Forbidden subgraphs and the hamiltonian index of a 2-connected graph

Přemysl Holub [1]

Abstract.
Hamiltonian index of a graph G is the smallest positive integer k, for which the k-th iterated line graph $L^k(G)$ is hamiltonian. Bedrossian characterized all pairs of forbidden induced subgraphs that imply hamiltonicity in 2-connected graphs. In this paper, some upper bounds on the hamiltonian index of a 2-connected graph in terms of forbidden not necessarily induced subgraphs are presented.

Keywords: Forbidden subgraphs, Hamiltonicity, Hamiltonian index

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1 Introduction

In this paper, we consider only finite undirected graphs without loops and multiple edges. We use [2] for terminology and notations not defined here. For $x, y \in V(G)$, an $x,y-$*path* is a path between vertices x and y in G. A *hamiltonian cycle* is a cycle in G on $|V(G)|$ vertices. A graph G is said to be *hamiltonian*, if $c(G) = |V(G)|$. For a nonempty set $A \subseteq V(G)$, the induced subgraph on A in G is denoted by $\langle A \rangle_G$. We denote by P_i the path on i vertices and we say that the *length* of a path P is the number of edges of P. Similarly we denote C_i the cycle on i vertices. For any $A \subset V(G)$, the graph G–A stands for $\langle V(G) \setminus A \rangle_G$. For a connected graph H, a graph G is said to be H-*free*, if G does not contain a copy of H as an induced subgraph; the graph H will be also referred to in this context as a *forbidden subgraph*. The graph $K_{1,3}$ will be called the *claw* and in

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the special case $H = K_{1,3}$ we say that G is *claw-free*. List of frequently used forbidden subgraphs is shown in Fig. 1.

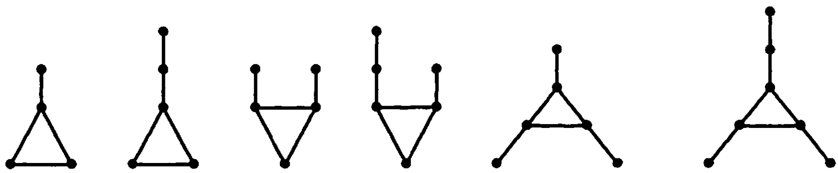


Fig. 1

The graphs Z_2, B and N were generalized in [4] as follows. We denote by :

Z_i, $(i \geq 1)$	- the graph which is obtained by identifying a vertex of a triangle with an end-vertex of a path of length i
$B_{i,j}$, $(i, j \geq 1)$	- the generalized (i, j)-bull, i.e., the graph which is obtained by identifying two distinct vertices of a triangle with an end-vertex of one of two vertex-disjoint paths of lengths i, j
$N_{i,j,k}$, $(i, j, k \geq 1)$	- the generalized (i, j, k)-net, i.e., the graph which is obtained by identifying each vertex of a triangle with an end-vertex of one of three vertex-disjoint paths of lengths i, j, k.

One of the motivations for studying hamiltonicity in the class of line graphs was given by Harary and Nash-Williams in [7]. A dominating closed trail (DCT) in a graph G is a connected eulerian subgraph F of G such that $u \in V(F)$ or $v \in V(F)$ for every edge $uv \in E(G)$. Note that the trivial DCT (consisting of only one vertex) is allowed.

Theorem A [7]. *Let G be a graph with at least three edges. Then $L(G)$ is hamiltonian if and only if G contains a DCT.*

It is easy to see that the line graph of a hamiltonian graph is also hamiltonian. The concept of the hamiltonian index of a graph was introduced by Chartrand in [6].

Let G be a graph and let k be a positive integer. The k-th iterated line graph of G, denoted by $L^k(G)$, is defined recursively in the following way:

$$L^0(G) = G, \quad L^k(G) = L\left(L^{k-1}(G)\right).$$

The *hamiltonian index* of a graph G, denoted by $h(G)$, is the smallest number k such that $L^k(G)$ is hamiltonian. Chartrand [6] showed that for every connected graph that is not a path the hamiltonian index exists.

An induced path P in G such that both end vertices of P have degree different from two and all internal vertices of P (if any) have degree exactly two in G, is called a *branch* of G. Let $B(G)$ denote the set of all branches of G. Let $S \subset B(G)$ and $G_S = (V_S, E_S)$ be the graph with $V_S = V(G)$ and $E_S = E(G) - E(S)$. Then $G - S$ denotes the subgraph obtained from G_S by deleting all internal vertices of all branches of S. A subset S of G is called a *branch cut* if $G - S$ has more components than G. A minimal branch cut is called a *branch-bond*. It is easy to see that, for a connected graph G, a subset S of G is a branch-bond if and only if $G - S$ has exactly two components. We denote by $BB(G)$ the set of all branch-bonds of G. A branch-bond is called *odd* if it consists of an odd number of branches. The *length of a branch-bond* S, denoted by $l(S)$, is the length of a shortest branch of S. Let $BB_1(G)$ denote the set of all branch-bonds S such that $|S| = 1$ and the only branch b of S has one end-vertex of degree 1, let $BB_2(G)$ denote the set of all branch-bonds S such that $|S| = 1$ and the only branch b of S has both end-vertices of degree at least 3 and let $BB_3(G)$ denote the set of all odd branch-bonds consisting of at least three branches. Now we define, for $i = 1, 2, 3$,

$$h_i(G) = \begin{cases} \max\{l(S) \mid S \in BB_i(G)\} & \text{if } BB_i(G) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Xiong and Wu [3] proved the following bound for the hamiltonian index of a graph.

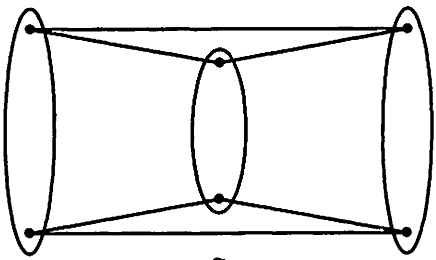
Theorem B [3]. *Let G be a 2-connected graph. Then*

$$h_3(G) - 1 \leq h(G) \leq h_3(G) + 1.$$

For studying hamiltonian properties in terms of forbidden induced subgraphs we give the following motivation.

Theorem C [1]. *Let X and Y be connected graphs with $X, Y \neq P_3$, and let G be a 2-connected graph that is not a cycle. Then, G being X, Y-free implies G is hamiltonian if and only if (up to symmetry) $X = K_{1,3}$ and $Y \in \{P_4, P_5, P_6, C_3, Z_1, Z_2, B, N, W\}$.*

Let $\mathfrak{F}$ denote the class of graphs shown in Fig. 2 (where elliptical parts represent cliques of order at least 3).



$\mathfrak{F}$

Fig. 2

Brousek, Ryjáček and Schiermeyer in [5] proved the following:

Theorem D [5]. *Let G be 2-connected graph. If G is $K_{1,3}, E$-free graph, then G is hamiltonian or $G \in \mathfrak{F}$.*

Clearly, if $G \in \mathfrak{F}$, then G has a DCT. Hence we obtain the following consequence.

Corollary 1. *If G is a 2-connected $K_{1,3}, E$-free graph, then G has a DCT, i.e., $h(G) \leq 1$.*

In this paper we present upper bounds for the hamiltonian index of a 2-connected graph in terms of forbidden, not necessarily induced graphs. We will use the following notation. We denote $L^{-1}(G)$ a graph H such that $L(H) = G$, where G is a line graph.
Note that the line graph of a graph is claw-free, the line graph of a 2-connected graph is 2-connected, and if G does not contain a connected graph X as a subgraph (not necessarily induced), then $L(G)$ is $L(X)$-free.

