Height Probabilities in the Abelian Sandpile Model on the Generalized Trees

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Abstract

This paper deals with the Abelian sandpile model on the generalized trees with certain given boundary condition. Using a combinatorial method, we obtain the exact expressions for all single-site probabilities and some two-site joint probabilities. Also, we prove that the sites near the boundary have a different height probability from those away from it in bulk for the Bethe lattice with the boundary condition, which is the same as those results found by Grassberger and Manna ["Some more sandpiles," J.Phys.(France)51,1077-1098(1990)] and proved by Haiyan chen and Fuji Zhang ["Height probabilities in the Abelian sandpile on the generalized finite Bethe lattice" J. Math. Phys. 54, 083503 (2013)].

1 Introduction

In 1987, the concept of self-organized criticality was put forward by Bak, Tang and Wiesenfeld (BTW) as an attempt to explain the occurrence of power laws in various and many natural phenomena. The sandpile model is the paradigm of a self-organized critical system in physics (see [3, 4, 5, 6]). It is a discrete model defined on a lattice and possesses a cellular automaton type of dynamics. A general analysis of the original sandpile model was undertaken by Dhar (see [7]). Dhar and Creutz (see [7, 8]) showed that the general sandpile model features an Abelian group, hence refer to this model as the Abelian sandpile model (ASM). It has been considered by many combinatorists as a game on a graph called the chip firing game or the dollar game (see [9, 10, 11, 12, 13, 14]).

The Abelian sandpile model can be described informally as the dynamics on a connected multigraph with a special vertex, called the *sink*. Let $G = (V, q, E)$ be a connected multigraph with the vertex set

$$V(G) = \{q, v_1, v_2, \cdots, v_n\},$$

the sink q and the edge set $E(G)$. In the model, a *configuration* on G is a non-negative integer vector

$$x = (x(v_1), x(v_2), \cdots, x(v_n)),$$

where the non-negative integer $x(v_i)$ is considered as the height of the vertex v_i. A vertex v_i with $0 \leq x(v_i) < d_G(v_i)$ (the degree of v_i in G) is called *stable* vertex. A configuration x is said to be stable when all the vertices different from q are stable. In an unstable configuration x, for a vertex v_i with $x(v_i) \geq d_G(v_i)$, we may perform a *toppling* to obtain a new configuration y such that

$$y(v_j) = \begin{cases} x(v_i) - d_G(v_i) & \text{if } j = i \\ x(v_i) + e_{ij} & \text{if } j \neq i \end{cases},$$

where e_{ij} denotes the number of edges between vertices v_i and v_j in G. The adjacency relation between the sink q and the rest vertices of G is called the boundary condition. The time evolution of a configuration of the ASM is defined by the following rules:

1. taking a stable configuration
2. choosing randomly a vertex v_i and setting $x(v_i) \to x(v_i) + 1$
3. performing topplings until a new stable configuration is obtained.

Starting with any unstable configuration and toppling unstable vertices repeatedly, we finally obtain a new stable configuration after a finite sequence of topplings since the graph G is connected. It is well known (see [7]) that the new stable configuration does not depend on the possible choice of the order of the finite sequence of topplings.

A configuration is *recurrent* in the ASM if it is a stable configuration which is met infinitely often in the dynamics. Dhar (see [7]) showed that the set of recurrent configurations of the ASM on a graph has a group structure (the Abelian group) with a natural addition and the recurrent configurations occur with equal probability under the invariant measure. It is well known that the number of recurrent configurations is equal to the number of spanning trees of the underlying graph (see [7]). Furthermore, in [7, 12, 13], several explicit one-to-one mappings between spanning trees and recurrent configurations have been constructed.

Let R be the set of all recurrent configurations, R_i^k the set of the recurrent configurations with height $x(v_i) = k$, $|R|$ the cardinal of the set R, then the single-site height probabilities are

$$P(x(v_i) = k) = \frac{|R_i^k|}{|R|}$$

and the two-site joint height probabilities are

$$P(x(v_i) = k, x(v_j) = l) = \frac{|R_i^k \cap R_j^l|}{|R|}.$$

There are many works to study these height probabilities in the ASM on lattices. In [15], Priezzhev developed a technique to obtain the height probabilities on the plane square lattice. Using basically this technique, many more infinite lattices, such as the plane square lattice [16, 17], the Bethe lattice [18], the d-dimensional hypercubic lattice [19], the triangular lattice [20], the honeycomb lattice [21], the upper-half plane square lattice [22, 23, 24], are also considered by many researchers. As for the finite lattices, it is a challenge to present exact expressions for the height probabilities, in view of effects of boundary conditions and lack of good symmetry. So up to date, only a few results [1, 25, 2] are obtained. In [1], Grassberger and Manna studied the height probabilities on the finite Bethe lattices with two different boundary conditions (see Figures 1(a) and 1(b)).

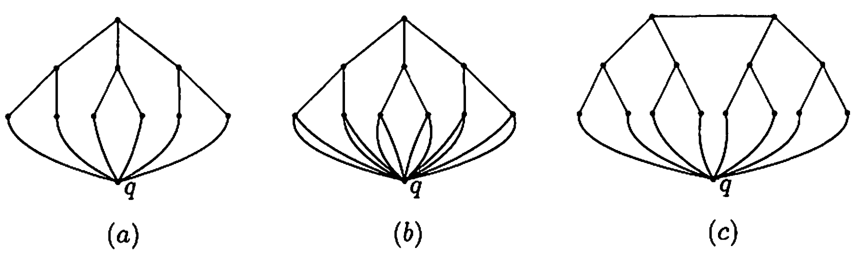


Fig. 1.Finite Bethe lattices with boundary conditions

For the finite Bethe lattices with boundary condition (b), they found that the height probabilities depend on the generation g (the distance from the sink q), which agrees with the result obtained in [18]. For the finite Bethe lattices with boundary condition (a), they found that for the total generation g from 4 to 20, $P(x(v) = 0) = \frac{1}{12}, P(x(v) = 1) = \frac{4}{12}, P(x(v) = 2) = \frac{7}{12}$ for all vertices with $g \geq 2$, while $P(x(v) = 0) =$

$\frac{1}{3}, P(x(v) = 1) = \frac{2}{3}$ for leaves ($g = 1$), which agrees with the result proved by Haiyan chen and Fuji Zhang in [2]. Here, under the boundary condition of (a), we consider the finite Bethe lattice (see Figures 1(c)). For some special event about this finite Bethe lattice, by direct calculation, it is not difficult to get that for the total generation g from 4 to 6, $P(x(v) = 0) = \frac{1}{12}, P(x(v) = 1) = \frac{4}{12}, P(x(v) = 2) = \frac{7}{12}$ for all vertices with $g \geq 2$ and $P(x(v) = 0) = \frac{1}{3}, P(x(v) = 1) = \frac{2}{3}$ for leaves ($g = 1$). This example is interesting, we can see that the result of the example is just the result which was found by Grassberger and Manna in [1] and proved by Haiyan chen and Fuji Zhang [2]. For the finite Bethe lattices with boundary condition (c), we found that there are many more sites near the boundary than that on the finite Bethe lattices with boundary condition (a) and (b). The motivation for this paper is the above fact, that is, whether such the finite Bethe lattices with boundary condition (c) affects the height probabilities or not. Thus, in this paper, we characterize height probabilities of the generalized finite Bethe lattice with the boundary condition (see Figures 1(c)). Based on the one-to-one mapping given by Biggs [12] and the technique developed by Haiyan chen and Fuji Zhang, first we determine spanning trees corresponding to recurrent configurations with given height. Then we count the total number of spanning trees and the numbers of spanning trees corresponding to recurrent configurations with the constrained height.

