$$\begin{aligned} B_u(e) &= \{(x,y) \in V(G) \times V(H) \mid d_X((x,y),(u_1,u_2)) < d_X((x,y),(v_1,u_2))\} \\ &= \{(x,y) \in V(G) \times V(H) \mid d_G(x,u_1) < d_G(x,v_1)\}. \end{aligned}$$

Therefore

$$n_u(e) = n_{u_1}(u_1v_1)|V(H)|.$$

By a similar argument we obtain that

$$n_v(e) = n_{v_1}(u_1v_1)|V(H)|.$$

Since $n_u(e) = n_v(e) = \frac{1}{2}|V(G)||V(H)$, we have

$$n_{u_1}(u_1v_1) = n_{v_1}(u_1v_1) = \frac{1}{2}|V(G)|,$$

where $u_1v_1 \in E(G)$. Therefore by Theorem 2.3 G is A-graph.
Conversely, let G be A-graph and $u_1v_1 \in E(G)$ that $n_{u_1}(u_1v_1) = n_{v_1}(u_1v_1) = \frac{1}{2}|V(G)|$. Suppose that u_2 is a vertex of H. Then by similar argument as above we obtain that

$$\begin{aligned} n_{(u_1,u_2)}((u_1,u_2)(v_1,u_2)) &= n_{u_1}(u_1v_1)|V(H)|, \\ n_{(v_1,u_2)}((u_1,u_2)(v_1,u_2)) &= n_{v_1}(u_1v_1)|V(H)|. \end{aligned}$$

Therefore $n_{(u_1,u_2)}((u_1,u_2)(v_1,u_2)) = n_{(v_1,u_2)}((u_1,u_2)(v_1,u_2)) = \frac{1}{2}|G+H|$ and $G+H$ is A-graph. ■

We can extend the previous theorem to the sum of n connected graphs.

Corollary 2.6. Let $G_1, G_2, \dots, G_n$ be connected graphs with even number of vertices. Then $G = \bigotimes_{i=1}^{n} G_i$ is A-graph if and only if G_i is an A-graph, for some i.

Now we extend the concept of A-graphs.

Definition 2.7. Let G be a connected graph. Then G is an A^*-graph if there exists an edge $e = uv$ such that $n_u(e) = n_v(e)$.

Note that, by Lemma 2.2, if there exists an edge $e = uv$ such that $d(u) = d(v)$, then G is an A^*-graph. Recall that if for each $e = uv \in E(G)$, $n_u(e) = n_v(e)$, then G is called a distance-balanced graph. (See for example [4])

Proposition 2.8. Let G and H be two connected graph. Then $G+H$ is A^*-graph if and only if G or H is A^*-graph.

Proof. Let $X = G+H$. Assume X is A-graph and let $e = uv$ be an edge of X that $d_X(u) = d_X(v)$. We may assume, without loss of generality, that $u = (u_1, u_2)$, $v = (v_1, u_2)$, where $u_1v_1 \in E(G)$. For each $(u_1,u_2), (x,y) \in V(X)$ we have

$$d_X((x,y),(u_1,u_2)) = d_G(x,u_1) + d_H(y,u_2).$$

Hence

$$\begin{aligned} d_X((u_1,u_2)) &= \sum_{x\in V(G)} \sum_{y\in V(H)} d_X((x,y),(u_1,u_2)) \\ &= \sum_{x\in V(G)} \sum_{y\in V(H)} \Big(d_G(x,u_1) + d_H(y,u_2)\Big) \\ &= \sum_{x\in V(G)} \sum_{y\in V(H)} d_G(x,u_1) + \sum_{x\in V(G)} \sum_{y\in V(H)} d_H(y,u_2) \\ &= \sum_{x\in V(G)} |V(H)| d_G(x,u_1) + \sum_{x\in V(G)} d_H(u_2) \\ &= |V(H)| d_G(u_1) + |V(G)| d_H(u_2). \end{aligned}$$

By a similar argument we obtain that

$$d_X((v_1,u_2)) = |V(H)|d_G(v_1) + |V(G)|d_H(u_2).$$

Since $d_X(u) = d_X(v)$, we have $d_G(u_1) = d_G(v_1)$, where $u_1v_1 \in E(G)$. Therefore G is A^*-graph.