2 Upper bounds for hamiltonian index in terms of forbidden subgraphs

Let G be a graph and H a subgraph of G. We denote by $G - H$ a graph G' which is obtained from G by deleting all edges of H and subsequently deleting all isolated vertices in the resulting graph.

Some auxiliary statements will be shown. Let $\mathcal{P}$ denote a set of paths. Then $l(\mathcal{P})$ stands for the length of a shortest path of $\mathcal{P}$.

Lemma 1. *Let x, y be a pair of nonadjacent vertices. Let $k \geq 2$ be an odd integer. Let G be a graph consisting of the vertices x, y and k vertex-disjoint x, y-paths $P_1, \ldots, P_k$. Let $\mathcal{P} = \{P_1, \ldots, P_k\}$. If $l(\mathcal{P}) > 1$, then $h(G) = l(\mathcal{P}) - 1$.*

Proof. Using the fact that the line graph operator decreases the length of any branch of G by one, $L^{l(\mathcal{P})-2}(G)$ has a DCT, implying that $h(G) \leq l(\mathcal{P}) - 1$. The equality follows from the following fact. The graph $L^{l(\mathcal{P})-2}(G)$ consists of two vertex disjoint graphs joined by three vertex disjoint paths $\mathcal{P}'$ with $l(\mathcal{P}') = 2$. And obviously this graph is not hamiltonian. ■

Lemma 2. *Let $k_1, k_2 \geq 2$ be integers such that $k = k_1 + k_2$ is odd. Let G be a graph consisting of vertices x, y_1, y_2 such that $y_1y_2 \in E(G)$, x is not adjacent to any of the vertices y_1, y_2, G contains k vertex-disjoint paths between x and one of the vertices y_1, y_2 such that there are k_1 x, y_1-paths $P_1, \ldots, P_{k_1}$ and k_2 x, y_2-paths $P_{k_1+1}, \ldots, P_k$. Let $\mathcal{P} = \{P_1, \ldots, P_k\}$. If $l(\mathcal{P}) > 1$, then $h(G) \leq l(\mathcal{P}) - 1$.*

Proof. Let G be a graph satisfying the hypothesis. Let P be a shortest path of $P_1, \ldots, P_k$. The path P has length $l(\mathcal{P})$. Up to symmetry suppose that P is an x, y_1-path. We prove this lemma by induction on $l(\mathcal{P})$.

i) Suppose that $l(\mathcal{P}) = 2$. If k_1 is odd, then the graph $H = G - P - y_1y_2$ is a DCT in G. If k_1 is even, then the graph $H = G - P$ is a DCT in G. Hence G has a DCT, implying that $h(G) \leq 1$ by Theorem A.

ii) Suppose that $h(G) \leq l - 2$ holds for each graph with the given structure, for which $l(\mathcal{P}) = l - 1$. Let G' be a graph obtained from G by replacing P by a path P' of length $l - 1$. Hence $h(G') \leq l - 2$ by the induction hypothesis. This yields that $L^{l-2}(G')$ is hamiltonian. Now we denote by G'' the graph obtained from $L^{l-2}(G')$ by replacing the edge, which corresponds to P' in G', by a path of length two. Clearly $G'' = L^{l-2}(G)$ and $L^{l-2}(G)$ has a DCT, implying that $h(G) \leq l - 1$. ■

Lemma 3. *Let $k_1, k_2 \geq 2$ be integers such that $k = k_1 + k_2$ is odd. Let G be a graph consisting of vertices x, y_1, y_2 such that x is not adjacent to any of the vertices y_1, y_2, G contains k vertex-disjoint paths between x and one of the* *vertices y_1, y_2 such that there are k_1 x, y_1-paths $P_1, \dots, P_{k_1}$ and k_2 x, y_2-paths $P_{k_1+1}, \dots, P_k$. Moreover there are l vertices adjacent to both vertices y_1, y_2 but not to x, $l \in \mathbb{N}$. Let $\mathcal{P} = \{P_1, \dots, P_k\}$. If $l(\mathcal{P}) > 1$, then $h(G) \le l(\mathcal{P}) - 1$.*

Proof. Let G be a graph satisfying the hypothesis. Let P be a shortest path of $P_1, \dots, P_k$. The path P has length $l(\mathcal{P})$. Up to symmetry suppose that P is an x, y_1-path. Let z denote any of the vertices adjacent to both vertices y_1, y_2 but not to x and Q the path induced by $\{y_1, z, y_2\}$. We prove this lemma by induction on $l(\mathcal{P})$.

i) Suppose that $l(\mathcal{P}) = 2$. If k_1 is odd, then the graph $H = \mathcal{P} - P$ is a DCT in G. If k_1 is even, then the graph $H = (\mathcal{P} - P) \cup Q$ is a DCT in G. Hence G has a DCT, implying that $h(G) \le 1$ by Theorem A.

ii) Suppose that $h(G) \le l-2$ holds for each graph with the given structure, for which $l(\mathcal{P}) = l-1$. Let G' be a graph obtained from G by replacing P by a path P' of length $l-1$. Hence $h(G') \le l-2$ by the induction hypothesis. Thus $L^{l-2}(G')$ is hamiltonian. Now we denote by G'' the graph obtained from $L^{l-2}(G')$ by replacing the edge, which corresponds to P' in G', by a path of length two. Clearly $G'' = L^{l-2}(G)$ and $L^{l-2}(G)$ has a DCT, implying that $h(G) \le l-1$. ∎

Lemma 4. *Let $k_1, k_2, k_3 \ge 1$ be integers such that $k = k_1 + k_2 + k_3$ is an odd number. Let G be a graph consisting of vertices x, y_1, y_2, y_3 such that $y_1y_2 \in E(G)$, $y_2y_3 \in E(G)$ but x is not adjacent to any of the vertices y_1, y_2, y_3, G contains k vertex-disjoint paths between x and one of the vertices y_1, y_2, y_3 such that there is k_1 x, y_1-paths $P_1, \dots, P_{k_1}$, k_2 x, y_2-paths $P_{k_1+1}, \dots, P_{k_1+k_2}$ and k_3 xy_3-paths $P_{k_1+k_2+1}, \dots, P_k$. Let $\mathcal{P} = \{P_1, \dots, P_k\}$. If $l(\mathcal{P}) > 1$, then $h(G) \le l(\mathcal{P}) - 1$.*

Proof. Let G be a graph satisfying the hypothesis. Let P be a shortest path of $P_1, \dots, P_k$. The path P has length $l(\mathcal{P})$. We prove this lemma by induction on $l(\mathcal{P})$.

i) Suppose that $l(\mathcal{P}) = 2$. First suppose that P is an x, y_2-path. If all the numbers k_1, k_2, k_3 are odd, then the graph $(\mathcal{P} - P) \cup \{y_1y_2, y_1y_3\}$ is a DCT in G. If k_2 is odd and both numbers k_1, k_3 are even, then the graph $\mathcal{P} - P$ is a DCT in G. Now suppose that k_2 is even. Since k is odd, exactly one of the numbers k_1, k_3, say k_1, is odd. Then $(\mathcal{P} - P) \cup \{y_1y_2\}$

is a DCT in G.