The remainder of this paper is organized as follows. In section 2, we do some preliminaries, including the notations and some useful results. In sections 3 and 4, we present the exact expressions for the height probabilities. In section 5, in view of the theorems in sections 3 and 4, we derive the height probabilities of the ASM on the finite Bethe lattice with the boundary condition (c).

2 Preliminaries

The purpose of this section is to define the finite Bethe lattice we shall study and introduce some useful results.

Let $\overline{T}_{k,n}$ denote a rooted tree in which the root has k children, the other vertex has two children and the leaves are at a distance n from the root. Let $T_{k,n}$ be one generalized finite Bethe lattice obtained from $\overline{T}_{k,n}$ by adding a sink q to $\overline{T}_{k,n}$ and joining the sink to every leaf vertex of $\overline{T}_{k,n}$, which is defined by Haiyan chen et al. [2]. Let $\overline{T}^2_{k,n}$ be a two-rooted tree constructed from two rooted trees $\overline{T}_{k,n}$ by using one edge to join the root vertices of the two rooted trees. Then another generalized finite Bethe lattice $T^2_{k,n}$ is obtained from $\overline{T}^2_{k,n}$ by adding a sink q to $\overline{T}^2_{k,n}$ and joining the sink to every leaf vertex of $\overline{T}^2_{k,n}$ (see Fig. 2).

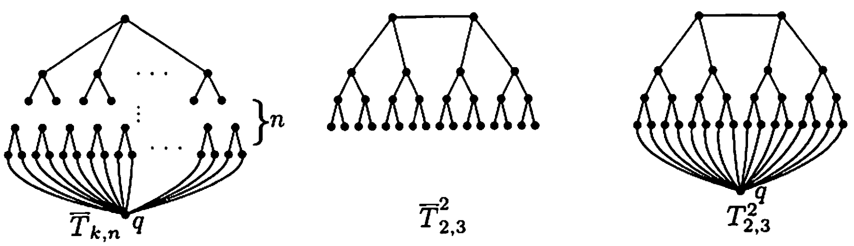


Fig. 2. The generalized Bethe lattice

In the next subsection, we present the one-to-one mapping [12] between spanning trees and recurrent configurations. Let $G = (V, q, E)$ be a connected multigraph with $|V(G)| = n + 1$ and $|E(G)| = m$, q be the fixed sink. We fix once and for all a vertex $q \in V(G)$ as the sink and an ordering $e_1, e_2, \cdots, e_m$ of $E(G)$. Let P be the set of paths in G which begin at q, and there is a lexicographic ordering $\prec$ of P induced by the ordering of edges defined as follows. If

$$\alpha = (\alpha_1, \alpha_2, \cdots \alpha_j) \text{ and } \beta = (\beta_1, \beta_2, \cdots \beta_k)$$

are sequences of edges corresponding to paths in P and i is the least index for which $\alpha_i \neq \beta_i$, then $\alpha \prec \beta$ if and only if α_i comes before β_i in the order on $E(G)$; if there is no such index then $\alpha \prec \beta$ if and only if $j < k$. (For completeness the empty set of edges is regarded as a path from q to q, and it comes first in the $\prec$ order.)

Given a spanning tree T of G, the construction

$$T \mapsto (\prec_T, x_T)$$

produces a total ordering $\prec_T$ of $V(G)$ and a recurrent configuration x_T. The total ordering depends on the simple observation that given T, for any vertex $v \in V(G)$, there is a unique path in T from q to v. We can therefore define an order $\prec_T$ on the vertices of G by using the $\prec$ order on the paths. For each vertex $v \neq q$ let v' be the unique vertex adjacent to v on the path from q to v in T. Define

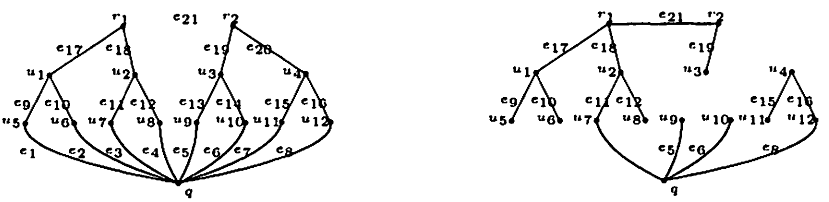
$$N_T(v) = \{u \in V(G) | u \prec_T v' \text{ and } uv \in E(G)\} \cup \{v'\}$$

The corresponding recurrent configuration x_T is defined by $x_T(v) = d_G(v) - |N_T(v)|$. Since the mapping from the set of spanning trees to the set of recurrent configurations is a bijection [12], this implies , about the height variables of recurrent configurations, that we just need to determine the corresponding spanning trees.

Throughout this paper, we fix once and for all an ordering of $E(T^2_{k,n})$ as follows.

Starting from the edges adjacent to the sink q, from left to right, down to up, label the edges of $T^2_{k,n}$. An example about the above ordering and the mapping is shown in figure 3, where the configuration $x_T = (x(r_1), x(r_2), x(u_1), x(u_2), \cdots, x(u_{12}))$ and the ordering $\prec_T$ is

$$q, u_7, u_2, u_8, r_1, u_1, u_5, u_6, r_2, u_{3,}, u_9, u_{10}, u_{12}, u_4, u_{11}.$$



$T \longleftrightarrow x_T = (2,2,2,2,2,1,0,0,1,0,1,1,0,1)$

(a) The way of the edge-labeling of $E(T^2_{2,2})$ (b) The mapping from T to x_T

Fig. 3

We denote the number of spanning trees of a simple graph G by $\tau(G)$. An edge e of G is said to be contracted if it is deleted and its ends are identified, then the resulting graph is denoted by $G \cdot e$, and $G - e$ denotes the graph obtained by deleting e. We shall apply the well-known formula $\tau(G) = \tau(G - e) + \tau(G \cdot e)$.

In order to calculate the number of spanning trees corresponding to recurrent configurations with the constrained height, we need the following lemmas.