Conversely, suppose that G is an A^*-graph and $d_G(u_1) = d_G(v_1)$, where $u_1v_1 \in E(G)$. Suppose that u_2 is an arbitrary vertex of H. Then by a similar argument as above we obtain that

$$\begin{aligned} d_X((u_1,u_2)) &= |V(H)|d_G(u_1) + |V(G)|d_H(u_2), \\ d_X((v_1,u_2)) &= |V(H)|d_G(v_1) + |V(G)|d_H(u_2) \end{aligned}$$

Therefore $d_X(u) = d_X(v)$ and $G + H$ is A^*-graph. ∎

Recall that the composition $G_1[G_2]$ of two connected graphs G_1 and G_2 has the vertex set $V(G_1[G_2]) = V_1 \times V_2$, and two vertices (u_1,u_2) and (v_1,v_2) of $G_1[G_2]$ are adjacent if and only if $[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$ or $u_1v_1 \in E(G_1)$.

Proposition 2.9. Let G_1 and G_2 be two connected graphs. If G_1 is A^*-graph or G_2 has an edge whose ends points have the same degree, then $G_1[G_2]$ is A^*-graph.

Proof. Let $X = G_1[G_2]$. Suppose that $u = (u_1,u_2)$ and $v = (v_1,v_2)$ are vertices of $G_1[G_2]$. Then

$$d_X(u,v) = \begin{cases} d_{G_1}(u_1,v_1) & \text{if} \quad u_1 \neq v_1 \\ 1 & \text{if} \quad u_1 = v_1 \quad \text{and} \quad u_2v_2 \in E(G_2) \\ 2 & \text{if} \quad u_1 = v_1 \quad \text{and} \quad u_2v_2 \notin E(G_2). \end{cases}$$

So

$$\begin{aligned} d_X((u_1,u_2)) &= \sum_{(v_1,v_2)\in V(X)} d_X((u_1,u_2),(v_1,v_2)) \\ &= \sum_{v_2\in V_2}\sum_{v_1\in V_1} d_{G_1}(u_1,v_1) + \sum_{v_2\in V(G_2),u_2v_2\notin E(G_2)} 2 + \sum_{v_2\in V(G_2),u_2v_2\in E(G_2)} 1 \\ &= |V(G_2)|d_{G_1}(u_1) + 2\Big(|V(G_2)| - \deg(u_2)\Big) + \deg(u_2) \\ &= |V(G_2)|d_{G_1}(u_1) + 2|V(G_2)| - \deg(u_2). \end{aligned} \tag{4}$$

Suppose that G_1 is A^*-graph and u_1v_1 is an edge of G_1 which $d_{G_1}(u_1) = d_{G_1}(v_1)$. Then for an arbitrary vertex u_2 of G_2 we have $(u_1,u_2)(v_1,u_2) \in E(X)$ and by (4)

$$\begin{aligned} d_X((u_1,u_2)) &= |V(G_2)|d_{G_1}(u_1) + 2|V(G_2)| \deg(u_2) \\ &= |V(G_2)|d_{G_1}(v_1) + 2|V(G_2)| - \deg(u_2) \\ &= d_X((v_1,u_2)). \end{aligned}$$

Thus X is A^*-graph.
Now suppose that $u_2v_2 \in E(G_2)$ which $\deg(u_2) = \deg(v_2)$. Then for an arbitrary vertex u_1 of G_1 we have $(u_1, u_2)(u_1, v_2) \in E(X)$ and by (4)

$$\begin{aligned} d_X((u_1,u_2)) &= |V(G_2)|d_{G_1}(u_1) + 2|V(G_2)| - \deg(u_2) \\ &= |V(G_2)|d_{G_1}(u_1) + 2|V(G_2)| - \deg(v_2) \\ &= d_X((u_1,v_2)). \end{aligned}$$

Therefore X is A^*-graph. ■

Let G be a bipartite connected graph with n vertices and n is even. Then G is an A-graph if and only if G is an A^*-graph. To see this fact we argue as follows. If G is an A-graph, then by Corollary 2.4, G is an A^*-graph. Conversely if G is an A^*-graph, then there exists an edge $e = uv$ of $E(G)$ such that $n_u(e) = n_v(e)$. But since G is a bipartite graph, it is easy to see that $n = n_u(e) + n_v(e)$. Therefore $n_u(e) = n_v(e) = \frac{1}{2}n$ and so G is an A-graph, by Corollary 2.4. The following question naturally arises.