Now suppose that P is not an x, y_2-path. Up to symmetry suppose that P is an x, y_1-path. If all the numbers k_1, k_2, k_3 are odd, then the graph $(\mathcal{P} - P) \cup \{y_2y_3\}$ is a DCT in G. If k_1 is odd and both numbers k_2, k_3 are even, then the graph $\mathcal{P} - P$ is a DCT in G. Now suppose that k_1 is even. Since k is odd, exactly one of the numbers k_2, k_3 is odd. If k_2 is odd, then the graph $(\mathcal{P} - P) \cup \{y_1y_2\}$ is a DCT in G. If k_3 is odd, then the graph $(\mathcal{P} - P) \cup \{y_1y_2, y_2y_3\}$ is a DCT in G. Hence, in any possibility, G has a DCT, implying that $h(G) \leq 1$ by Theorem A.

ii) Suppose that $h(G) \leq l-2$ holds for each graph with the given structure, for which $l(\mathcal{P}) = l-1$. Let G' be a graph obtained from G by replacing P by a path P' of length $l-1$. Hence $h(G') \leq l-2$ by the induction hypothesis. Therefore $L^{l-2}(G')$ is hamiltonian. Now we denote by G'' the graph obtained from $L^{l-2}(G')$ by replacing the edge, which corresponds to P' in G', by a path of length two. Clearly $G'' = L^{l-2}(G)$ and $L^{l-2}(G)$ has a DCT, implying that $h(G) \leq l-1$. ■

The following lemma gives a lower bound for the length of a path in a graph G involving the hamiltonian index of G.

Lemma 5. *Let G be a 2-connected graph with $h(G) > 2$. Then G contains a path of length at least $3h(G) - 2$.*

Proof. Let G be a 2-connected graph with hamiltonian index $h(G)$. Since G is 2-connected, every branch-bond of G has at least two branches. By Theorem B, there is a branch-bond S of G such that S contains an odd number of branches and each branch of S has length at least $h(G) - 1$. Since $h(G) > 2$, $l(S) \geq 2$. By the definition of a branch-bond, there are exactly two components of the graph $G - S$. Let G_1, G_2 be the components of $G - S$. Let b_1, b_2, b_3 be a triple of branches of S, $B = \{b_1, b_2, b_3\}$, let x_i denote the end-vertex of b_i in G_1, y_i the end-vertex of b_i in G_2, $i = 1, 2, 3$. Let $g_1 = |V(G_1) \cap (V(b_1) \cup V(b_2) \cup V(b_3))|$ and $g_2 = |V(G_2) \cap (V(b_1) \cup V(b_2) \cup V(b_3))|$. Choose branches b_1, b_2, b_3 with maximum $g_1 + g_2$. Consider the following cases:

Case 1: $g_1 = 3$ and $g_2 = 3$. Since G_1 is connected, there is a x_1, x_2-path P_1 in G_1 and a x_1, x_3-path P_2 in G_1. Choose P_1, P_2 shortest possible. By minimality of the paths P_1 and P_2, P_1 does not contain x_3 or P_2 does not contain x_2. Up to symmetry suppose that P_1 does not contain x_3.

Since G_2 is connected, there is a y_2, y_3-path Q in G_2. Let y' denote the neighbour of y_1 on b_1. Then the path $y', b_1, x_1, P_1, x_2, b_2, y_2, Q, y_3, b_3, x_3$ has length at least $3h(G) - 2$.

Case 2: $g_1 = 2$, $g_2 = 3$ or $g_1 = 3$, $g_2 = 2$. By symmetry suppose that $g_1 = 2$ and $g_2 = 3$. Two of the branches of B have a common end-vertex in G_1, say branches b_1, b_2. Hence $x_1 = x_2$. Since G_2 is connected, there is a y_2, y_3-path Q in G_2. If $y_1 \notin V(Q)$, then the path $y_1, b_1, x_1, b_2, y_2, Q, y_3, b_3, x_3$ is a path of length at least $3h(G) - 2$ in G. Now suppose that $y_1 \in V(Q)$. Hence Q has length at least two. Then the path $x_3, b_3, y_3, Q, y_2, b_2, x_1, b_1 - y_1$ has length at least $3h(G) - 2$.

Case 3: $g_1 = 2$ and $g_2 = 2$. Since $g_1 = 2$, two of the branches of B have a common end-vertex in G_1, say branches b_1, b_2. Thus $x_1 = x_2$. Since $d_G(x_3) \geq 3$, there is a vertex $z \in V(G)$ such that $z \notin V(b_3)$, $z \neq x_1$. Since $l(S) > 1$, $z \notin V(G_2)$. Since $g_2 = 2$, $y_1 \neq y_3$ or $y_2 \neq y_3$. Up to symmetry suppose that $y_1 \neq y_3$. Since G_2 is connected, there is a y_1, y_3-path Q in G_2. Then the path $z, x_3, b_3, y_3, Q, y_1, b_1, x_1, b_2 - y_2$ has length at least $3h(G) - 2$.

Case 4: $g_1 = 1$, $g_2 = 3$ or $g_1 = 3$, $g_2 = 1$. Up to symmetry suppose that $g_1 = 1$ and $g_2 = 3$. Suppose that $|V(G_1)| > 1$. Then at least one of the vertices of G_1 different from x_1, say vertex u, is an end-vertex of some branch of S, since otherwise G is not 2-connected. But then the branches b_1, b_2, b_3 can be chosen in such a way that there are at least two different end-vertices of the triple of branches in G_1 and at least two different end-vertices of the triple of branches in G_2. Hence we are in one of the previous three cases.

Hence $|V(G)| = 1$. Let x' denote the neighbour of x_1 on b_1. Since $l(S) > 1$, $x' \notin V(G_2)$. Now we suppose that $|V(G_2)| = 3$. By Lemma 4, $l(S) \geq h(G) + 1$. Since G_2 is connected, there is a y_1, y_2-path Q is G_2. Then the path $x', b_1, y_1, Q, y_2, b_2, x_2, b_3 - y_3$ has length at least $3h(G) - 2$. Now suppose that $|V(G_3)| > 3$. There is a vertex z in G_2 different from y_1, y_2, y_3 such that z is a neighbour of at least one of the vertices y_1, y_2, y_3. Up to symmetry suppose that $zy_1 \in E(G)$. Since G_2 is connected, there is a y_2, y_3-path Q in G_2.

Subcase 4.1: There is a path Q in G_2 such that Q does not contain y_1. If $z \notin V(Q)$, then the path $z, y_1, b_1, x_1, b_2, y_2, Q, y_3, b_3 - x_3$ has length at least $3h(G) - 2$. If Q contains z but not y_1, then the path $y_1, b_1, x_1, b_2, y_2, Q, y_3, b_3 - x_3$ has length at least $3h(G) - 2$

since Q has length at least two.

Subcase 4.2: Every y_2, y_3-path Q contains vertex y_1. Choose Q shortest possible. Since $d_G(y_3) \geq 3$ and $y_2y_3 \notin E(G)$, there is a vertex z' different from y_1, y_2 in G such that $z' \notin V(b_3)$. By minimality of Q there is a y_1, y_2-path Q_1 in G_2 such that Q_1 does not contain y_3. Clearly $z' \notin V(Q_1)$, since otherwise there is a y_2, y_3-path which does not contain y_1. Then the path $z', y_3, b_3, x_3, b_2, y_2, Q, y_1, b_1 - x_1$ has length at least $3h(G) - 2$.

