

Bijections between bicoloured ordered trees and non-crossing partitions

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Abstract: Using new ways to label edges in an ordered tree, this paper introduces two bijections between bicoloured ordered trees and non-crossing partitions. Consequently, enumeration results of non-crossing partitions specified with several parameters are derived.

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1 Introduction

An ordered tree can be defined inductively as an unlabelled rooted tree whose principal subtrees (the subtrees obtained by removing the root) are ordered trees and have been assigned a linear order (from left to right) among themselves. A bicoloured ordered tree is an ordered tree in which even height vertices are assigned by one colour and odd height ones by the other, where height of a vertex is the distance from it to the root ([3]). Namely, the notion of “bicoloured ordered tree” is essentially the same as the notion of “ordered tree”, with the only difference being that vertices in the former shall be treated distinguishingly according to the parity of their heights.

A partition of the set $[n] = \{1, 2, \dots, n\}$ is a collection $\pi = \{B_1, B_2, \dots, B_k\}$ of non-empty disjoint subsets of $[n]$, called blocks, whose union is $[n]$. A partition is called non-crossing if there do not exist four numbers $a < b < c < d$ such that a and c are in one block of the partition and b and

d are in another block.

Bijections between non-crossing partitions of $[n]$ into k blocks and ordered trees on $n+1$ vertices with k leaves were presented ([2, 4]), meanwhile the number of such ordered trees equals that of bicoloured ordered trees with $n+1-k$ even height vertices and k odd height ones ([1, 3, 5]), which implies that there could be a connection between bicoloured ordered trees and non-crossing partitions. This paper was motivated by such observation and aims to establish bijections between them. Consequently, enumeration results of non-crossing partitions specified with several parameters are obtained.

2 Bijections

In an ordered tree T , the number of subtrees of a vertex u is called the degree $d(u)$ of u . A vertex is called a leaf if its degree is 0, otherwise an internal vertex. Suppose any edge in an ordered tree has a direction leading away from the root. Then the in-degree of any vertex u is 1 (with the root as exception) and the out-degree of u is $d(u)$. We denote the directed edge flowing from u to v by $e = \langle u, v \rangle$. To a vertex u with linearly ordered (from left to right) subtrees T_1, T_2, \dots, T_m , whose roots are v_1, v_2, \dots, v_m respectively, we call u the parent of $v_i (1 \leq i \leq m)$, $v_i (1 \leq i \leq m)$ the children of u and v_1 the leftmost child; define the claw subtree of T centered by u (denoted by $CT(u)$) to be the subgraph of T induced by the edge set $\{\langle v_0, u \rangle\} \cup \{\langle u, v_1 \rangle, \langle u, v_2 \rangle, \dots, \langle u, v_m \rangle\}$, where $\langle v_0, u \rangle$ is the possible edge flowing to u , that is $\{\langle v_0, u \rangle\} = \emptyset$ when u is the root of T . Denote by $T(u)$ the subgraph of T induced by the vertex set $\{w \mid \text{the path from the root to } w \text{ contains } u\}$.

To a block B in a non-crossing partition π , we denote by $|B|$ the number of elements in B . B is called singleton if $|B| = 1$, otherwise non-singleton. The smallest element in B is called the leader of B (denoted by $l(B)$) and let the leader set of a non-crossing partition π be $l(\pi) = \{l(B_i) \mid B_i \in \pi\}$. Two different blocks B_i and B_j are said to be adjacent if $|l(B_i) - l(B_j)| = 1$. A block run is a maximal sequence of blocks $B_{i_1}, B_{i_2}, \dots, B_{i_t}$ such that any two consecutive blocks are adjacent, i.e. $|l(B_{i_p}) - l(B_{i_{p+1}})| = 1$ for $1 \leq p \leq t-1$. For example, $\pi = \{\{1, 2\}, \{3\}, \{4, 12, 14\}, \{5\}, \{6, 8, 10\}, \{7\}, \{9\}, \{11\}, \{13\}\}$ is a non-crossing partition of $[14]$ into 9 blocks 3 of which are non-singleton, with 5 block runs.