Question. Determine the class $\mathcal{K}$ of graphs such that for each $G \in \mathcal{K}$, G is an A-graph if and only if G is an A^*-graph.

Acknowledgement The second author was partially supported by the Center of Excellence of Mathematics of Isfahan University of Technology (CEAMA).

References

[1] A. R. Ashrafi, B. Manoochehrian, H. Yousefi-Azari, *On Szeged polynomial of a graph*, Bul. Iranian Math. Soc. **33** (2007) 37-46.

[2] A. A. Dobrynin, I. Gutman, *On a graph invariant related to the sum of all distances in a graph*, Publications De L'Institut mathématiques, Nouvelle série, **56** (1994) 18-22.

[3] I. Gutman, *A formula for the Wiener number of trees and its extension to graphs cycles*, Graph theory Notes of New York **27** (1994) 9-15.

[4] S. Klavzăr, J. Jerebic, D. F. Rall, *Distance-balanced graphs*, Anna. Comb. **12** (2008) 71-79.

[5] S. Klavzăr, A. Rajapakse, I. Gutman, *The Szeged and Wiener index of graphs*, Appl. Math. Lett. **9** (1996) 45-49.

[6] D. Stevanović, *Hosoya polynomial of composite graphs*, Discrete Math. **235** (2001) 237-244.

Bijections between bicoloured ordered trees and non-crossing partitions

Chunlin Liu[a], Zhenghua Wang[b] and Baodi Li[a]
[a] Department of Mathematics and System Science,
College of Science, National University of Defense Technology,
Changsha, Hunan 410073 P. R. China
[b] National Laboratory for Parallel and Distributed Processing,
School of Computer, National University of Defense Technology,
Changsha, Hunan 410073 P. R. China
E-mail: liuchunlin77@eyou.com

Abstract: Using new ways to label edges in an ordered tree, this paper introduces two bijections between bicoloured ordered trees and non-crossing partitions. Consequently, enumeration results of non-crossing partitions specified with several parameters are derived.

MSC: 05C05
Keywords: tree, bijection, non-crossing partition

1 Introduction

An ordered tree can be defined inductively as an unlabelled rooted tree whose principal subtrees (the subtrees obtained by removing the root) are ordered trees and have been assigned a linear order (from left to right) among themselves. A bicoloured ordered tree is an ordered tree in which even height vertices are assigned by one colour and odd height ones by the other, where height of a vertex is the distance from it to the root ([3]). Namely, the notion of "bicoloured ordered tree" is essentially the same as the notion of "ordered tree", with the only difference being that vertices in the former shall be treated distinguishingly according to the parity of their heights.

A partition of the set $[n] = \{1, 2, \ldots, n\}$ is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ of non-empty disjoint subsets of $[n]$, called blocks, whose union is $[n]$. A partition is called non-crossing if there do not exist four numbers $a < b < c < d$ such that a and c are in one block of the partition and b and

d are in another block.

Bijections between non-crossing partitions of $[n]$ into k blocks and ordered trees on $n+1$ vertices with k leaves were presented ([2, 4]), meanwhile the number of such ordered trees equals that of bicoloured ordered trees with $n+1-k$ even height vertices and k odd height ones ([1, 3, 5]), which implies that there could be a connection between bicoloured ordered trees and non-crossing partitions. This paper was motivated by such observation and aims to establish bijections between them. Consequently, enumeration results of non-crossing partitions specified with several parameters are obtained.

2 Bijections

In an ordered tree T, the number of subtrees of a vertex u is called the degree $d(u)$ of u. A vertex is called a leaf if its degree is 0, otherwise an internal vertex. Suppose any edge in an ordered tree has a direction leading away from the root. Then the in-degree of any vertex u is 1 (with the root as exception) and the out-degree of u is $d(u)$. We denote the directed edge flowing from u to v by $e = <u, v>$. To a vertex u with linearly ordered (from left to right) subtrees $T_1, T_2, \ldots, T_m$, whose roots are $v_1, v_2, \ldots, v_m$ respectively, we call u the parent of $v_i (1 \le i \le m)$, $v_i (1 \le i \le m)$ the children of u and v_1 the leftmost child; define the claw subtree of T centered by u (denoted by $CT(u)$) to be the subgraph of T induced by the edge set $\{<v_0, u>\} \bigcup \{<u, v_1>, <u, v_2>, \ldots, <u, v_m>\}$, where $<v_0, u>$ is the possible edge flowing to u, that is $\{<v_0, u>\} = \emptyset$ when u is the root of T. Denote by $T(u)$ the subgraph of T induced by the vertex set $\{w |$the path from the root to w contains $u\}$.

