A COMPUTATIONAL CRITERION FOR THE SUPERSOLVABILITY OF LINE ARRANGEMENTS

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ABSTRACT. In this note we find a necessary and sufficient condition for the supersolvability of an essential, central arrangement of rank 3 (i.e., line arrangement in the projective plane). We present an algorithmic way to decide if such an arrangement is supersolvable or not that does not require an ordering of the lines as the Björner-Ziegler's and Peeva's criteria require. The method uses the duality between points and lines in the projective plane in the context of coding theory.

1. INTRODUCTION

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central essential hyperplane arrangement in $\mathbb{K}^3$, where $\mathbb{K}$ is a field of characteristic 0. This means that $\{(0,0,0)\} = \bigcap H_i$ and therefore the H_i's are lines in $\mathbb{P}^2$ not all passing through the same point and each H_i is the kernel of a linear form $l_i \in \mathbb{K}[x, y, z]$.

Denote with $Sing(\mathcal{A}) = \{P_1, \ldots, P_s\} \subset \mathbb{P}^2$ the set of the intersection points of the lines H_i in $\mathcal{A}$. These points are the rank 2 elements in the lattice of intersection $L_{\mathcal{A}}$ of $\mathcal{A}$ and the Möbius function value at each P_i is $\mu(P_i) = m_i - 1$, where $m_i = m_{P_i}$ is the number of lines from $\mathcal{A}$ passing through P_i (see [10] for background and more details). Denote with $M = \max\{m_P | P \in Sing(\mathcal{A})\}$ and let $maxSing(\mathcal{A}) = \{P \in Sing(\mathcal{A}) | m_P = M\}$.

By [10] a hyperplane arrangement is called *supersolvable* if $L_{\mathcal{A}}$ has a maximal chain of modular elements. For the case of our interest (line arrangements), this translates into the existence of a point (that we will call *modular point*) $P \in Sing(\mathcal{A})$ that is connected to all the other points in $Sing(\mathcal{A})$ by lines in $\mathcal{A}$.

Supersolvable arrangements are a special class of free arrangements ([8]). A hyperplane arrangement is called free if a certain derivations module is free. By Saito's Criterion ([13]), if f is the defining polynomial of the arrangement $\mathcal{A} \subset \mathbb{K}^{n+1}$ (i.e., f is the product of the equations of the hyperplanes in $\mathcal{A}$), then $\mathcal{A}$ is free if and only if the Jacobian ideal of f, J_f, is Cohen-Macaulay. In this situation *the exponents of* $\mathcal{A}$ are $exp(\mathcal{A}) = \{1, a_1, \ldots, a_{n-1}\}$, where $a_1, \ldots, a_{n-1}$ are the degrees of basis for the syzygies module of J_f.

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Example 1.1. Let $\mathcal{A}_1$ be the line arrangement in $\mathbb{P}^2$ with defining polynomial $f_1 = xyz(x-y)(x-z)(y-z)(x+y-z)$ (the non-Fano arrangement).

Let $\mathcal{A}_2$ be the line arrangement in $\mathbb{P}^2$ with defining polynomial $f_2 = xyz(x-z)(x+z)(y-z)(y+z)$.

Both arrangements are free with exponents $\{1,3,3\}$, but $\mathcal{A}_1$ is not supersolvable and $\mathcal{A}_2$ is supersolvable; it is clear that the free resolution of the Jacobian ideal of the defining polynomial does not give such information.

Supersolvable arrangements are an important class of hyperplane arrangements and they have been studied quite a lot from different points of view (combinatorial, algebraic or topological): [16], [17], [8], [15], [9], [5], just to cite here a few. Also testing the supersolvability of hyperplane arrangements has been done before by Björner and Ziegler ([2]): a hyperplane arrangement is supersolvable if and only if the Orlik-Solomon algebra has a quadratic minimal broken circuit basis. From this criterion, Peeva in [12] gives a more computational test, using Gröbner bases: a hyperplane arrangement is supersolvable if and only if the Orlik-Solomon ideal has a quadratic Gröbner basis. In both cases a good ordering of the hyperplanes is necessary. For the case of line arrangements this means that once a modular point is found, order the lines passing through the modular point to be the smallest in the lexicographic order than the other lines (see [12] and [11]).