Case 5: $g_1 = 1$, $g_2 = 2$ or $g_1 = 2$, $g_2 = 1$. Up to symmetry suppose that $g_1 = 1$ and $g_2 = 2$. First we suppose that $|V(G_1)| > 1$. Since G is 2-connected, at least one of the vertices of G_1 different from x_1, say vertex u, is an end-vertex of some branch of S. But then the branches b_1, b_2, b_3 can be chosen in such a way that there are at least two different end-vertices of the triple of branches in G_1 and at least two different end-vertices of the triple of branches in G_2. Hence we are in one of the first three cases.
Hence $|V(G_1)| = 1$. Since $g_2 = 2$, two of the branches of B have a common end-vertex in G_2, say branches b_1, b_2. Hence $y_1 = y_2$. Let x' denote the neighbour of x_1 on b_1. Clearly $x' \notin V(G_2)$ since $l(S) > 1$. Suppose that $|V(G_2)| = 2$. Then, by Lemma 2, $l(S) \geq h(G) + 1$. Then the path $x', b_1, y_1, b_2, x_2, b_3, y_3$ has length at least $3h(G) - 2$.
Now we suppose that $|V(G_2| > 2$ and each vertex $z \in V(G_2) \setminus \{y_1, y_3\}$ is adjacent to both vertices y_1, y_3. By Lemma 3, each branch of S has length at least $h(G) + 1$. Then the path $x', b_1, y_1, b_2, x_2, b_3, y_3$ has length at least $3h(G) - 2$.
Finally we suppose that $|V(G_2| > 2$ and there is a vertex $z \in V(G_2) \setminus \{y_1, y_3\}$ such that z is adjacent to exactly one of the vertices y_1, y_3, say $y_3z \notin E(G)$. (The case $y_1z \notin E(G)$ can be shown analogously). By maximality of g_2, there is no vertex $y \in V(G_2) \setminus \{y_1, y_3\}$ such that y is an end-vertex of some branch of S. Using this fact and since G is 2-connected, there is a z, y_3-path Q in G_2 such that Q does not contain y_1. Since $y_3z \notin E(G)$, the path Q has length at least two. Then the path $b_3 - x_3, y_3, z, Q, y_1, b_1, x_1, b_2 - y_2$ has length at least $3h(G) - 2$.

Case 6: $g_1 = 1$ and $g_2 = 1$. Clearly $|V(G_1)| = |V(G_2)| = 1$. Then, by Lemma 1, each branch of S has length at least $h(G) + 1$. Then the path $b_1 - x_1, y_1, b_2, x_2, b_3 - y_3$ has length at least $3h(G) - 2$. ■

By Theorem C, if G is 2-connected $K_{1,3}$, P_6-free, then $h(G) < 1$. Now we consider a 2-connected graph H with no P_7 as a subgraph. Then $L(H)$ is 2-connected and

$K_{1,3}, P_6$-free. This implies that $h(H) < 2$. As an immediate consequence of the previous lemma we obtain the following upper bound for the hamiltonian index of a graph in terms of maximum path length.

Theorem 1. *Let k be a positive integer, let G be a 2-connected graph such that G does not contain a path of length k. Then $h(G) < \frac{k+2}{3}$.*

Corollary 2. *Let G be a 2-connected graph which does not contain $L^{-1}(P_7)$. Then $h(G) < 3$.*

The following lemma is an analogue to Lemma 5.

Lemma 6. *Let G be a 2-connected graph with $h(G) > 2$. Then G contains a graph $L^{-1}(Z_{2h(G)-3})$ as an induced subgraph.*

Proof. Let G be a 2-connected graph with hamiltonian index $h(G)$. Since G is 2-connected, every branch-bond of G has at least two branches. By Theorem B, there is a branch-bond S of G such that S contains an odd number of branches and each branch of S has length at least $h(G) - 1$. Since $h(G) > 2$, $l(S) \geq 2$. By the definition of a branch-bond, there are exactly two components of the graph $G - S$. Let G_1, G_2 be the components of $G - S$. Let b_1, b_2, b_3 be a triple of branches of S, $B = \{b_1, b_2, b_3\}$, let x_i denote the end-vertex of b_i in G_1, y_i the end-vertex of b_i in G_2, $i = 1, 2, 3$. Let $g_1 = |V(G_1) \cap (V(b_1) \cup V(b_2) \cup V(b_3))|$ and $g_2 = |V(G_2) \cap (V(b_1) \cup V(b_2) \cup V(b_3))|$. Choose branches b_1, b_2, b_3 in such a way that $g_1 + g_2$ is maximum. Consider the following cases:

Case 1: $g_1 = 3$ and $g_2 = 3$. Since G_2 is connected, there is a y_1, y_2-path P_1 in G_2. Since $d_G(x_2) \geq 3$, there are at least two neighbours x', x'' of x_2 such that none of them belongs to b_2 nor $V(G_2)$. Then the graph $x_2x', x_2x'', b_2, P_1, b_1 - x_1$ is isomorphic to $L^{-1}(Z_{2h(G)-3})$.

Case 2: $g_1 = 2$, $g_2 = 3$ or $g_1 = 3$, $g_2 = 2$. Up to symmetry suppose that $g_1 = 2$ and $g_2 = 3$. Two of the branches of B have exactly one end-vertex in G_1, say branches b_1 and b_2. Hence $x_1 = x_2$. Since G_2 is connected, there is a y_2, y_3-path Q in G_2. Since $d_G(x_2) \geq 3$, there are at least two neighbours x', x'' of x_1 in G such that none of them belongs to b_2. Since $h(G) > 2$, $x' \notin V(G_2)$ and $x'' \notin V(G_2)$. Then the graph $x_1x', x_1x'', b_2, Q, b_3 - x_3$ is isomorphic to $L^{-1}(Z_{2h(G)-3})$.

Case 3: $g_1 = 2$ and $g_2 = 2$. Since $g_1 = 2$, two of the branches of B, say b_1,

b_2, have a common end-vertex in G_1. Hence $x_1 = x_2$. Since $g_2 = 2$, then $y_1 \neq y_2$ or $y_1 \neq y_3$. Up to symmetry suppose that $y_1 \neq y_3$. Since G_2 is connected, there is a y_1, y_3-path Q in G_2. By the definition of a branch, $d_G(x_3) \geq 3$. Hence there are at least two neighbours x', x'' of x_3 in G such that none of them belongs to b_3 nor $V(G_2)$. Then the graph $x_3x', x_3x'', b_3, Q, b_1 - x_1$ is isomorphic to $L^{-1}(Z_{2h(G)-3})$.

Case 4: $g_1 = 1$, $g_2 = 3$ or $g_1 = 3$, $g_2 = 1$. Up to symmetry suppose that $g_1 = 1$ and $g_2 = 3$. Suppose that $|V(G_1)| > 1$. Then at least one of the vertices of G_1 different from x_1, say vertex u, is an end-vertex of some branch of S, since otherwise G is not 2-connected. But then the branches b_1, b_2, b_3 can be chosen in such a way that there are at least two different end-vertices of the triple of branches in G_1 and at least two different end-vertices of the triple of branches in G_2. Hence we are in one of the previous three cases.