To a block B in a non-crossing partition π, we denote by $|B|$ the number of elements in B. B is called singleton if $|B| = 1$, otherwise non-singleton. The smallest element in B is called the leader of B (denoted by $l(B)$) and let the leader set of a non-crossing partition π be $l(\pi) = \{l(B_i) | B_i \in \pi\}$. Two different blocks B_i and B_j are said to be adjacent if $|l(B_i) - l(B_j)| = 1$. A block run is a maximal sequence of blocks $B_{i_1}, B_{i_2}, \ldots, B_{i_t}$ such that any two consecutive blocks are adjacent, i.e. $|l(B_{i_p}) - l(B_{i_{p+1}})| = 1$ for $1 \le p \le t-1$. For example, $\pi = \{\{1,2\},\{3\},\{4,12,14\},\{5\},\{6,8,10\},\{7\},\{9\},\{11\},\{13\}\}$ is a non-crossing partition of [14] into 9 blocks 3 of which are non-singleton, with 5 block runs.

In order to present bijections between the set of bicoloured ordered trees and the set of non-crossing partitions, first we introduce labelling algorithms which are different from that contained in [4].

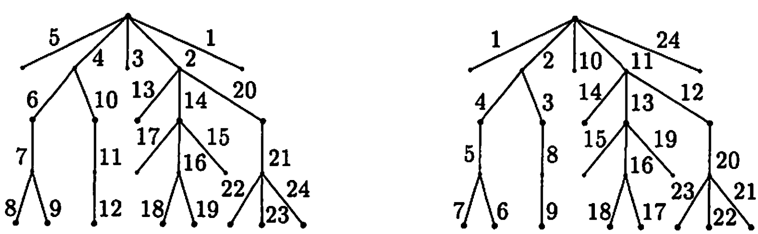
Given a bicoloured ordered tree T with $n+1-k$ even height vertices and k odd height ones, two different ways to attach numbers $1, 2, \ldots, n$ to the edges in T are described as follows, where consecutive numbers are assigned to edges which may constitute a claw subtree.

Even-Height-Vertex-Centered-Labelling (E-Labelling): traverse the tree in preorder (visit the root, then traverse its subtrees from left to right), whenever encountering an even height vertex u the first time, we label the edges in the claw subtree $CT(u)$ in a clockwise direction by beginning at the edge flowing to u, with the smallest not yet used number consecutively. That is, if the possible father of u is v_0 and the linearly ordered (from left to right) children are $v_1, v_2, \ldots, v_m$, then first assign the smallest not yet used number, say i, to edge $< v_0, u >$ and consequently assign $i+1, i+2, \ldots, i+m$ to $< u, v_m >, < u, v_{m-1} >, \ldots, < u, v_1 >$ respectively.

Obviously, $d(u) + 1$ labels are needed for the edges in $CT(u)$ if u is a non-root vertex, otherwise $d(u)$ labels are needed.

Odd-Height-Vertex-Centered-Labelling (O-Labelling) can be defined almost similarly as the above, with the only difference being that: whenever encountering an odd height vertex u the first time, we label those edges in $CT(u)$ in a clockwise direction.

Fig.1 is an illustration of the above two different labelling algorithms.



(a) E-Labelling (b) O-Labelling

Fig .1. Two labelling algorithms on edges of a tree T.

It's worth noting a property of the above labelling algorithms that after an E-Labelling (resp. O-Labelling) procedure having been finished, to any even (resp. odd) height internal vertex u with linearly ordered (from left to right) children $v_1, v_2, \ldots, v_m$, the number assigned to edge $< u, v_i >$ $(2 \leq i \leq m)$ is smaller than that assigned to $< u, v_j >$ $(1 \leq j \leq i-1)$, meanwhile the numbers assigned to edges in $T(v_i)$ are bigger than those assigned to edges in $T(v_j)$, and to any odd (resp. even) height internal vertex u with linearly ordered (from left to right) children $v_1, v_2, \ldots, v_m$, the number assigned to $< u, v_j >$ $(1 \leq j \leq m-1)$ is smaller than that assigned to $< u, v_i >$ $(j < i \leq m)$.