Our note is about an alternative criterion to test the supersolvability of line arrangements, that does not require an ordering of the lines. First we show that points of maximum multiplicity for a supersolvable line arrangements are modular points (Lemma 2.1), and once we have this, from the definition of supersolvability we get the theoretical result (Theorem 2.2). The theory of hyperplane arrangements interacts very well with coding theory, and we are going to speculate this as well in Section 3 for the effective computations and algorithm: the Macaulay 2 ([6]) code to implement this algorithm is available at `http://homepages.uc.edu/~tohanesn/`, where we also compare our criterion to Peeva's.

2. The theoretical result

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a line arrangement in $\mathbb{P}^2$. Let $f = l_1 \cdots l_n \in \mathbb{K}[x,y,z]$ be its defining polynomial. For a point $P \in Sing(\mathcal{A})$, let f_P be the product of the linear forms of the lines in $\mathcal{A}$ passing through P. So $\deg(f_P) = m_P$.

Lemma 2.1. *If $\mathcal{A}$ is supersolvable, then any point in $maxSing(\mathcal{A})$ is a modular point.*

Proof. If $\mathcal{A}$ is supersolvable, then there exist a modular point P. Let m_P is the number of lines in $\mathcal{A}$ passing through P.

Let $Q \in maxSing(\mathcal{A})$ with $m_Q = M$ (so $M \geq m_P$). If Q is not a modular point, then there exists a point $R \in Sing(\mathcal{A})$ not connected to Q.

Since P is modular point, then P is connected to Q by a line $H_Q \in \mathcal{A}$ and is connected to R by a line $H_R \in \mathcal{A}$. Since $R \in Sing(\mathcal{A})$, there should exist a second line $H \in \mathcal{A}$ passing through R (but not through P nor Q).

In these conditions the line H intersects the lines passing through Q in M points $Q_1, \ldots, Q_M$. Since P is modular point it should connect with M lines from $\mathcal{A}$ with all of these points. Now counting, we get that through P should pass these M lines (here we include H_Q) plus the one line H_R. So $m_P \geq M + 1$. Contradiction. □

Let $I_{\mathcal{A}} = \langle l_2 \cdots l_n, l_1 l_3 \cdots l_n, \ldots, l_1 \cdots l_{n-1} \rangle \subset \mathbb{K}[x, y, z]$. This ideal has been studied in [14], in connection to the blow up of $\mathbb{P}^2$ at $Sing(\mathcal{A})$ and the Orlik-Terao algebra.

Theorem 2.2. *Let $\mathcal{A}$ be an essential line arrangement in $\mathbb{P}^2$. Let $P \in maxSing(\mathcal{A})$ with $m_P = M$. Then $\mathcal{A}$ is supersolvable if and only if $f_P^{M-1} \in I_{\mathcal{A}}$.*

Proof. First of all, by [14], the primary decomposition of $I_{\mathcal{A}}$ is

$$I_{\mathcal{A}} = I_1^{m_1 - 1} \cap \cdots \cap I_s^{m_s - 1},$$

where each I_i is the ideal of the point $P_i \in Sing(\mathcal{A})$.

From the definition, $\mathcal{A}$ is supersolvable if there exists $Q \in Sing(\mathcal{A})$ such that all the other intersection points of $\mathcal{A}$ lie on the lines passing through Q (i.e., Q is a modular point). This means that $f_Q \in I_1 \cap \cdots \cap I_s = \sqrt{I_{\mathcal{A}}}$.

Let $P \in maxSing(\mathcal{A})$ and assume $I_1 = I_P$ is the ideal of P, so

$$I_{\mathcal{A}} = I_P^{M-1} \cap J,$$

where $J = I_2^{m_2 - 1} \cap \cdots \cap I_s^{m_s - 1}$ and all $m_i \leq M$.

If $\mathcal{A}$ is supersolvable, then from Lemma 2.1, P is a modular point and therefore

$$f_P \in I_P \cap I_2 \cap \cdots \cap I_s.$$

Of course $f_P \in I_P$ and so $f_P^{M-1} \in I_P$. Since $M - 1 \geq m_i - 1$, then we also have $f_P^{M-1} \in J$ and so

$$f_P^{M-1} \in I_{\mathcal{A}}.$$

For the other direction, if P is an intersection point with $f_P^{M-1} \in I_{\mathcal{A}}$, then $f_P \in \sqrt{I_{\mathcal{A}}}$ and hence P is a modular point for $\mathcal{A}$. Note that for this implication P need not to be in $maxSing(\mathcal{A})$. □

This method works only for line arrangements in $\mathbb{P}^2$: the decomposition of $I_{\mathcal{A}}$ in the proof above is known to work only for this situation (see [1], [7] and [14] for analysis of ideals of fat points and blowups of $\mathbb{P}^2$ at the singularities of line arrangements). Also, working in $\mathbb{P}^2$ gives a nice pictorial geometric intuition for supersolvable arrangements that can lead to results similar to Lemma 2.1.