In order to present bijections between the set of bicoloured ordered trees and the set of non-crossing partitions, first we introduce labelling algorithms which are different from that contained in [4].

Given a bicoloured ordered tree T with $n+1-k$ even height vertices

and k odd height ones, two different ways to attach numbers $1, 2, \dots, n$ to the edges in T are described as follows, where consecutive numbers are assigned to edges which may constitute a claw subtree.

Even-Height-Vertex-Centered-Labeling (E-Labeling): traverse the tree in preorder (visit the root, then traverse its subtrees from left to right), whenever encountering an even height vertex u the first time, we label the edges in the claw subtree $CT(u)$ in a clockwise direction by beginning at the edge flowing to u , with the smallest not yet used number consecutively. That is, if the possible father of u is v_0 and the linearly ordered (from left to right) children are v_1, v_2, \dots, v_m , then first assign the smallest not yet used number, say i , to edge $\langle v_0, u \rangle$ and consequently assign $i+1, i+2, \dots, i+m$ to $\langle u, v_m \rangle, \langle u, v_{m-1} \rangle, \dots, \langle u, v_1 \rangle$ respectively.

Obviously, $d(u) + 1$ labels are needed for the edges in $CT(u)$ if u is a non-root vertex, otherwise $d(u)$ labels are needed.

Odd-Height-Vertex-Centered-Labeling (O-Labeling) can be defined almost similarly as the above, with the only difference being that: whenever encountering an odd height vertex u the first time, we label those edges in $CT(u)$ in a clockwise direction.

Fig.1 is an illustration of the above two different labelling algorithms.

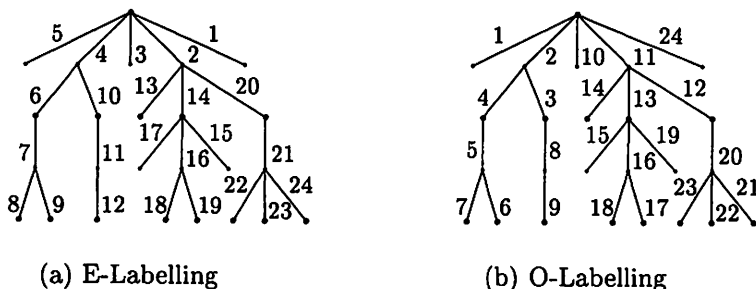


Fig .1. Two labelling algorithms on edges of a tree T .

It's worth noting a property of the above labelling algorithms that after an E-Labeling (resp. O-Labeling) procedure having been finished, to any even (resp. odd) height internal vertex u with linearly ordered (from left to right) children v_1, v_2, \dots, v_m , the number assigned to edge $\langle u, v_i \rangle$ ($2 \leq i \leq m$) is smaller than that assigned to $\langle u, v_j \rangle$ ($1 \leq j \leq i - 1$), meanwhile the numbers assigned to edges in $T(v_i)$ are bigger than those assigned to edges in $T(v_j)$, and to any odd (resp. even) height internal vertex u with linearly ordered (from left to right) children v_1, v_2, \dots, v_m , the number assigned to $\langle u, v_j \rangle$ ($1 \leq j \leq m - 1$) is smaller than that assigned to $\langle u, v_i \rangle$ ($j < i \leq m$).

Theorem 2.1 *There is a bijection between the set of bicoloured ordered trees with k odd height vertices r of which are internal and $n + 1 - k$ even height ones s of which are internal and the set of non-crossing partitions of $[n]$ into k blocks r of which are non-singleton with s block runs.*

Proof. We first give the procedure to construct a non-crossing partition π from a bicoloured ordered tree T .

(1) Label the edges in T with numbers $1, 2, \dots, n$ by the E-Labeling algorithm.

(2) To each odd height vertex u with degree t , let the set of labels of those $t + 1$ edges in $CT(u)$ be a block of the partition π .