Lemma 2.1 [2]. *Let $t_{k,n}$ be the number of spanning trees of $T_{k,n}$ and $s_{k,n}$ denote the number of spanning forests of $T_{k,n}$ with two components in* *which the root belongs to one and the sink the other, then for* $k, n \geq 1$,

$$t_{k,n} = k2^{k2^n-(k+1)} \textit{ and } s_{k,n} = 2^{k2^n-k}.$$

According to the structure of spanning trees of $T_{k,n}$ and induction on n, Lemma 2.1 can be obtained. By the formula $\tau(G) = \tau(G-e)+\tau(G\cdot e)$, we have $\tau(T^2_{k,n}) = (t_{k,n})^2 + t_{2k,n}$. Combining Lemma 2.1, we can get the following lemma.

Lemma 2.2 *For* $T^2_{k,n}$, $k, n \geq 1$, *then* $\tau(T^2_{k,n}) = k2^{k2^{n+1}-2k}(k2^{-2}+1)$.

3 The Single-site Probabilities of the ASM on $T^2_{k,n}$

In this section we concentrate on the single-site probabilities of heights of vertices in recurrent configurations.

First we give the properties of recurrent configurations and spanning trees of $T^2_{k,n}$. For convenience, we label the k children of the root r_1 of $T^2_{k,n}$ from left to right as $1, 2, \cdots, k$ (see Fig. 4(a)). Denote $\{1, 2, \cdots, k, r_2\} = U_1 \bigcup U_2$, where r_2 can be viewed as the $k+1$-th child of r_1 and U_1 denotes the set of the children of the root r_1 lying before it in the ordering $\prec_T$ of $V(T^2_{k,n})$ corresponding to the recurrent configuration x_T, U_2 behind r_1. Writing B_j for the subgraph of $T^2_{k,n}$ induced by a vertex j and all its descendants together with the sink q (see Fig. 4(b)).

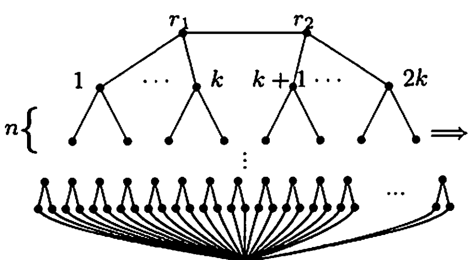
By the bijection between spanning trees and recurrent configurations defined in Section 2, for x_T, we have the property as follows:

(a) if $x_T(r_1) = i$, $i = 0, 1, \cdots, k$, then $|\ U_1\ | = k+1-i$.

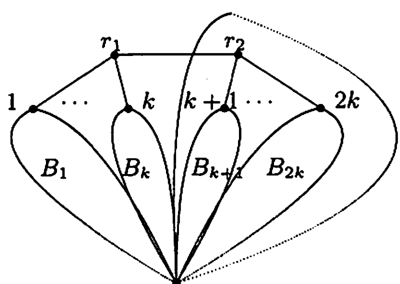
Moreover, if the unique path from sink q to the root r_1 in T is $q \sim \cdots \sim v \sim r_1$, then T has the following structural properties:

(b)$vr_1 \in E(T)$ and v is the element in U_1 with the largest label; if $j \in U_1$ and $j < v$, then $jr_1 \notin E(T)$ and T restricted B_j is a spanning tree of B_j;

(c)if $j \in U_2$ and $j < v$, then $jr_1 \in E(T)$ and T restricted B_j is a spanning forest of B_j which has two components with j and r_1 in different components; if $j \in U_2$ and $j > v$, then T restricted B_j is either a spanning tree or a spanning forest with two components according to $jr_1 \notin E(T)$ or not.



(a) To label the children of the root r_1



(b) Illustration of the definition B_j and the right of $T^2_{k,n} - r_1r_2$

Fig. 4

Now we can calculate the probability distribution of heights of root vertices in recurrent configurations.

Theorem 3.1. *For $T^2_{k,n}$, $n \geq 1$ and $k > 1$*

the height probabilities of the roots are

$$P(x(r_i) = 0) = \frac{1}{k2^{k-1} + 2^{k+1}},$$

$$P(x(r_i) = j) = \frac{(k+2)\sum_{a=0}^{j-1} C^a_{k-j+a} 2^{j-a-1} + kC^j_k}{k^2 2^{k-1} + k2^{k+1}},$$

where $i = 1, 2,\ j = 1, 2, \cdots, k$.

Proof. By analysis as above, we just need to calculate the number of spanning trees T of $T^2_{k,n}$ such that corresponding recurrent configuration x_T satisfies $x_T(r_1) = i$ $(i = 0, 1, \cdots, k)$. By property (a), for such any spanning tree T, we get that $|U_1| = k + 1 - i$. According to properties (b) and (c), T also satisfies the label of v on the unique path from q to r_1 adjacent to r_1 is $k+1-i+a$, $a = 0, 1, \cdots, i-1$ and r_2, that is, the largest label of U_1 is $k+1-i+a$, $a = 0, 1, \cdots, i-1$ and r_2. So there are two cases in the following.

(i)Suppose the largest label of U_1 be $k+1-i+a$, $a = 0, 1, \cdots, i-1$, by (b), for $j \in U_1$ and $j \neq k+1-i+a$, a spanning tree of B_j can be assigned independently and $jr_1 \notin E(T)$, there are $(t_{2,n-1})^{k+1-i}$ possibilities. For $j \in U_2$ and $j < k+1-i+a$, a spanning forest of B_j which has two components with j and q in different components can be assigned independently and $jr_1 \in E(T)$, there are $(s_{2,n-1})^a$ possibilities. For the rest $j \in U_2$, either a spanning tree of B_j can be assigned independently and $jr_1 \notin E(T)$ or a spanning forest of B_j which has two components with j and q in different components independently and $jr_1 \in E(T)$, there are $\sum_{l=0}^{i-1-a} C^l_{i-1-a}(t_{2,n-1})^{i-1-a-l}(s_{2,n-1})^l$ possibilities. If $r_1r_2 \notin E(T)$, then for the right of $T^2_{k,n} - r_1r_2$, we may assign independently a spanning tree of $T_{k,n}$, there are $t_{k,n}$ possibilities. If $r_1r_2 \in E(T)$, then the unique path from q to r_2 in T is $q \sim \cdots \sim k+1-i+a \sim r_1 \sim r_2$. For the right of $T^2_{k,n} - r_1r_2$ and $k+1 \leq j \leq 2k$, either a spanning tree of B_j or a spanning forest of B_j with two components according to $jr_2 \notin E(T)$ or not, there are $\sum_{l=0}^{k} C^l_k(t_{2,n-1})^{k-l}(s_{2,n-1})^l$ possibilities. Combining Lemma 2.1, we can get that the number of spanning trees satisfying above property is

$$(\sum_{a=0}^{i-1} C_{k-i+a}^{a} 2^{i-1-a})(t_{2,n-1})^k(t_{k,n} + (2t_{2,n-1})^k), \quad i = 1, 2, \cdots, k.$$