Hence $|V(G_1)| = 1$. Let x denote the only vertex of G_1. Since G_2 is connected, there is a path Q of length at least two in G_2 joining two of the vertices y_1, y_2, y_3. Without loss of generality suppose that Q is a y_1, y_2-path. Let x' denote the neighbour of x on b_1 and x'' the neighbour of x on b_3. Clearly $x' \notin V(G_2)$, $x'' \notin V(G_2)$. The graph $xx', xx'', b_2, Q, b_1 - \{x, x'\}$ is isomorphic to $L^{-1}(Z_{2h(G)-3})$.

Case 5: $g_1 = 1$, $g_2 = 2$ or $g_1 = 2$, $g_2 = 1$. Up to symmetry suppose that $g_1 = 1$ and $g_2 = 2$. First we suppose that $|V(G_1)| > 1$. Since G is 2-connected, at least one of the vertices of G_1 different from x_1, say vertex u, is an end-vertex of some branch of S. But then the branches b_1, b_2, b_3 can be chosen in such a way that there are at least two different end-vertices of the triple of branches in G_1 and at least two different end-vertices of the triple of branches in G_2. Hence we are in one of the first three cases.

Hence $|V(G_1)| = 1$. Since $g_2 = 2$, two of the branches of B have a common end-vertex in G_2, say branches b_1, b_2. Thus $y_1 = y_2$. Let x be the vertex of G_1. Let x' denote the neighbour of x on b_1 and x'' the neighbour of x on b_2. It is easy to see that none of the vertices x', x'' belongs to G_2.

First suppose that $|V(G_2)| = 2$. By Lemma 2, each branch of B has length at least $h(G) + 1$. Clearly $y_1y_3 \in E(G)$ by connectivity of G_2. The graph $xx', xx'', b_3, y_1y_3, b_1 - \{x, x'\}$ has a subgraph isomorphic to $L^{-1}(Z_{2h(G)-3})$.

Now we suppose that $|V(G_2)| > 2$. There is a vertex $z \in V(G_2) \setminus \{y_1, y_3\}$

such that z is adjacent to at least one of the vertices y_1, y_3, say $y_1 z \in E(G)$. (The case $y_3 z \in E(G)$ can be shown analogously). By maximality of g_2, there is no vertex $y \in V(G_2) \setminus \{y_1, y_3\}$ such that y is an end-vertex of some branch of S. Using this fact and since G is 2-connected, there is a z, y_3-path Q in G_2 such that Q does not contain y_1. The graph $xx', xx'', b_3, Q, y_1 z, b_1 - \{x, x'\}$ is isomorphic to $L^{-1}(Z_{2h(G)-3})$.

Case 6: $g_1 = 1$ and $g_2 = 1$. Clearly $|V(G_1)| = |V(G_2)| = 1$. Then each branch of B has length at least $h(G) + 1$ by Lemma 1. Let x' be the neighbour of x on b_1 and x'' the neighbour of x on b_2. Clearly none of the vertices x', x'' belongs to G_2. The graph $xx', xx'', b_3, b_1 - \{x, x'\}$ has a subgraph isomorphic to $L^{-1}(Z_{2h(G)-3})$. ■

By Bedrossian's characterization (see Theorem C), if G is 2-connected $K_{1,3}, Z_2$-free, then $h(G) < 1$. Now we consider a 2-connected graph H with no $L^1(Z_2)$ as a subgraph. Then $L(H)$ is 2-connected and $K_{1,3}, Z_2$-free, implying that $h(H) < 2$. Using this fact and the previous lemma we obtain the following theorem.

Theorem 2. *Let $k \geq 1$ be an integer, let G be a 2-connected graph such that G does not contain a subgraph isomorphic to $L^{-1}(Z_k)$. Then $h(G) < \frac{k+3}{2}$.*

Corollary 3. *Let G be a 2-connected graph such that G does not contain $L^{-1}(Z_3)$. Then $h(G) < 3$.*

For graphs $B_{i,j}$ we prove the following lemma.

Lemma 7. *Let G be a 2-connected graph with $h(G) > 2$ and let i, j be positive integers at least one. Then G contains a graph $L^{-1}(B_{i,j})$ as an induced subgraph, where $i + j \geq 3h(G) - 5$.*

Proof. Let G be a 2-connected graph with hamiltonian index $h(G)$. Since G is 2-connected, every branch-bond of G contains at least two branches. By Theorem B, there is a branch-bond S of G such that S contains an odd number of branches and $l(S) \geq h(G) - 1$. Since $h(G) > 2$, $l(S) \geq 2$. By the definition of a branch-bond, there are exactly two components of the graph $G - S$. Let G_1, G_2 denote the components of $G - S$. Let b_1, b_2, b_3 denote a triple of branches of S, $B = \{b_1, b_2, b_3\}$, let x_i denote the end-vertex of b_i in G_1, y_i the end-vertex of b_i in G_2, $i = 1, 2, 3$. Let $g_j = |V(G_j) \cap (V(b_1) \cup V(b_2) \cup V(b_3))|$, $j = 1, 2$. Choose branches b_1, b_2, b_3 in such a way that $g_1 + g_2$ is maximum. The following possibilities can occur:

Case 1: $g_1 = 3$ and $g_2 = 3$. Since G_1 is connected, there is a x_1, x_2-path P_1 and x_1, x_3-path P_2 in G_1. Choose P_1 and P_2 shortest possible. Clearly P_1 does not contain x_3 or P_2 does not contain x_2. By symmetry suppose that P_1 does not contain x_3. Since $d_G(x_1) \geq 3$, there is a neighbouring vertex x' of x_1 such that x' does not belong to any of b_1, P_1. Since $l(S) \geq 2$, $x' \notin V(G_2)$. Since G_2 is connected, there is a y_1y_3-path Q in G_2. Choose Q shortest possible. Consider the following paths: $B_1 = x_1x'$, $B_2 = x_1, b_1, y_1, Q, y_3, b_3 - x_3$, $B_3 = x_1, P_1, x_2, b_2 - y_2$. A subgraph of G consisting of the paths B_1, B_2, B_3 is isomorphic to $L^{-1}(B_{2h(G)-3,h(G)-2})$.

Case 2: $g_1 = 2$, $g_2 = 3$ or $g_1 = 3$, $g_2 = 2$. Up to symmetry suppose that $g_1 = 2$ and $g_2 = 3$. Two of the branches of B have exactly one end-vertex in G_1, say branches b_1 and b_2. Hence $x_1 = x_2$. Since G_1 is connected, there is a x_1, x_3-path P in G_1. Analogously, since G_2 is connected, there is a y_2, y_3-path Q in G_2. Choose P and Q shortest possible. Since $d_G(x_3) \geq 3$, there is a vertex x' such that $x' \notin V(P)$, $x' \notin V(b_3)$. Since $l(S) \geq 2$, $x' \notin V(G_2)$. Consider the following paths: $B_1 = x_3, x'$, $B_2 = x_3, b_3, y_3, Q, y_2, b_2 - x_2$, $B_3 = x_3, P, x_1, b_1 - y_1$. The subgraph of G consisting of the paths B_1, B_2, B_3 is isomorphic to $L^{-1}(B_{2h(G)-3,h(G)-2})$.