3. Brief intro to coding theory

From computational point of view, the main issue in Theorem 2.2 is to find a point $P \in maxSing(\mathcal{A})$. In theory, to answer this question, one should compute the associated primes of $I_{\mathcal{A}}$ to get the ideals $I_1, \ldots, I_s$, and therefore the set $Sing(\mathcal{A})$. Then localize $I_{\mathcal{A}}$ at all these ideals and select those I_j with $\deg(I_{\mathcal{A}})_{I_j}$ maximal. These selected ideals will be the ideals of the points in $maxSing(\mathcal{A})$. The main difficulty of this approach is that s can be very large and we have to do lots of localizations and degree of ideal computations. We can avoid this by doing the following approach derived from basic coding theory.

Let C be a linear code of length n and dimension k ($k \leq n$), given by the matrix A of rank k

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix},$$

$a_{ij} \in \mathbb{K}$, where $\mathbb{K}$ is a field.

A codeword $w \in C$ is an element in the vector subspace generated by the rows of A.

The *minimal (minimum) distance (or Hamming distance)* of C is, by definition,

$$d = \min_{w \in C \setminus \{0\}} |w|,$$

where $|w|$ = the number of nonzero entries in w = the *weight* of w. By projective codewords of weight d we will understand the equivalence class, under multiplication by nonzero scalars, of codewords of weight d.

The linear code C has minimal distance d if and only if $n - d$ is the maximum number of columns in A that span a $k - 1$ dimensional vector space ([19]). In other words, if we think of the columns of A as the coordinates of n points in $\mathbb{P}^{k-1}$, $n - d$ is the maximum number of these points lying in a hyperplane. The coefficients of the equation of such a hyperplane give the scalars in the linear combination of the rows of A to obtain a projective codeword of weight exactly d.

Another way to find the minimal distance is due to de Boer and Pellikaan ([3]): In $\mathbb{P}^{k-1}$ consider the hyperplanes given by the vanishing of the linear forms (with coefficients the entries of each column of the matrix A):

$$\begin{aligned} L_1 &= x_1 a_{11} + \cdots + x_k a_{k1} \\ L_2 &= x_1 a_{12} + \cdots + x_k a_{k2} \\ &\vdots \\ L_n &= x_1 a_{1n} + \cdots + x_k a_{kn} \end{aligned}$$

and in $R = \mathbb{K}[x_1, \ldots, x_k]$, consider J_j to be the ideal generated by all the j products of linear forms $L_1, \ldots, L_n$. Then

$$d = \min\{j | Z(J_{j+1}) \neq \emptyset \text{ as algebraic set in } \mathbb{P}^{k-1}\}.$$

Note that d is the maximum value of j such that $codim(J_j) = k$; therefore $codim(J_{j+1}) \leq k - 1$.

Let p be a minimal prime containing J_j. Then any product $L_{m_1} \cdots L_{m_j}$ is an element in p. This means that at least one of the linear forms $L_{m_1}, \ldots, L_{m_j}$ is an element in p. Inductively we get that

$$J_j \subseteq \langle L_{i_1}, \ldots, L_{i_{n-j+1}} \rangle \subseteq p,$$

for some $1 \leq i_1 < \cdots < i_{n-j+1} \leq n$ and hence $p = \langle L_{i_1}, \ldots, L_{i_{n-j+1}} \rangle$ (note that p need not to be minimally generated by all of the $n - j + 1$ linear forms above).