(ii)Suppose the largest label of U_1 be r_2, then $r_1r_2 \in E(T)$ and the unique path from q to r_2 in T is $q \sim v_j \sim r_2$. So we can construct spanning trees of $T_{k,n}^2$ corresponding to configuration x_T with $x_T(r_1) = i$, $i = 0, 1, \cdots, k$ in the following. For the left of $T_{k,n}^2 - r_1r_2$, for $j \in \{1, 2, \cdots, k\}$, B_j may be assigned a spanning tree or a spanning forest independently. For the right of $T_{k,n}^2 - r_1r_2$, we may assign independently a spanning tree of $T_{k,n}$. So the number of spanning trees satisfying above property is

$$C_k^i(t_{2,n-1})^{k-i}(s_{2,n-1})^i(t_{k,n}), \qquad i = 0, 1, \cdots, k.$$

By Lemma 2.2, we have the probability distributions

$$P(x(r_1) = 0) \;=\; \frac{C_k^0(t_{2,n-1})^k(s_{2,n-1})^0 t_{k,n}}{k2^{k2^{n+1}-2k}(k2^{-2}+1)} = \frac{1}{k2^{k-1} + 2^{k+1}},$$

$$\begin{aligned}
&P(x(r_1) = j) \\
&= \frac{(t_{2,n-1})^k((t_{k,n} + 2^k(t_{2,n-1})^k)\sum_{a=0}^{j-1} C_{k-j+a}^{a} 2^{j-a-1} + t_{k,n}C_k^j)}{k2^{k2^{n+1}-2k}(k2^{-2}+1)} \\
&= \frac{(k+2)\sum_{a=0}^{j-1} C_{k-j+a}^{a} 2^{j-a-1} + kC_k^j}{k^2 2^{k-1} + k2^{k+1}},
\end{aligned}$$

where $j = 1, 2, \cdots, k$.

Analyzing similarly as above, combining the symmetry of the structure of $T_{k,n}^2$, we also prove that the probability distributions of the root r_2

$$P(x(r_2) = 0) = \frac{1}{k2^{k-1} + 2^{k+1}},$$

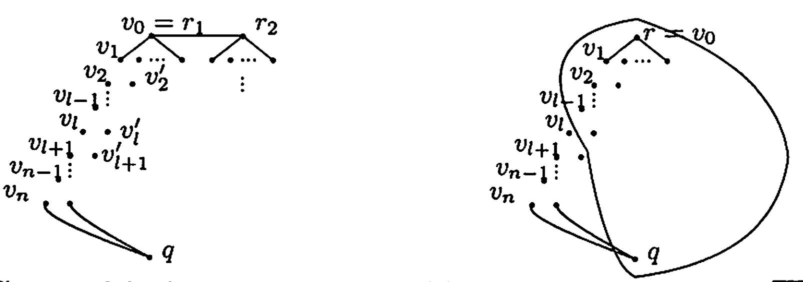
$$P(x(r_2) = j) = \frac{(k+2)\sum_{a=0}^{j-1} C_{k-j+a}^{a} 2^{j-a-1} + kC_k^j}{k^2 2^{k-1} + k2^{k+1}},$$

where $j = 1, 2, \cdots, k$.

So the proof of the theorem is complete.

■

In order to obtain the height probabilities of v_l, $l = 1, 2, \cdots, n$, in the next subsection, we first do some preparation work. For convenience, for $T^2_{k,n}$, we label the left-most descendants from up to down as $v_0 = r_1, v_1, v_2, \cdots, v_n$, the sibling of v_l as v'_l (see Fig.5(a)). Let $\overline{B_{v_l}}$ be the subgraph of $T_{k,n}$ by deleting the vertex v_l and all its descendants, and let $m_{k,l}$ be the number of spanning trees of $\overline{B_{v_l}}$, let $n_{k,l}$ be the number of spanning forest with two components of $\overline{B_{v_l}}$ in which v_l and the sink q belong to different components (see Fig.5(b)).



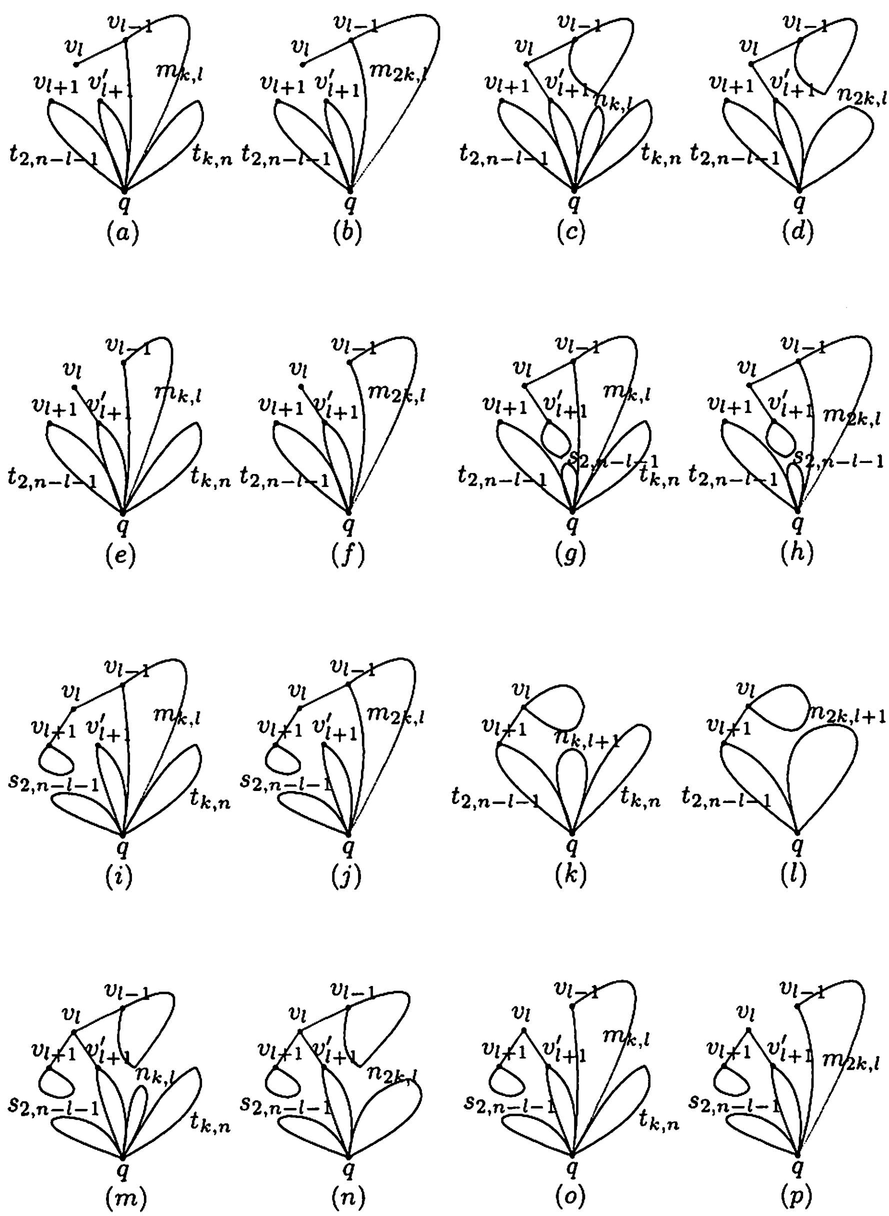
(a) The way of the descendant-labelling (b) Illustration of the definition $\overline{B_{v_l}}$