Theorem 2.1 *There is a bijection between the set of bicoloured ordered trees with k odd height vertices r of which are internal and $n+1-k$ even height ones s of which are internal and the set of non-crossing partitions of $[n]$ into k blocks r of which are non-singleton with s block runs.*

Proof. We first give the procedure to construct a non-crossing partition π from a bicoloured ordered tree T.
(1) Label the edges in T with numbers $1, 2, \ldots, n$ by the E-Labelling algorithm.
(2) To each odd height vertex u with degree t, let the set of labels of those $t+1$ edges in $CT(u)$ be a block of the partition π.

Since in T there are k odd height vertices r of which are internal, we obtain a partition of $[n]$ into k blocks r of which are non-singleton. To a block $\{p_0, p_1, \ldots, p_t\}$ $(p_0 < p_1 < \cdots < p_t)$ in the partition π, suppose the edges in the corresponding claw subtree $CT(u)$ are $< v_0, u >, < u, v_1 >, \ldots, < u, v_t >$, where $v_1, \ldots, v_t$ are linearly ordered (from left to right) children of u. Then $< v_0, u >$ must have been labelled by p_0, $< u, v_i >$ $(1 \le i \le t)$ by p_i, $p_1 = p_0+1$ if and only if u is the leftmost child of v_0, and $p_{i+1} = p_i+1$ $(1 \le i \le t-1)$ if and only if v_i is a leaf. When $p_1 > p_0+1$, suppose the children of v_0 which locate on the left-hand side of u are $u_1, \ldots, u_l$. By the E-Labelling algorithm, we have that (i) numbers $p_0+1, \ldots, p_1-1$ must have been assigned to $< v_0, u_1 >, \ldots, < v_0, u_l >$ and edges in $T(u_1), \ldots, T(u_l)$; and (ii) to each even height internal vertex v_i, numbers $p_i+1, \ldots, p_{i+1}-1$ must have been assigned to edges in $T(v_i)$. This means that any numbers q_0 and q_1, satisfying $p_i < q_0 < p_j < q_1$ $(i, j \in \{0, 1, \ldots, t\})$, must belong to different blocks in π, that is, π is a non-crossing partition. Moreover, to an even height internal vertex u with linearly ordered (from left to right) children $v_1, v_2, \ldots, v_m$, we have that if $< u, v_m >$ is labelled by some number i, then $< u, v_{m-1} >, < u, v_{m-2} >, \ldots, < u, v_1 >$ must have been labelled by $i+1, i+2, \ldots, i+m-1$ respectively, which will be leaders of different blocks $B_{j_1}, B_{j_2}, \ldots, B_{j_m}$ respectively, which result in a block run. Therefore the non-crossing partition obtained from T contains s block runs, which is the desired.

Conversely, to each block B in a partition π, a claw tree centered by some vertex (say u) with $|B|$ edges (denoted by T_B) shall be constructed where the edge flowing to u is labelled by $l(B)$ and other edges are labelled by the left elements in B increasingly from left ro right. That is, if $B = \{p_0, p_1, \ldots, p_t\}(p_0 < p_1 < \cdots < p_t)$, a claw tree T_B centered by u which has a father v_0 and linearly ordered (from left to right) children $v_1, v_2, \ldots, v_t$ would be constructed, where $< v_0, u >$ is labelled by p_0 and $< u, v_i >$ $(1 \le i \le t)$ by p_i. These corresponded claw trees may be put inductively in the suitable places to get the bicoloured ordered tree T as follows.

(1) Find the block B_1 in π containing number 1 and construct the corresponding claw tree T_{B_1}. Let T_{B_1} be a claw subtree of T such that their roots are identical.
(2) Find the block B_2 in π that contains the smallest remaining element, say x, and construct the claw tree T_{B_2}.
(A) If $x - 1$ has been assigned to some edge $< u, v >$ where u is an even height vertex, merge T_{B_2} and T_{B_1} by putting T_{B_2} on the left-hand side of $< u, v >$ and identifying the root of T_{B_2} and u. We shall call this operation a left-horizontal merge to T_{B_2}.
(B) Otherwise, i.e. $x - 1$ has been assigned to $< u, v >$ where v is an even height vertex, merge T_{B_2} and T_{B_1} by putting T_{B_2} underneath v and identifying the root of T_{B_2} and v. We shall call this operation a vertical merge to T_{B_2}.
(3) Repeat (2) until all blocks in π are considered.