Suppose $codim(J_{d+1}) = k - a, a \geq 2$. Then there exists a minimal prime $p = \langle L_{d+1}, \ldots, L_n \rangle$ with $\dim_{\mathbb{K}} Span\{L_{d+1}, \ldots, L_n\} = k - a$. But

$$J_d \subset \langle L_1 \cdots L_d, L_{d+1}, \ldots, L_n \rangle \subset \langle L_1, L_{d+1}, \ldots, L_n \rangle.$$

Therefore $\dim_{\mathbb{K}} Span\{L_1, L_{d+1}, \ldots, L_n\} = k$, which contradicts the assumption: if we add a vector to the spanning set of a vector subspace, we increase the dimension with at most 1. Hence $codim(J_{d+1}) = k - 1$. So the primary decomposition of J_{d+1} is:

$$J_{d+1} = r_1 \cap \cdots \cap r_m \cap J,$$

where r_i are q_i−primary ideals of codimension $k - 1$ and J is a $\langle x_1, \ldots, x_k \rangle$−primary ideal.

The associated primes of J_{d+1} are $\{q_1, \ldots, q_m, \langle x_1, \ldots, x_k \rangle\}$. To obtain a projective codeword of weight exactly d, we only have to pick any of the q_i above (so a minimal prime): $V(q_i) \subset \mathbb{P}^{k-1}$ is the point whose coordinates are the scalars in the linear combination of the rows of A that gives the corresponding projective codeword; this is true from the simple observation that a vector has no more than d non-zero entries if the products of any of the $d + 1$ entries in the vector vanish.

4. THE COMPUTATIONAL APPROACH

Fix the equations of the lines in $\mathcal{A}$:

$$l_i = a_i x + b_i y + c_i z, i = 1, \ldots, n.$$

To find a point in $maxSing(\mathcal{A})$, first we need to find M.

Suppose we know that $\mathcal{A}$ is free with exponents $\{1, u, v\}$. If $\mathcal{A}$ is not free, then it will not be supersolvable either. If $\mathcal{A}$ is supersolvable, then by Lemma 2.1, if $P \in maxSing(\mathcal{A})$ with $m_P = M$, then P is a modular point. So deleting lines from $\mathcal{A}$, not passing through P, from Terao's Addition-Deletion Theorem ([18]), we must have $exp(\mathcal{A}) = \{1, M - 1, n - M\}$. So u or v must be equal to $M - 1$.

Consider $S \subset \mathbb{P}^2$ the set of points $(a_i : b_i : c_i)$. It is clear that M is exactly the maximum number of collinear points in S. If the line where these points lie has equation $ax + by + cz$, then $P = (a : b : c) \in maxSing(\mathcal{A})$.

Create the matrix with entries in $\mathbb{K}$:

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{bmatrix}.$$

From Section 3, the linear code with generating matrix A has minimal distance $d = n - M$.

Consider the ideals J_k generated by all the k products, $k = 1, \dots, n$, of the linear forms l_i. For example, $I_{\mathcal{A}} = J_{n-1}$. Also from Section 3, the minimal distance is the number d such that $codim(J_d) = 3$ and $codim(J_{d+1}) = 2$.

To find M, one should do the following computations: create the ideals $J_{n-(u+1)}, J_{n-u}$ and check if $codim(J_{n-(u+1)}) = 3$ and $codim(J_{n-u}) = 2$. If not, do the same computations for v. If we get a negative answer as well, then $\mathcal{A}$ is not supersolvable. Otherwise, $M = u + 1$ or $M = v + 1$. Suppose $M = u + 1$.

To obtain a $P \in maxSing(\mathcal{A})$, from Section 3, just pick any of the associated primes of the saturation of J_{n-u}; so it should be a q_i and hence $P = V(q_i)$. In most of the cases the cardinality of $maxSing(\mathcal{A})$ is much smaller than the cardinality of $Sing(\mathcal{A})$.

Example 4.1. We are going to check this algorithm for the Example 1.1 at the beginning of these notes.

For both arrangements we have $u = v = 3$ and $n = 7$. For $\mathcal{A}_1$ we get $codim(J_3) = codim(J_4) = 3$, so $d = 7 - M \geq 4$ which means that $M - 1 \leq 2$ and therefore $\mathcal{A}_1$ is not supersolvable.

For $\mathcal{A}_2$ we get $codim(J_3) = 3$ and $codim(J_4) = 2$. So $7 - M = 3$ and hence $M = 4$. If we saturate J_4 we get the ideal

$$\langle xy, z \rangle = \langle x, z \rangle \cap \langle y, z \rangle.$$

So $P = (0 : 1 : 0) \in maxSing(\mathcal{A})$. We have $f_P = xz(x - z)(x + z)$. Since $f_P^3 \in I_{\mathcal{A}_2}$, from Theorem 2.2, we get that $\mathcal{A}_2$ is supersolvable.

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