Fig. 5

Lemma 3.2. [2]. *With the notation as above, for $l = 1, 2, \cdots, n$, $k \geq 2$,*

(i) $m_{k,l} = 2^{k2^n - 2^{n-l+1} - 2(l-1) - k}[\frac{k}{3}(2^{2l-1} + 1) - 1]$,

(ii) $n_{k,l} = 2^{k2^n - 2^{n-l+1} - 2(l-1) - k}[\frac{k}{3}(2^{2l-1} - 2) + 2]$.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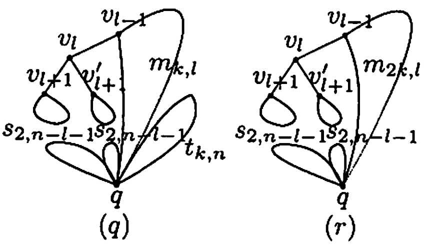

Fig. 6 Illustration of the proof of Theorem 3.3

Theorem 3.3. *For $T^2_{k,n}$, $n \geq 1$ and $k > 1$,*

the height probabilities of v_l, $l = 1, 2, \cdots, n-1$ are

$$P(x(v_l) = 0) = (2^{-2l-3}(\tfrac{k}{3}(2^{2l-1} + 1) - 1)k + 2^{-2(l+1)}(\tfrac{2k}{3}(2^{2l-1} + 1) - 1))(k^2 2^{-2} + k)^{-1};$$

$$P(x(v_l) = 1) = (2^{-2l-3}(\tfrac{k}{3}(2^{2l-1} - 2) + 2)k + 2^{-2(l+1)}(\tfrac{2k}{3}(2^{2l-1} - 2) + 2) + 2^{-2l-3}(k(2^{2l-1} + 1) - 3)k + 2^{-2(l+1)}(2k(2^{2l-1} + 1) - 3))(k^2 2^{-2} + k)^{-1};$$

$$P(x(v_l) = 2) = (2^{-2l-3}(\tfrac{k}{3}(2^{2l+1} - 2) + 2)k + 2^{-2(l+1)}(\tfrac{2k}{3}(2^{2l+1} - 2) + 2) + 2^{-2l-3}(\tfrac{k}{3}(2^{2l-1} - 2) + 2)k + 2^{-2(l+1)}(\tfrac{k}{3}(2^{2l-1} + 1) - 1)k + 2^{-2l-1}(\tfrac{2k}{3}(2^{2l-1} + 1) - 1) + 2^{-2l-2}(\tfrac{2k}{3}(2^{2l-1} - 2) + 2))(k^2 2^{-2} + k)^{-1};$$

the height probabilities of v_n are

$$P(x(v_n) = 0) = (2^{-2n-1}(\tfrac{k}{3}(2^{2n-1} + 1) - 1)k + 2^{-2n}(\tfrac{2k}{3}(2^{2n-1} + 1) - 1))(k^2 2^{-2} + k)^{-1};$$

$$P(x(v_n) = 1) = (2^{-2n-1}(\tfrac{k}{3}(2^{2n-1} + 1) - 1)k + 2^{-2n}(\tfrac{2k}{3}(2^{2n-1} + 1) - 1) + 2^{-2n-1}(\tfrac{k}{3}(2^{2n-1} - 2) + 2)k + 2^{-2n}(\tfrac{2k}{3}(2^{2n-1} - 2) + 2))(k^2 2^{-2} + k)^{-1}.$$

Proof. By analysis as Theorem 3.1, we just need to calculate the number of spanning trees T of $T^2_{k,n}$ such that corresponding recurrent configuration x_T satisfies $x_T(v) = i$ $(0 \le i < d(v))$. Note that to each spanning tree T of $T^2_{k,n}$ that contains r_1r_2, there corresponds a spanning tree $T \cdot r_1r_2$ of $T^2_{k,n} \cdot r_1r_2$. This correspondence is clearly a bijection. Therefore $\tau(T^2_{k,n} \cdot r_1r_2)$ is precisely the number of spanning trees of $T^2_{k,n}$ that contain r_1r_2.

For v_l, $1 \le l < n$, the three vertices adjacent to v_l are v_{l-1}, v_{l+1}, v'_{l+1}. For recurrent configurations with $x(v_l) = 0$, the corresponding spanning trees must contain the edge $v_{l-1}v_l$ but not v_lv_{l+1}, $v_lv'_{l+1}$. So there are two cases:

(i)If the corresponding spanning trees T of $T^2_{k,n}$ do not contain the edge r_1r_2, then the number of spanning trees satisfying the above condition is $(t_{2,n-l-1})^2 m_{k,l} t_{k,n}$ (see Figure 6(a)).

(ii)If the corresponding spanning trees T of $T^2_{k,n}$ contain the edge r_1r_2, then the number of spanning trees satisfying the above condition is $(t_{2,n-l-1})^2 m_{2k,l}$ (see Figure 6(b)). Thus the total number of corresponding spanning trees is $(t_{2,n-l-1})^2 m_{k,l} t_{k,n} + (t_{2,n-l-1})^2 m_{2k,l}$. Hence
$P(x(v_l) = 0)$

$$\begin{aligned}
&= \frac{(t_{2,n-l-1})^2 m_{k,l} t_{k,n} + (t_{2,n-l-1})^2 m_{2k,l}}{(t_{k,n})^2 + t_{2k,n}} \\
&= (2^{-2l-3}(\frac{k}{3}(2^{2l-1}+1)-1)k + 2^{-2(l+1)}(\frac{2k}{3}(2^{2l-1}+1) \\
&\quad -1))(k^2 2^{-2} + k)^{-1}.
\end{aligned}$$

For recurrent configurations with $x(v_l) = 1$, the corresponding spanning trees must satisfy that only two of v_{l-1}, v_{l+1}, v'_{l+1} lie before v_l in the vertex order $\prec_T$. So there are three cases:

(i)If $v_{l+1} \prec_T v'_{l+1} \prec_T v_l$, then the number of spanning trees satisfying the above condition is $(t_{2,n-l-1})^2(n_{k,l}t_{k,n} + n_{2k,l} + m_{k,l}t_{k,n} + m_{2k,l})$ (see

Figure 6(c,d,e,f)).

(ii)If $v_{l+1} \prec_T v_{l-1} \prec_T v_l$, then the number of spanning trees satisfying the above condition is $(t_{2,n-l-1})(s_{2,n-l-1})(m_{k,l}t_{k,n} + m_{2k,l})$ (see Figure 6(g,h)).

(iii)If $v'_{l+1} \prec_T v_{l-1} \prec_T v_l$, then the number of spanning trees satisfying the above condition is $(t_{2,n-l-1})(s_{2,n-l-1})(m_{k,l}t_{k,n} + m_{2k,l})$ (see Figure 6(i,j)).