Case 3: $g_1 = 2$ and $g_2 = 2$. Two of the branches of B, say b_1, b_2, have a common end-vertex in G_1. Thus $x_1 = x_2$. Similarly, there are exactly two different end-vertices of the branches of B in G_2. Let y_a, y_b denote these vertices and, up to symmetry, suppose that $y_b \in V(b_3)$. Hence $y_a \in V(b_1)$ or $y_a \in V(b_2)$. Since G_1 is connected, there is a x_1, x_3-path P in G_1. Analogously, since G_2 is connected, there is a y_a, y_b-path Q in G_2. Choose P and Q shortest possible. Since $d_G(x_3) \geq 3$, there is a vertex x' such that $x' \notin V(P)$, $x' \notin V(b_3)$. Clearly $x' \notin V(G_2)$. First we suppose that $y_a \in V(b_1)$. Consider the following paths $B_1 = x_3, x'$, $B_2 = x_3, b_3, y_b, Q, y_a, b_1 - x_1$, $B_3 = x_3, P, x_1, b_2 - y_i$, where y_i denote the end-vertex of b_2 in G_2. Then the triple B_1, B_2, B_3 forms a subgraph of G isomorphic to $L^{-1}(B_{2h(G)-3,h(G)-2})$.

Finally we suppose that $y_a \in V(b_2)$. Consider the following paths $B_1 = x_3, x'$, $B_2 = x_3, b_3, y_b, Q, y_a, b_2 - x_1$, $B_3 = x_3, P, x_1, b_1 - y_i$, where y_i is the end-vertex of b_1 in G_2. Then the triple B_1, B_2, B_3 forms a subgraph of G isomorphic to $L^{-1}(B_{2h(G)-3,h(G)-2})$.

Case 4: $g_1 = 1$, $g_2 = 3$ or $g_1 = 3$, $g_2 = 1$. Up to symmetry suppose that $g_1 = 1$ and $g_2 = 3$. First we suppose that $|V(G_1)| > 1$. Then at least one of the vertices of G_1 different from x_1, say vertex u, is an end-vertex of some branch of S, since otherwise G is not 2-connected. But then the branches b_1, b_2, b_3 can be chosen in such a way that there are at least two different end-vertices of the branches of S in G_1 and at least two different end-vertices of the branches of S in G_2. Therefore we are in one of the previous three cases.

Hence $|V(G_1)| = 1$ and $x_1 = x_2 = x_3$. Suppose that $|V(G_2)| = 3$. Let x' denote the neighbour of x_1 on b_1. Clearly $x' \notin V(G_2)$. Since G_2 is connected, $y_1y_2 \in E(G)$ or $y_1y_3 \in E(G)$. Up to symmetry suppose that $y_1y_2 \in E(G)$. By Lemma 4, each branch of S has length at least $h(G)+1$. The following triple of paths $B_1 = x_1, x'$, $B_2 = x_1, b_2, y_2, y_1, b_1 - \{x, x'\}$, $B_3 = x_1, b_3, y_3$ forms a subgraph of G isomorphic to $L^{-1}(B_{2h(G),h(G)})$.

Now suppose that $|V(G_2)| > 3$. There is a vertex $z \in V(G_2)$ such that z is different from y_1, y_2, y_3 and z is a neighbour of at least one of the vertices y_1, y_2, y_3, say vertex y_1. Since G_2 is connected, there is a y_2, y_3-path Q in G_2. Choose Q shortest possible. Let u denote the neighbour of x_2 on b_2 and let v denote the neighbour of x_3 on b_3.

Subcase 4.1: The path Q does not contain any of the vertices z, y_1. Then the triple of paths $B_1 = x_3, v$, $B_2 = x_2, b_2, y_2, Q, y_3, b_3 - \{x_3, v\}$, $B_3 = x_1, b_1, y_1, z$ forms a subgraph of G isomorphic to $L^{-1}(B_{2h(G)-4,h(G)-1})$.

Subcase 4.2: The path Q contains z but not y_1. The graph consisting of the paths $B_1 = x_3, v$, $B_2 = x_2, b_2, y_2, Q, y_3, b_3 - \{x_3, v\}$, $B_3 = x_1, b_1, y_1$ is isomorphic to $L^{-1}(B_{2h(G)-3,h(G)-2})$ since Q has length at least two.

Subcase 4.3: The path Q contains y_1 but not z. By minimality of Q, there is a y_1, y_3-path Q_1 in G_2 such that Q_1 does not contain y_2. (Note that Q_1 is a subpath of Q.) The graph consisting of the paths $B_1 = y_1, z$, $B_2 = y_1, b_1, x_1, b_2, y_2$, $B_3 = y_1, Q_1, y_3, b_3 - x_3$ is isomorphic to $L^{-1}(B_{2h(G)-3,h(G)-2})$.

Subcase 4.4: The path Q contains both vertices z, y_1. By minimality of Q, there is a y_1, y_3-path Q_1 in G_2 such that Q_1 does not contain y_2. If $z \in V(Q_1)$, then the triple of paths $B_1 = y_1, z$, $B_2 = y_1, b_1, x_1, b_3, y_3$, $B_2 = y_1, Q - Q_1, y_2, b_2 - x_2$ forms a subgraph of G isomorphic to $L^{-1}(B_{2h(G)-3,h(G)-2})$. If $z \notin V(Q_1)$, then the graph consisting of the paths $B_1 = y_1, z$, $B_2 = y_1, b_1, x_1, b_2, y_2$, $B_3 = y_1, Q_1, y_3, b_3 - x_3$ is isomorphic to

$L^{-1}(B_{2h(G)-3,h(G)-2})$.

Case 5: $g_1 = 1$, $g_2 = 2$ or $g_1 = 2$, $g_2 = 1$. By symmetry suppose that $g_1 = 1$ and $g_2 = 2$. First we suppose that $|V(G_1)| > 1$. Since G is 2-connected, at least one of the vertices of G_1 different from x_1, say vertex u, is an end-vertex of some branch of S. But then the branches b_1, b_2, b_3 can be chosen in such a way that there are at least two different end-vertices of the triple of branches in G_1 and at least two different end-vertices of the triple of branches in G_2. Hence we are in one of the first three cases.

Hence $|V(G_1)| = 1$. Since $g_2 = 2$, two of the branches of B have a common end-vertex in G_2, say branches b_1, b_2. Thus $y_1 = y_2$. Let x' denote the neighbour of x_1 on b_1. Since $l(S) \geq 2$, $x' \notin V(G_2)$. The following possibilities can occur:

- $|V(G_2)| = 2$. By Lemma 2, every branch of S has length at least $h(G)+1$. The triple of paths $B_1 = x_1, x'$, $B_2 = x_1, b_2, y_2 = y_1, b_1 - \{x_1, x'\}$, $B_3 = x_1, b_3, y_3$ forms a subgraph of G isomorphic to $L^{-1}(B_{2h(G)-1,h(G)})$.
- $|V(G_2)| > 2$. First we suppose that every vertex of $V(G_2) \setminus \{y_1, y_3\}$ is adjacent to both vertices y_1, y_3. Then, by Lemma 3, each branch of S has length at least $h(G)+1$. The triple of paths $B_1 = x_1, x'$, $B_2 = x_1, b_2, y_2 = y_1, b_1 - \{x_1, x'\}$, $B_3 = x_1, b_3, y_3$ forms a subgraph of G isomorphic to $L^{-1}(B_{2h(G)-1,h(G)})$.