Since a claw subtree with t edges is added corresponding to a block of t elements and an odd height vertex is added after either a left-horizontal merge or a vertical merge, when all k blocks in π are considered, we get a bicoloured ordered tree with k odd height vertices and $n+1-k$ even height vertices. Moreover, a singleton block leads to an odd height leaf and a non-singleton block to an odd height internal vertex. Since a left-horizontal merge is conducted to a claw subtree T_{B_i} if and only if $l(B_i) - 1$ is a leader of some another block $B_j (i \neq j)$, a block run corresponds to consecutive left-horizontal merges. Furthermore, to s different block runs, $s-1$ vertical merges are needed, which lead to s even height internal vertices. This means that after $s-1$ vertical merges and $k-s$ left-horizontal merges, the eventually obtained bicoloured ordered tree is the required. ∎

Example 2.2 *Fig. 2 shows a bicoloured ordered tree T with 16 even height vertices 7 of which are internal and 19 odd height vertices 8 of which are internal and the corresponding non-crossing partition π of* [34] *into 19 blocks 8 of which are non-singleton with 7 block runs, where the edges in T are assigned with numbers by the E-Labelling algorithm.*

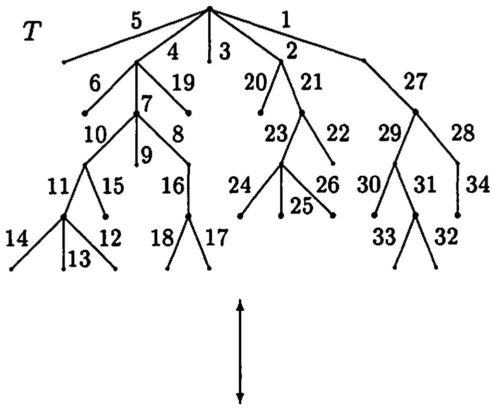
Lemma 2.3 *([1]) The number of bicoloured ordered trees with k odd height vertices r of which are internal and $n+1-k$ even height ones s of which are internal equals*

$$\frac{1}{n+1-k}\binom{k}{r}\binom{n+1-k}{s}\binom{n-1-k}{r-1}\binom{k-1}{s-1}.$$

From Theorem 2.1 and Lemma 2.3, we have

Corollary 2.4 *The number of non-crossing partitions of $[n]$ into k blocks r of which are non-singleton with s block runs equals*

$$\frac{1}{n+1-k}\binom{k}{r}\binom{n+1-k}{s}\binom{n-1-k}{r-1}\binom{k-1}{s-1}.$$



π={{1,27},{2,20,21},{3},{4,6,7,19},{5},{8,16},{9},{10,11,15},{12},
{13},{14},{17},{18},{22},{23,24,25,26},{28,34},{29,30,31},{32},{33}}

Fig.2.

Summing the number in Corollary 2.4 over s and over r respectively, we obtain

Corollary 2.5 *The number of non-crossing partitions of $[n]$ into k blocks r of which are non-singleton equals*

$$\frac{1}{n+1-k}\binom{k}{r}\binom{n-1-k}{r-1}\binom{n}{k}.$$

Corollary 2.6 *The number of non-crossing partitions of $[n]$ into k blocks with s block runs equals*

$$\frac{1}{n+1-k}\binom{k-1}{s-1}\binom{n+1-k}{s}\binom{n-1}{k-1}.$$

Almost using the procedures similar to that in Theorem 2.1, we now give another bijection between bicoloured ordered trees and non-crossing partitions, with the main difference being that O-Labelling algorithm will be used instead of E-Labelling algorithm.