Combining Lemma 2.1, we can get that the total number of corresponding spanning trees is $(t_{2,n-l-1})^2(n_{k,l}t_{k,n}+n_{2k,l}+m_{k,l}t_{k,n}+m_{2k,l}+2(m_{k,l}t_{k,n}+m_{2k,l}))$. Thus

$$
\begin{aligned}
&P(x(v_l) = 1)\\
=\ & \frac{(t_{2,n-l-1})^2(n_{k,l}t_{k,n} + n_{2k,l} + m_{k,l}t_{k,n} + m_{2k,l} + 2(m_{k,l}t_{k,n} + m_{2k,l}))}{(t_{k,n})^2 + t_{2k,n}}\\
=\ & (2^{-2l-3}(\frac{k}{3}(2^{2l-1} - 2) + 2)k + 2^{-2(l+1)}(\frac{2k}{3}(2^{2l-1} - 2) + 2)\\
& +2^{-2l-3}(k(2^{2l-1} + 1) - 3)k + 2^{-2(l+1)}(2k(2^{2l-1} + 1) - 3))(k^2 2^{-2}\\
& +k)^{-1}.
\end{aligned}
$$

For recurrent configurations with $x(v_l) = 2$, the corresponding spanning trees must satisfy that only one of v_{l-1}, v_{l+1}, v'_{l+1} lies before v_l in the vertex order $\prec_T$. So there are three cases:

(i)If $v_{l+1} \prec_T v_l$, then the number of spanning trees satisfying the above condition is

$t_{2,n-l-1}(n_{k,l+1}t_{k,n} + n_{2k,l+1})$ (see Figure 6(k,l)).

(ii)If $v'_{l+1} \prec_T v_l$, then the number of spanning trees satisfying the above condition is $t_{2,n-l-1}s_{2,n-l-1}(n_{k,l} + n_{2k,l} + m_{k,l} + m_{2k,l})$ (see Figure 6(m,n,o,p)).

(iii)If $v_{l-1} \prec_T v_l$, then the number of spanning trees satisfying the above condition is $(s_{2,n-l-1})^2(m_{k,l}+m_{2k,l})$ (see Figure 6(q,r)).

So the total number of corresponding spanning trees is

$t_{2,n-l-1}(n_{k,l+1}t_{k,n}+n_{2k,l+1})+(t_{2,n-l-1})^2((n_{k,l}+2m_{k,l})t_{k,n}+n_{2k,l}+2m_{2k,l})$.

Thus

$$\begin{aligned} &P(x(v_l)=2)\\ &= (2^{-2l-3}(\frac{k}{3}(2^{2l+1}-2)+2)k+2^{-2(l+1)}(\frac{2k}{3}(2^{2l+1}-2)+2)\\ &\quad +2^{-2l-3}(\frac{k}{3}(2^{2l-2}-2)+2)k+2^{-2(l+1)}(\frac{k}{3}(2^{2l-1}+1)-1)k\\ &\quad +2^{-2l-1}(\frac{2k}{3}(2^{2l-1}+1)-1)+2^{-2l-2}(\frac{2k}{3}(2^{2l-1}-2)+2))(k^2 2^{-2}\\ &\quad +k)^{-1}. \end{aligned}$$

Analyzing similarly as above, note that v_n has only two neighbors, we also prove that the height probabilities of v_n are

$$\begin{aligned} &P(x(v_n)=0)\\ &= \frac{m_{k,n}t_{k,n}+m_{2k,n}}{(t_{k,n})^2+t_{2k,n}}\\ &= (2^{-2n-1}(\frac{k}{3}(2^{2n-1}+1)-1)k+2^{-2n}(\frac{2k}{3}(2^{2n-1}+1)-1))(k^2 2^{-2}\\ &\quad +k)^{-1}. \end{aligned}$$

$$\begin{aligned} &P(x(v_n)=1)\\ &= \frac{m_{k,n}t_{k,n}+m_{2k,n}+n_{k,n}t_{k,n}+n_{2k,n}}{(t_{k,n})^2+t_{2k,n}}\\ &= (2^{-2n-1}(\frac{k}{3}(2^{2n-1}+1)-1)k+2^{-2n}(\frac{2k}{3}(2^{2n-1}+1)-1)\\ &\quad +2^{-2n-1}(\frac{k}{3}(2^{2n-1}-2)+2)k+2^{-2n}(\frac{2k}{3}(2^{2n-1}-2+2))(k^2 2^{-2}\\ &\quad +k)^{-1}. \end{aligned}$$

So the proof of the theorem is complete. ∎

By the symmetry of the structure of $T^2_{k,n}$, we get all single site height probability distributions.

4 The Two-site Joint Probabilities of the ASM on $T^2_{k,n}$

In this section we will give some two-site joint probabilities of the ASM on $T^2_{k,n}$. First we consider two adjacent root vertices r_1, r_2. Note that if in a recurrent configuration, there are two adjacent sites with height 0, then the recurrent configuration occurs with zero probability. For convenience, let $P_r(i,j)$ and $P_l(i,j)$ denote $P(x(r_1) = i, x(r_2) = j)$ and $P(x(v_l) = i, x(v_{l+1}) = j), l = 0, 1, 2, \cdots, n-1 (v_0 = r_1)$, respectively. So, for $i = j = 0$, $P_r(0,0) = P_l(0,0) = 0$.

Theorem 4.1. *For $T^2_{k,n}$, $n \geq 1$ and $k > 1$,*

$P_r(0,i) = P_r(i,0) = \frac{\sum_{a=0}^{i-1} C^a_{k-i+a} 2^{i-1-a}}{k^2 2^{2(k-1)} + k 2^{2k}}, \quad 1 \leq i \leq k;$

$P_r(i,k) = \frac{\sum_{a=0}^{i-1} C^a_{k-i+a} 2^{k+i-1-a} + C^i_k \sum_{a=0}^{k-1} 2^{k-a-1}}{k^2 2^{2(k-1)} + k 2^{2k}}, \quad 1 \leq i \leq k;$

$P_r(i,j) = \frac{\sum_{a=0}^{i-1} C^a_{k-i+a} 2^{i-1-a} \sum_{a=0}^{j} C^a_{k-j+a-1} 2^{j-a} + C^i_k \sum_{a=0}^{j-1} C^a_{k-j+a} 2^{j-a-1}}{k^2 2^{2(k-1)} + k 2^{2k}},$