Since in T there are k odd height vertices r of which are internal, we obtain a partition of $[n]$ into k blocks r of which are non-singleton. To a block $\{p_0, p_1, \dots, p_t\}$ ($p_0 < p_1 < \dots < p_t$) in the partition π , suppose the edges in the corresponding claw subtree $CT(u)$ are $\langle v_0, u \rangle, \langle u, v_1 \rangle, \dots, \langle u, v_t \rangle$, where v_1, \dots, v_t are linearly ordered (from left to right) children of u . Then $\langle v_0, u \rangle$ must have been labelled by p_0 , $\langle u, v_i \rangle$ ($1 \leq i \leq t$) by p_i , $p_1 = p_0 + 1$ if and only if u is the leftmost child of v_0 , and $p_{i+1} = p_i + 1$ ($1 \leq i \leq t - 1$) if and only if v_i is a leaf. When $p_1 > p_0 + 1$, suppose the children of v_0 which locate on the left-hand side of u are u_1, \dots, u_l . By the E-Labeling algorithm, we have that (i) numbers $p_0 + 1, \dots, p_1 - 1$ must have been assigned to $\langle v_0, u_1 \rangle, \dots, \langle v_0, u_l \rangle$ and edges in $T(u_1), \dots, T(u_l)$; and (ii) to each even height internal vertex v_i , numbers $p_i + 1, \dots, p_{i+1} - 1$ must have been assigned to edges in $T(v_i)$. This means that any numbers q_0 and q_1 , satisfying $p_i < q_0 < p_j < q_1$ ($i, j \in \{0, 1, \dots, t\}$), must belong to different blocks in π , that is, π is a non-crossing partition. Moreover, to an even height internal vertex u with linearly ordered (from left to right) children v_1, v_2, \dots, v_m , we have that if $\langle u, v_m \rangle$ is labelled by some number i , then $\langle u, v_{m-1} \rangle, \langle u, v_{m-2} \rangle, \dots, \langle u, v_1 \rangle$ must have been labelled by $i + 1, i + 2, \dots, i + m - 1$ respectively, which will be leaders of different blocks $B_{j_1}, B_{j_2}, \dots, B_{j_m}$ respectively, which result in a block run. Therefore the non-crossing partition obtained from T contains s block runs, which is the desired.

Conversely, to each block B in a partition π , a claw tree centered by some vertex (say u) with $|B|$ edges (denoted by T_B) shall be constructed where the edge flowing to u is labelled by $l(B)$ and other edges are labelled by the left elements in B increasingly from left to right. That is, if $B = \{p_0, p_1, \dots, p_t\}$ ($p_0 < p_1 < \dots < p_t$), a claw tree T_B centered by u which has a father v_0 and linearly ordered (from left to right) children v_1, v_2, \dots, v_t would be constructed, where $\langle v_0, u \rangle$ is labelled by p_0 and $\langle u, v_i \rangle$ ($1 \leq i \leq t$) by p_i . These corresponded claw trees may be put inductively in the suitable places to get the bicoloured ordered tree T as follows.

(1) Find the block B_1 in π containing number 1 and construct the corresponding claw tree T_{B_1} . Let T_{B_1} be a claw subtree of T such that their roots are identical.

(2) Find the block B_2 in π that contains the smallest remaining element, say x , and construct the claw tree T_{B_2} .

(A) If $x - 1$ has been assigned to some edge $\langle u, v \rangle$ where u is an even height vertex, merge T_{B_2} and T_{B_1} by putting T_{B_2} on the left-hand side of $\langle u, v \rangle$ and identifying the root of T_{B_2} and u . We shall call this operation a left-horizontal merge to T_{B_2} .

(B) Otherwise, i.e. $x - 1$ has been assigned to $\langle u, v \rangle$ where v is an even height vertex, merge T_{B_2} and T_{B_1} by putting T_{B_2} underneath v and identifying the root of T_{B_2} and v . We shall call this operation a vertical merge to T_{B_2} .