  Finally we suppose that $|V(G_2)| > 2$ and that there is a vertex $z \in V(G_2) \setminus \{y_1, y_3\}$ such that z is adjacent to exactly one of the vertices y_1, y_3, say $y_3 z \notin E(G)$. (The case $y_1 z \notin E(G)$ can be shown analogously). By maximality of g_2, there is no vertex $y \in V(G_2)$ such that y is an end-vertex of some branch of S. Using this fact and since G is 2-connected, there is a z, y_3-path Q in G_2 such that Q does not contain y_1. Since $y_3 z \notin E(G)$, the path Q has length at least two. Now we consider the paths $B_1 = x_1, x'$, $B_2 = x_1, b_2, y_2 = y_1, b_1 - \{x_1, x'\}$, $B_3 = x_1, b_3, y_3, Q, z$. The subgraph of G consisting of the triple B_1, B_2, B_3 is isomorphic to $L^{-1}(B_{2h(G)-5,h(G)})$.

Case 6: $g_1 = 1$, $g_2 = 1$. Clearly $|V(G_1)| = |V(G_2)| = 1$. By Lemma 1, each branch of S has length at least $h(G) + 1$. Let x' be the neighbour of x_1 on b_1 and clearly $x' \notin V(G_2)$. Consider the following triple of paths: $B_1 = x_1, x'$, $B_2 = x_1, b_2, y_2 = y_1, b_1 - \{x_1, x'\}$, $B_3 = x_1, b_3 - y_3$. The graph consisting of B_1, B_2, B_3 is isomorphic to $L^{-1}(B_{2h(G)-1,h(G)-1})$.■

By Theorem C, if G is 2-connected $K_{1,3}, B_{1,2}$-free, then $h(G) < 1$. Now we consider a 2-connected graph H with no $L^1(B_{1,2})$ as a subgraph. Then $L(H)$ is 2-connected and $K_{1,3}, B_{1,2}$-free. This implies that $h(H) < 2$. As an immediate consequence of Lemma 7 and Theorem C we obtain the following theorem.

Theorem 3. *Let i, j be positive integers such that $i, j \geq 1$. Let G be a 2-connected graph. If G does not contain any of graphs $L^{-1}(B_{i,j})$ as a subgraph (not necessarily induced), then $h(G) < \frac{i+j+5}{3}$.*

Corollary 4. *Let G be a 2-connected graph such that G does not contain any of graphs $L^{-1}(B_i, j)$, where i, j are positive integers such that $i + j = 4$. Then $h(G) < 3$.*

The following lemma gives an upper bound on the hamiltonian index in terms of the preimage of a line graph $N_{i,j,k}$.

Lemma 8. *Let G be a 2-connected graph with $h(G) > 3$ and let i, j, k be positive integers at least one. Then G contains a graph $L^{-1}(N_{i,j,k})$ as an induced subgraph, where $i + j + k \geq 3h(G) - 5$.*

Proof. Let G be a 2-connected graph with hamiltonian index $h(G)$. Since G is 2-connected, every branch-bond of G contains at least two branches. By Theorem B, there is a branch-bond S of G such that S contains an odd number of branches and $l(S) \geq h(G) - 1$. Since $h(G) > 2$, $l(S) \geq 2$. By the definition of a branch-bond, there are exactly two components of the graph $G - S$. Let G_1, G_2 denote the components of $G - S$. Let b_1, b_2, b_3 denote a triple of branches of S, $B = \{b_1, b_2, b_3\}$, let x_i denote the end-vertex of b_i in G_1, y_i the end-vertex of b_i in G_2, $i = 1, 2, 3$. Let $g_j = |V(G_j) \cap (V(b_1) \cup V(b_2) \cup V(b_3))|$, $j = 1, 2$. Choose branches b_1, b_2, b_3 in such a way that $g_1 + g_2$ is maximum. The following possibilities can occur:

Case 1: $g_1 = 3$ and $g_2 = 3$. Since G_1 is connected, there is a x_1, x_2-path P_1 in G_1 and x_2, x_3-path P_2 in G_2. Choose P_1 and P_2 shortest possible. Then P_1 does not contain x_3 or P_2 does not contain x_1. Up to symmetry suppose that P_1 does not contain x_3. If P_2 does not contain x_1, then the triple of paths $B_1 = x_2, P_1, x_1, b_1$, $B_2 = x_2, b_2, y_2$, $B_3 = x_2, P_2, x_3, b_3, y_3$ forms a subgraph of G isomorphic to $L^{-1}(N_{h(G)-1,h(G)-2,h(G)-1})$, since y_1, y_2, y_3 are mutually different. Now suppose that P_2 contains vertex

x_1. Then there is a path P_3 in G_1, a subpath of P_2, such that P_3 does not contain x_2. Hence we switch vertices x_1 and x_2, y_1 and y_2, branches b_1 and b_2, and we are in the previous possibility.

Case 2: $g_1 = 2, g_2 = 3$ or $g_1 = 3, g_2 = 2$. Up to symmetry suppose that $g_1 = 2$ and $g_2 = 3$. Two branches of B have a common end-vertex in G_1, say branches b_1 and b_2. Hence $x_1 = x_2$. Since G_1 is connected, there is a x_1, x_3-path P in G_1. Consider the following paths $B_1 = x_1, b_1, y_1$, $B_2 = x_1, b_2, y_2$ and $B_3 = x_1, P, x_3, b_3, y_3$. Then the triple B_1, B_2, B_3 forms a subgraphs of G isomorphic to $L^{-1}(N_{h(G)-2,h(G)-2,h(G)-1})$.

Case 3: $g_1 = 2$ and $g_2 = 2$. Two branches of B, say b_1 and b_2, have a common end-vertex in G_1. Thus $x_1 = x_2$. Similarly, there are exactly two end-vertices of the branches of B in G_2. Since G_1 is connected, there is a x_1, x_3-path P in G_1. First we suppose that $y_1 = y_2$. Since $d_G(y_3) \geq 3$, there is a vertex $z \in [V(G) \setminus V(G_1) \setminus V(B)]$. The triple of paths $B_1 = x_1, b_1, y_1$, $B_2 = x_1, b_2 - y_1$, $B_3 = x_1, P, x_3, b_3$ forms a subgraph of G isomorphic to $L^{-1}(N_{h(G)-2,h(G)-3,h(G)})$.

Now suppose that $y_1 \neq y_2$. Since $g_2 = 2$, $y_1 = y_3$ or $y_2 = y_3$. Up to symmetry suppose that $y_2 = y_3$. Since $d_G(y_1) \geq 3$, there is a vertex $z \in [V(G) \setminus V(G_1) \setminus V(B)]$. Then the triple of paths $B_1 = x_1, b_1, y_1, z$, $B_2 = x_1, b_2, y_3$, $B_3 = x_1, P, x_3, b_3 - y_3$ forms a subgraph of G isomorphic to $L^{-1}(N_{h(G)-1,h(G)-2,h(G)-2})$.

Case 4: $g_1 = 1, g_2 = 3$ or $g_1 = 3, g_2 = 1$. Up to symmetry suppose that $g_1 = 1$ and $g_2 = 3$. First we suppose that $|V(G_1)| > 1$. Then at least one vertex of G_1 different from x_1, say vertex u, is an end-vertex of some branch of S, since otherwise G is not 2-connected. But then the branches b_1, b_2, b_3 can be chosen in such a way that $g_1 \geq 2$ and $g_2 \geq 2$. Hence we are in one of the previous three cases.

Hence $|V(G_1)| = 1$. This yields that $x_1 = x_2 = x_3$. Suppose that $|V(G_2)| = 3$. By Lemma 4, each branch of S has length at least $h(G)+1$. Then the subgraph of G consisting of the paths $B_1 = x_1, b_1, y_1$, $B_2 = x_1, b_2, y_2$, $B_3 = x_1, b_3, y_3$ contains $L^{-1}(N_{h(G),h(G),h(G)})$.