$1 \leq i \leq k, 1 \leq j \leq k-1.$

Proof. Let T be the spanning tree of $T^2_{k,n}$ corresponding to the recurrent configuration x with $x(r_1) = i$ and $x(r_2) = j$. So, in the order $\prec_T$, $k+1-i$ children of the root r_1 must lie before itself, and $k+1-j$ children of the root r_2 must lie before itself. Note that $\{1, 2, \cdots, k, r_2\} = U_1 \bigcup U_2$, where U_1 denotes the set of the children of the root r_1 lying before it in the ordering $\prec_T$ of $V(T^2_{k,n})$, U_2 behind r_1 (see Fig. 4(a)). By property (a) in section 3, we get that $|U_1| = k+1-i$, $i = 0, 1, \cdots, k$. According to properties (b) and (c) in section 3, T also satisfies the label of v on the unique path from q to r_1 adjacent to r_1 is $k+1-i+a$, $a = 0, 1, \cdots, i-1$, and r_2, that is, the largest label of the element in U_1 is $k+1-i+a$, $a = 0, 1, \cdots, i-1$, and r_2. Then there are two different cases as follows:

(i)Suppose the largest label of U_1 be $k+1-i+a$, $a=0,1,\cdots,i-1$, by (b), for $l\in U_1$ and $l\neq k+1-i+a$, we may assign independently a spanning tree of B_l and $lr_1\notin E(T)$. For $l\in U_2$ and $l<k+1-i+a$, we may assign independently a spanning forest of B_l which has two components with l and q in different components and $lr_1\in E(T)$. For the rest $l\in U_2$, either a spanning tree of B_l or a spanning forest of B_l with two components according to $lr_1\notin E(T)$ or not. So there are $(\sum_{a=0}^{i-1}C_{k-i+a}^{a}(t_{2,n-1})^{k+1-i}(s_{2,n-1})^{a})(\sum_{l=0}^{i-1-a}C_{i-1-a}^{l}(t_{2,n-1})^{i-1-a-l}$ $(s_{2,n-1})^{l})$ possibilities, $i=1,2,\cdots,k$. If any spanning tree T of $T_{k,n}^2$ does not contain the edge r_1r_2, an analogous discussion for the root r_2, combining Lemma 2.1, then we can get the number of spanning tree corresponding to recurrent configurations with $x(r_1)=i$ and $x(r_2)=j$ is

$$(\sum_{a=0}^{i-1}C_{k-i+a}^{a}2^{i-1-a}(t_{2,n-1})^{k})(\sum_{a=0}^{j}C_{k-j+a-1}^{a}2^{j-a}(t_{2,n-1})^{k}),$$

$j=0,1,\cdots,k-1$. If any spanning tree T of $T_{k,n}^2$ contains the edge r_1r_2, then the unique path from q to r_2 in T is $q\sim\cdots\sim k+1-i+a\sim r_1\sim r_2$. For the right of $T_{k,n}^2-r_1r_2$ (see Fig. 4(b)), note that $x(r_2)=k$, we may assign independently either a spanning tree of B_j or a spanning forest of B_j with two components according to $jr_2\notin E(T)$ or not, $k+1\leq j\leq 2k$. So the number of spanning trees corresponding to recurrent configurations with $x(r_1)=i$ and $x(r_2)=k$ is

$$(\sum_{a=0}^{i-1}C_{k-i+a}^{a}2^{i-1-a}(t_{2,n-1})^{k})(\sum_{l=0}^{k}C_{k}^{l}(t_{2,n-1})^{k-l}(s_{2,n-1})^{l})$$
$$=(\sum_{a=0}^{i-1}C_{k-i+a}^{a}2^{i-1-a}(t_{2,n-1})^{k})(2t_{2,n-1})^{k}.$$

(ii)Suppose the largest label of U_1 be r_2, then $r_1r_2\in E(T)$ and the unique path from q to r_2 in T is $q\sim\cdots\sim v\sim r_2\sim r_1$, where v is the child of the root r_2. So we can construct spanning trees of $T^2_{k,n}$ corresponding to configuration x_T with $x(r_1) = i$ and $x(r_2) = j$ as follows. For the left of $T^2_{k,n} - r_1 r_2$, for $l \in \{1, 2, \cdots, k\}$, B_l may be assigned a spanning tree or a spanning forest with two components independently according to $lr_1 \notin E(T)$ or not. For the right of $T^2_{k,n} - r_1 r_2$, we may assign independently a spanning tree of $T_{k,n}$. So the number of spanning trees satisfying above property is

$$C^i_k (t_{2,n-1})^{k-i} (s_{2,n-1})^i (\sum_{a=0}^{j-1} C^a_{k-j+a} 2^{j-1-a}) (t_{2,n-1})^k,$$
$$i = 0, 1, \cdots, k, \quad j = 1, \cdots, k.$$

By the above analysis, we have

$P_r(0,i) = P_r(i,0) = \frac{\sum_{a=0}^{i-1} C^a_{k-i+a} 2^{i-1-a}}{k^2 2^{2(k-1)} + k 2^{2k}}, \quad 1 \le i \le k;$

$P_r(i,k) = \frac{\sum_{a=0}^{i-1} C^a_{k-i+a} 2^{k+i-1-a} + C^i_k \sum_{a=0}^{k-1} 2^{k-a-1}}{k^2 2^{2(k-1)} + k 2^{2k}}, \quad 1 \le i \le k;$

$P_r(i,j) = \frac{\sum_{a=0}^{i-1} C^a_{k-i+a} 2^{i-1-a} \sum_{a=0}^{j} C^a_{k-j+a-1} 2^{j-a} + C^i_k \sum_{a=0}^{j-1} C^a_{k-j+a} 2^{j-a-1}}{k^2 2^{2(k-1)} + k 2^{2k}},$

$1 \le i \le k, 1 \le j \le k-1.$

So the proof of the theorem is complete. ∎

For simplicity, we use $t_l(i,j)$ to denote the number of spanning trees corresponding to the recurrent configurations with $x(v_l) = i$ and $x(v_{l+1}) = j$, $l = 0, 1, 2, \cdots, n-1 (v_0 = r_1)$. By an argument similar to the above, we have the following results (Tables 1-6):

Table 1: $i = 0, 1, 2, \cdots, k-1$; $j = 0$.

$t_0(i+1,j)$	j=0
i=0	$(t_{2,n-2})^2 C^i_{k-1} (t_{2,n-1})^{k-i-1} (s_{2,n-1})^i t_{k,n}$
$1 \le i \le k-1$	$(t_{2,n-2})^2 (t_{2,n-1})^{k-1} ((t_{k,n} + (2t_{2,n-1})^k) \sum_{a=1}^{i} C^{a-1}_{k-i+a-2} 2^{i-a} + C^i_{k-1} t_{k,n})$

Then we can calculate all adjacent two-site joint probabilities. Along the same line, it is natural to consider any two-site joint probabilities. But the results are cumbersome.

Table 2: $i = -1, 0, 1, \cdots, k-1;\ j = 1$.