(3) Repeat (2) until all blocks in π are considered.

Since a claw subtree with t edges is added corresponding to a block of t elements and an odd height vertex is added after either a left-horizontal merge or a vertical merge, when all k blocks in π are considered, we get a bicoloured ordered tree with k odd height vertices and $n + 1 - k$ even height vertices. Moreover, a singleton block leads to an odd height leaf and a non-singleton block to an odd height internal vertex. Since a left-horizontal merge is conducted to a claw subtree T_{B_i} if and only if $l(B_i) - 1$ is a leader of some another block $B_j (i \neq j)$, a block run corresponds to consecutive left-horizontal merges. Furthermore, to s different block runs, $s - 1$ vertical merges are needed, which lead to s even height internal vertices. This means that after $s - 1$ vertical merges and $k - s$ left-horizontal merges, the eventually obtained bicoloured ordered tree is the required. ■

Example 2.2 *Fig. 2 shows a bicoloured ordered tree T with 16 even height vertices 7 of which are internal and 19 odd height vertices 8 of which are internal and the corresponding non-crossing partition π of $[34]$ into 19 blocks 8 of which are non-singleton with 7 block runs, where the edges in T are assigned with numbers by the E-Labeling algorithm.*

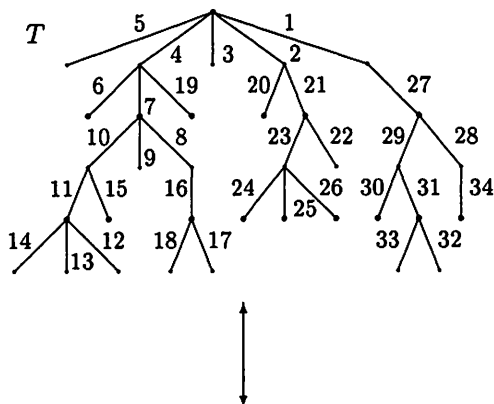
Lemma 2.3 ([1]) *The number of bicoloured ordered trees with k odd height vertices r of which are internal and $n + 1 - k$ even height ones s of which are internal equals*

$$\frac{1}{n + 1 - k} \binom{k}{r} \binom{n + 1 - k}{s} \binom{n - 1 - k}{r - 1} \binom{k - 1}{s - 1}.$$

From Theorem 2.1 and Lemma 2.3, we have

Corollary 2.4 *The number of non-crossing partitions of $[n]$ into k blocks r of which are non-singleton with s block runs equals*

$$\frac{1}{n+1-k} \binom{k}{r} \binom{n+1-k}{s} \binom{n-1-k}{r-1} \binom{k-1}{s-1}.$$



$$\pi = \{\{1,27\}, \{2,20,21\}, \{3\}, \{4,6,7,19\}, \{5\}, \{8,16\}, \{9\}, \{10,11,15\}, \{12\}, \{13\}, \{14\}, \{17\}, \{18\}, \{22\}, \{23,24,25,26\}, \{28,34\}, \{29,30,31\}, \{32\}, \{33\}\}$$

Fig.2.

Summing the number in Corollary 2.4 over s and over r respectively, we obtain

Corollary 2.5 *The number of non-crossing partitions of $[n]$ into k blocks r of which are non-singleton equals*

$$\frac{1}{n+1-k} \binom{k}{r} \binom{n-1-k}{r-1} \binom{n}{k}.$$

Corollary 2.6 *The number of non-crossing partitions of $[n]$ into k blocks with s block runs equals*

$$\frac{1}{n+1-k} \binom{k-1}{s-1} \binom{n+1-k}{s} \binom{n-1}{k-1}.$$

Almost using the procedures similar to that in Theorem 2.1, we now give another bijection between bicoloured ordered trees and non-crossing partitions, with the main difference being that O-Labeling algorithm will be used instead of E-Labeling algorithm.