Theorem 2.7 *There is a bijection between the set of bicoloured ordered trees with k odd height vertices and $n+1-k$ even height ones and the set of non-crossing partitions of $[n]$ into $n+1-k$ blocks.*

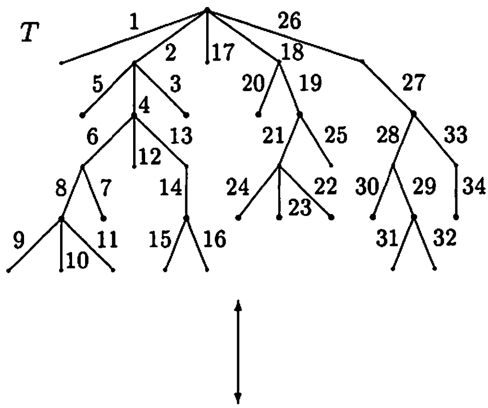
Proof. Let T be such a bicoloured ordered tree.
(1) Label the edges in T with numbers $1, 2, \ldots, n$ by the O-Labelling algorithm.
(2) To each even height vertex u, let the set of labels of those edges in $CT(u)$ be a block of the partition.

Since there are $n+1-k$ even height vertices, we eventually get the desired non-crossing partition π. Moreover, if the root's degree is 1, the number of non-singleton blocks in π is 1 less than the number of even height internal vertices in T, otherwise those two numbers are equal; if the leftmost child of the root is a leaf, the number of block runs in π is 1 more than the number of odd height internal vertices in T, otherwise those two numbers are equal.

To the reverse procedure, without loss of generality, suppose the blocks of a partition have been ordered increasingly with respect to their leaders, i.e. $\pi = \{B_1, B_2, \ldots, B_{n+1-k}\}$, $1 = l(B_1) < l(B_2) < \cdots < l(B_{n+1-k}) \leq n$. To B_1, a claw tree T_{B_1} centered by some vertex u with $|B_1|$ edges, where edges all flow from u and are labelled by the elements of B_1 increasingly from left to right, will be constructed; and to $B_i (2 \leq i \leq n+1-k)$ a corresponding claw tree T_{B_i} will be constructed similarly as in Theorem 2.1.
(1) Let the claw tree T_{B_1} corresponding to B_1 be a subtree of T such that their roots are identical.
(2) Consider the claw tree T_{B_2} corresponding to B_2. Let the edge labelled by $l(B_2) - 1$ is $< u, v >$. Merge T_{B_2} and T_{B_1} through a vertical merging operation: put T_{B_2} underneath v and identify the root of T_{B_2} and v.
(3) Consider the claw tree T_{B_3} corresponding to B_3.
(A) If $l(B_3) - 1$ has been assigned to some edge $< u, v >$ where u is a even height vertex, insert T_{B_3} into the constructed tree by a vertical merge.
(B) Otherwise, when $l(B_3) - 1$ has been assigned to $< u, v >$ where u is a odd height vertex, insert T_{B_3} into the constructed tree by a left-horizontal merging operation: put T_{B_3} on the left-hand side of $< u, v >$ and identify the root of T_{B_3} and u.
(4) Repeat (3) until $T_{B_4}, \ldots, T_{B_{n+1-k}}$ all are considered.

Since an even height vertex is added after either of the above two merging operations, when claw tress corresponding to those blocks in π all are considered, we get the desired bicoloured ordered tree at last. ∎

For example, from the tree T in Example 2.2, we get another non-crossing partition π', as shown in Fig. 3, where the edges in T are assigned with numbers by the O-Labelling algorithm.



$\pi'=\{\{1,2,17,18,26\},\{3\},\{4,6,12,13\},\{5\},\{7\},\{8,9,10,11\},\{14,15,16\},$

$\{19,21,25\},\{20\},\{22\},\{23\},\{24\},\{27,28,33\},\{29,31,32\},\{30\},\{34\}\}$

Fig.3.

Acknowledgements

The authors wish to express their sincere thanks to the referees for their very helpful suggestions that have resulted in great improvements that appear here. The first author was supported by the National Science Foundation of China.

References

[1] L. H. Clark, J. E. McCanna, L. A. Székely, A survey of counting bicoloured trees, Bull. Inst. Combin. Appl. 21 (1997) 33-45.

[2] N. Dershowitz, S. Zaks, Ordered trees and non-crossing partitions, Discrete Math. 62 (1986) 215-218.

[3] C. L. Liu, Z. H. Wang, A bijection between ordered trees and bicoloured ordered trees, Discrete Math. 309 (2009) 1417-1421.

[4] H. Prodinger, A correspondence between ordered trees and noncrossing partitions, Discrete Math. 46 (1983) 205-206.

[5] D. Zeilberger, Six etudes in generating functions, Intern. J. Computer Math. 29 (1989) 201-215.