Now suppose that $|V(G_2) > 3$. Then there is a vertex $z \in V(G_2) \setminus V(B)$ such that z is a neighbour of at least one of the vertices y_1, y_2, y_3. Up to symmetry suppose that $y_1 z \in E(G)$. Then the following triple of paths $B_1 = x_1, b_1, y_1, z$, $B_2 = x_1, b_2, y_2$, $B_3 = x_1, b_3, y_3$ forms a subgraph of G isomorphic to $L^{-1}(N_{h(G)-1,h(G)-2,h(G)-2})$.

Case 5: $g_1 = 1, g_2 = 2$ or $g_1 = 2, g_2 = 1$. Up to symmetry suppose that $g_1 = 1$

and $g_2 = 2$. Suppose that $|V(G_1)| > 1$. Since G is 2-connected, at least one of the vertices of G_1 different from x_1, say vertex u, is an end-vertex of some branch of S. But then the branches b_1, b_2, b_3 can be chosen in such a way that there are at least two different end-vertices of the triple of branches in G_1 and at least two different end-vertices of the triple of branches in G_2. Hence we are in one of the first three cases.

Hence $|V(G_1) = 1$. Since $g_2 = 2$, two branches of B have a common end-vertex in G_2, say branches b_1, b_2. Thus $y_1 = y_2$. Since $l(S) \geq 2$, $x_1 \notin V(G_2)$. The following possibilities can occur:

- $|V(G_2)| = 2$. By Lemma 2, every branch of S has length at least $h(G) + 1$. The triple of paths $B_1 = x_1, b_1, y_1$, $B_2 = x_1, b_2 - y_2$, $B_3 = x_1, b_3, y_3$ forms a subgraph of G isomorphic to $L^{-1}(N_{h(G),h(G)-1,h(G)})$.
- $|V(G_2)| > 2$. First we suppose that every vertex of $V(G_2) \setminus \{y_1, y_3\}$ is adjacent to both vertices y_1 and y_3. Then, by Lemma 3, every branch of S has length at least $h(G) + 1$. The triple of paths $B_1 = x_1, b_1, y_1$, $B_2 = x_1, b_2 - y_2$, $B_3 = x_1, b_3, y_3$ forms a subgraphs of G isomorphic to $L^{-1}(N_{h(G),h(G)-1,h(G)})$.

  Finally we suppose that $|V(G_2)| > 2$ and there is a vertex $z \in V(G_2) \setminus \{y_1, y_3\}$ such that z is adjacent to exactly one of the vertices y_1, y_3. Suppose that $y_3z \notin E(G)$. Since $d_G(y_3) \geq 3$, there is a vertex $z' \in V(G) \setminus V(B)$ different from z such that $z'y_3 \in E(G)$. The triple of paths $B_1 = x_1, b_1, y_1, z$, $B_2 = x_1, b_2 - y_2$, $B_3 = x_1, b_3, y_3, z'$ forms a subgraph of G isomorphic to $L^{-1}(N_{h(G)-1,h(G)-3,h(G)-1})$. Hence we suppose that $y_3z \in E(G)$, but $y_1z \notin E(G)$. By maximality of g_2, there is no vertex $y \in V(G_2)$ such that y is an end-vertex of some branch of S. Using this fact and since G is 2-connected, there is a z, y_1-path Q in G such that Q does not contain y_3. Since $y_1z \notin E(G)$, Q has length at least two. Consider the following paths $B_1 = x_1, b_1, y_1, Q, z$, $B_2 = x_1, b_2 - y_2$, $B_3 = x_1, b_3, y_3$. The subgraph consisting of B_1, B_2, B_3 is isomorphic to $L^{-1}(N_{h(G),h(G)-3,h(G)-2})$.

Case 6: $g_1 = 1$ and $g_2 = 1$. Clearly $|V(G_1)| = 1$ and $|V(G_2)| = 1$. By Lemma 1, every branch of S has length at least $h(G) + 1$. Then the triple of paths $B_1 = x_1, b_1, y_1$, $B_2 = x_1, b_2 - y_2$, $B_3 = x_1, b_3 - y_3$ forms a subgraph of G isomorphic to $L^{-1}(N_{h(G),h(G)-1,h(G)-1})$. ∎

By Corollary 1, if G is a 2-connected $K_{1,3}, E$-free graph, then $h(G) \leq 1$. Now we consider a 2-connected graph H such that H does not contain $L^{-1}(E)$ as a subgraph. Then $L(H)$ is 2-connected and $K_{1,3}, E$-free, implying that $h(H) \leq 2$. By Theorem C, if G is 2-connected $K_{1,3}, N$-free, then $h(G) < 1$. Now we consider a 2-connected graph H with no $L^{-1}(N)$ as a subgraph. Then $L(H)$ is 2-connected and $K_{1,3}, N$-free. This implies that $h(H) < 2$.
As an immediate consequence of Lemma 7, Theorem C and Corollary 1 we obtain the following theorem.

Theorem 4. *Let i, j be positive integers such that $i, j, k \geq 1$. Let G be a 2-connected graph. If G does not contain any of graphs $L^{-1}(N_{i,j,k})$ as a subgraph (not necessarily induced), then $h(G) < \frac{i+j+k+5}{3}$.*

Corollary 5. *Let G be a 2-connected graph such that G does not contain any of graphs $L^{-1}(N_{i,j,k})$, where i, j, k are positive integers such that $i + j + k = 4$. Then $h(G) < 3$.*

3 Sharpness

Now we consider the following example. Let k be a positive integer, let G be a graph consisting of paths P_1, P_2, P_3 each of length $k+1$ with common end-vertices x and y. By Lemma 1, $h(G) = k$ and G contains a path of length $3k+1$, a graph $L^{-1}(Z_{2k-1})$, a graph $L^{-1}(B_{k-1,2k-1})$, and a graph $L^{-1}(N_{k-1,i,j})$ for each positive integers i, j with $i + j = 2k - 1$.
Replacing any of the paths P_1, P_2, P_3 with a shorter one we obtain a graph H such that $h(H) < h(G)$ by Lemma 1. Hence the graph G is the graph with minimum number of vertices such that $h(G) = k$ and G is 2-connected, and the following conjectures could be true.

Conjecture 1. *Let $k \geq 1$ be a positive integer, let G be a 2-connected graph such that G does not contain a subgraph isomorphic to $L^{-1}(P_k)$. Then $h(G) < \frac{k-1}{3}$.*

Conjecture 2. *Let $k \geq 1$ be a positive integer, let G be a 2-connected graph such that G does not contain a subgraph isomorphic to $L^{-1}(Z_k)$. Then $h(G) < \frac{k+1}{2}$.*

Conjecture 3. *Let i, j be positive integers at least one, let G be a 2-connected graph such that G does not contain any of subgraphs isomorphic to $L^{-1}(B_{i,j})$. Then $h(G) < \frac{i+j+2}{3}$.*

Conjecture 4. *Let i, j, k be positive integers at least one, let G be a 2-connected graph such that G does not contain any of subgraphs isomorphic to $L^{-1}(N_{i,j,k})$. Then $h(G) < \frac{i+j+k+2}{3}$.*

Note that, for cases P_6, Z_2, $B_{1,2}$ and $N_{1,1,1}$, the previous conjectures holds by Theorem C.

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