Theorem 2.7 *There is a bijection between the set of bicoloured ordered trees with k odd height vertices and $n + 1 - k$ even height ones and the set of non-crossing partitions of $[n]$ into $n + 1 - k$ blocks.*

Proof. Let T be such a bicoloured ordered tree.

(1) Label the edges in T with numbers $1, 2, \dots, n$ by the O-Labeling algorithm.

(2) To each even height vertex u , let the set of labels of those edges in $CT(u)$ be a block of the partition.

Since there are $n + 1 - k$ even height vertices, we eventually get the desired non-crossing partition π . Moreover, if the root's degree is 1, the number of non-singleton blocks in π is 1 less than the number of even height internal vertices in T , otherwise those two numbers are equal; if the leftmost child of the root is a leaf, the number of block runs in π is 1 more than the number of odd height internal vertices in T , otherwise those two numbers are equal.

To the reverse procedure, without loss of generality, suppose the blocks of a partition have been ordered increasingly with respect to their leaders, i.e. $\pi = \{B_1, B_2, \dots, B_{n+1-k}\}$, $1 = l(B_1) < l(B_2) < \dots < l(B_{n+1-k}) \leq n$. To B_1 , a claw tree T_{B_1} centered by some vertex u with $|B_1|$ edges, where edges all flow from u and are labelled by the elements of B_1 increasingly from left to right, will be constructed; and to B_i ($2 \leq i \leq n + 1 - k$) a corresponding claw tree T_{B_i} will be constructed similarly as in Theorem 2.1.

(1) Let the claw tree T_{B_1} corresponding to B_1 be a subtree of T such that their roots are identical.

(2) Consider the claw tree T_{B_2} corresponding to B_2 . Let the edge labelled by $l(B_2) - 1$ is $\langle u, v \rangle$. Merge T_{B_2} and T_{B_1} through a vertical merging operation: put T_{B_2} underneath v and identify the root of T_{B_2} and v .

(3) Consider the claw tree T_{B_3} corresponding to B_3 .

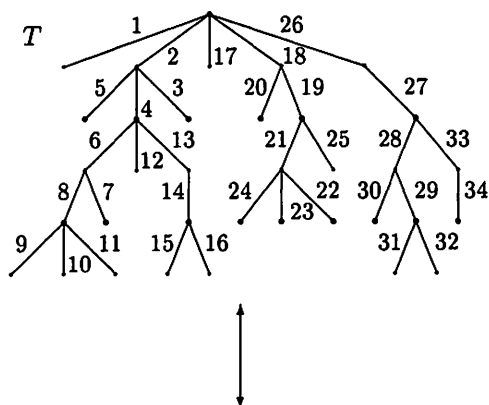
(A) If $l(B_3) - 1$ has been assigned to some edge $\langle u, v \rangle$ where u is an even height vertex, insert T_{B_3} into the constructed tree by a vertical merge.

(B) Otherwise, when $l(B_3) - 1$ has been assigned to $\langle u, v \rangle$ where u is an odd height vertex, insert T_{B_3} into the constructed tree by a left-horizontal merging operation: put T_{B_3} on the left-hand side of $\langle u, v \rangle$ and identify the root of T_{B_3} and u .

(4) Repeat (3) until $T_{B_4}, \dots, T_{B_{n+1-k}}$ all are considered.

Since an even height vertex is added after either of the above two merging operations, when claw trees corresponding to those blocks in π all are considered, we get the desired bicoloured ordered tree at last. ■

For example, from the tree T in Example 2.2, we get another non-crossing partition π' , as shown in Fig. 3, where the edges in T are assigned with numbers by the O-Labeling algorithm.



$$\pi' = \{\{1,2,17,18,26\}, \{3\}, \{4,6,12,13\}, \{5\}, \{7\}, \{8,9,10,11\}, \{14,15,16\}, \\ \{19,21,25\}, \{20\}, \{22\}, \{23\}, \{24\}, \{27,28,33\}, \{29,31,32\}, \{30\}, \{34\}\}$$

Fig.3.

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