$t_0(i+1, j)$	j=1
i=-1	$(t_{2,n-2})^2 (t_{2,n-1})^{k-1} C_{k-1}^{i+1} t_{k,n}$
i=0	$(t_{2,n-2})^2 (t_{2,n-1})^{k-1} ((k+2) t_{k,n} + (2t_{2,n-1})^k)$
$1 \leq i \leq k-2$	$(t_{2,n-2})^2 (t_{2,n-1})^{k-1} ((t_{k,n} + (2t_{2,n-1})^k)$ $(\sum_{a=0}^{i} C_{k-i+a-2}^{a} 2^{i-a} + 2\sum_{a=1}^{i} C_{k-i+a-2}^{a-1} 2^{i-a}) + (C_{k-1}^{i+1} + 2C_{k-1}^{i}) t_{k,n})$
i=k-1	$(t_{2,n-2})^2 (t_{2,n-1})^{k-1} ((t_{k,n} + (2t_{2,n-1})^k)(2^{k-1} + 2\sum_{a=1}^{i} C_{k-i+a-2}^{a-1} 2^{i-a})$ $+ 2t_{k,n})$

Table 3: $i = -1, 0, 1, \cdots, k-1;\ j = 2$.

$t_0(i+1, j)$	j=2
i=-1	$3(t_{2,n-2})^2 (t_{2,n-1})^{k-1} C_{k-1}^{i+1} t_{k,n}$
i=0	$(t_{2,n-2})^2 (t_{2,n-1})^{k-1} (3\sum_{a=0}^{i} C_{k-i+a-2}^{a} 2^{i-a} (t_{k,n} + (2t_{2,n-1})^k)$ $+ (3C_{k-1}^{i+1} + C_{k-1}^{i}) t_{k,n})$
$1 \leq i \leq k-2$	$(t_{2,n-2})^2 (t_{2,n-1})^{k-1} ((t_{k,n} + (2t_{2,n-1})^k)(\sum_{a=0}^{i} C_{k-i+a-2}^{a} 2^{i-a}$ $+ 3\sum_{a=1}^{i} C_{k-i+a-2}^{a-1} 2^{i-a}) + (3C_{k-1}^{i-1} + C_{k-1}^{i}) t_{k,n})$
i=k-1	$(t_{2,n-2})^2 (t_{2,n-1})^{k-1} ((t_{k,n} + (2t_{2,n-1})^k)(3 \cdot 2^{k-1}$ $+ \sum_{a=1}^{i} C_{k-i+a-2}^{a-1} 2^{i-a}) + C_{k-1}^{i} t_{k,n})$

Table 4: $l = 1, \cdots, n-2;\ j = 0, 1$.

$t_l(i, j)$	j=0	j=1
i=0	0	$\frac{(t_{2,n-l-1})^2}{4} (m_{k,l} t_{k,n} + m_{2k,l})$
i=1	$\frac{(t_{2,n-l-1})^2}{4} (m_{k,l} t_{k,n}$ $+ m_{2k,l})$	$\frac{(t_{2,n-l-1})^2}{4} ((4m_{k,l} + n_{k,l}) t_{k,n}$ $+ 4m_{2k,l} + n_{2k,l})$
i=2	$\frac{(t_{2,n-l-1})^2}{4} ((2m_{k,l} + n_{k,l}) t_{k,n}$ $+ 2m_{2k,l} + n_{2k,l})$	$\frac{(t_{2,n-l-1})^2}{4} ((6m_{k,l} + 4n_{k,l}) t_{k,n}$ $+ 6m_{2k,l} + 4n_{2k,l})$

Table 5: $l = 1, \cdots, n-2;\ j = 2$.

$t_l(i, j)$	j=2
i=0	$\frac{(t_{2,n-l-1})^2}{4} (3m_{k,l} t_{k,n} + 3m_{2k,l})$
i=1	$\frac{(t_{2,n-l-1})^2}{4} ((7m_{k,l} + 3n_{k,l}) t_{k,n} + 7m_{2k,l} + 3n_{2k,l})$
i=2	$\frac{(t_{2,n-l-1})^2}{4} ((8m_{k,l} + 7n_{k,l}) t_{k,n} + 8m_{2k,l} + 7n_{2k,l})$

Table 6: $l = n-1$.

$t_l(i, j)$	j=0	j=1
i=0	0	$m_{k,n-1} t_{k,n} + m_{2k,n-1}$
i=1	$m_{k,n-1} t_{k,n}$ $+ m_{2k,n-1}$	$(2m_{k,n-1} + n_{k,n-1}) t_{k,n}$ $+ 2m_{2k,n-1} + n_{2k,n-1}$
i=2	$(2m_{k,n-1} + n_{k,n-1}) t_{k,n}$ $+ 2m_{2k,n-1} + n_{2k,n-1}$	$(2m_{k,n-1} + 2n_{k,l}) t_{k,n}$ $+ 2m_{2k,n-1} + 2n_{2k,n-1}$

From the above results, we can see that the height probabilities of the ASM on the finite Bethe lattice $T_{k,n}^2$ depend on whether the site is near the boundary or not, but not on the size of the system. Hence we can get the corresponding probabilities of the ASM on infinite Bethe lattice $T_{2,n}^2 (n \to \infty)$.

5 The Height Probabilities of the ASM on the Finite Bethe Lattice

For the remainder of this paper, we shall now use the above theorems in sections 3 and 4 to derive the height probabilities of the ASM on the finite Bethe lattice $T_{2,n}^2$.

Theorem 5.1. *For the one-site probabilities of the finite Bethe lattice* $T_{2,n}^2$, $n \geq 1$, $j = 1, 2$, $l = 1, 2, \cdots, n-1$,

$$P(x(r_j) = 0) = \frac{1}{12}, P(x(r_j) = 1) = \frac{4}{12}, P(x(r_j) = 2) = \frac{7}{12};$$

$$P(x(v_l) = 0) = \frac{1}{12}, P(x(v_l) = 1) = \frac{4}{12}, P(x(v_l) = 2) = \frac{7}{12};$$

$$P(x(v_n) = 0) = \frac{1}{3}, P(x(v_n) = 1) = \frac{2}{3}.$$

Theorem 5.2. *For the two-site joint probabilities of the finite Bethe lattice* $T_{2,n}^2$,

$$P_r(0,0) = P_l(0,0) = 0, \quad P_r(0,1) = P_r(1,0) = P_l(0,1) = P_l(1,0) = \frac{1}{48},$$

$$P_r(1,1) = P_l(1,1) = \frac{5}{48}, \quad P_r(0,2) = P_r(2,0) = P_l(0,2) = P_l(2,0) = \frac{3}{48},$$

$$P_r(2,2) = P_l(2,2) = \frac{15}{48}, \quad P_r(1,2) = P_r(2,1) = P_l(1,2) = P_l(2,1) = \frac{10}{48},$$

$$l = 0, 1, \cdots, n-2;$$

$$P_{n-1}(0,0) = 0, \quad P_{n-1}(1,0) = \frac{1}{12}, \quad P_{n-1}(2,0) = \frac{3}{12},$$
$$P_{n-1}(0,1) = \frac{1}{12}, \quad P_{n-1}(1,1) = \frac{3}{12}, \quad P_{n-1}(2,1) = \frac{4}{12}.$$

The result of Theorem 5.1 is the same as those results found by Grassberger and Manna in [1] and proved by Haiyan chen and Fuji Zhang in [2].

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