On the endomorphism monoid of $K(n,4)$ *

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Abstract

In this paper, the endomorphism monoid of circulant complete graph $K(n,4)$ is explored explicitly. It is shown that $Aut(K(n,4)) \cong D_n$, the dihedral group of degree n. It is also shown that $K(n,4)$ is unretractive when $n = 4m+1, 4m+3$ for some $m \geq 2$, $End(K(n,4)) = qEnd(K(n,4))$, $sEnd(K(n,4)) = Aut(K(n,4))$ when $n = 4m, 4m+2$ for some $m \geq 2$, $End(K(4m,4))$ is regular and $End(K(4m+2,4))$ is completely regular. Some enumerative problems concerning $End(K(n,4))$ are solved.

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1 Introduction and preliminaries

Endomorphism monoids of graphs are generalizations of automorphism groups of graphs. In recent years much attention has been paid to endomorphism monoids of graphs and many interesting results concerning graphs and their endomorphism monoids have been obtained. The aim of this research is to establish the relationship between graph theory and algebra theory of semigroups and to apply the theory of semigroups to graph theory. Hou, Luo and Cheng [6] explored the endomorphism monoid of $\overline{P_n}$, the complement of a path P_n with n vertices. It was shown that $End(\overline{P_n})$ is an orthodox monoid. The endomorphism spectrum and the

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endomorphism type of $\overline{P_n}$ were given. The endomorphism monoids and the endomorphism-regularity of graphs were considered by several authors (see [1], [3], [7], [8] and [10]). If n and d are positive integers with $n \geq 2d$, then the circulant complete graph $K(n,d)$ is the graph with vertex set $V = \{1, 2, \cdots, n\}$ in which i is adjacent to j if and only if $d \leq |i-j| \leq n-d$. It is easy to see that $K(n,2)$ is $\overline{C_n}$ and its endomorphism monoid was studied in [10]. In [8], the endomorphism monoids of $K(n,3)$ was characterized. In this paper, the endomorphism monoid of circulant complete graph $K(n,4)$ is explored explicitly. We show that $End(K(4m,4))$ is regular, $End(K(4m+2,4))$ is completely regular, $End(K(n,4)) = qEnd(K(n,4))$ and $sEnd(K(n,4)) = Aut(K(n,4))$ when $n = 4m, 4m+2$ for some $m \geq 2$. We also show that $K(4m+1,4)$ and $K(4m+3,4)$ are unretractive.

The graphs considered in this paper are finite undirected graphs without loops and multiple edges. Let X be a graph. The vertex set of X is denoted by $V(X)$ and the edge set of X is denoted by $E(X)$. If two vertices x_1 and x_2 are adjacent in graph X, then the edge connecting x_1 and x_2 is denoted by $\{x_1, x_2\}$ and we write $\{x_1, x_2\} \in E(X)$ (or briefly $\{x_1, x_2\} \in E$). A subgraph H is called an *induced subgraph* of X if for any $a, b \in V(H)$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(X)$. A *clique* of a graph X is the maximal complete subgraph of X. The *clique number* of X, denoted by $\omega(X)$, is the maximal order among the cliques of X. Let $S \subseteq V(X)$. We denote by $< S >$ the subgraph of X induced by S.

Let X and Y be graphs. A mapping f from $V(X)$ to $V(Y)$ is called a *homomorphism* (from X to Y) if $\{x_1, x_2\} \in E(X)$ implies $\{f(x_1), f(x_2)\} \in E(Y)$. A homomorphism f is called *half-strong* if $\{f(a), f(b)\} \in E(Y)$ implies that there exist $x_1, x_2 \in V(X)$ with $f(x_1) = f(a)$ and $f(x_2) = f(b)$ such that $\{x_1, x_2\} \in E(X)$. A homomorphism f is called *locally-strong* if $\{f(a), f(b)\} \in E(Y)$ implies that for every preimage $x_1 \in V(X)$ of $f(a)$ there exists a preimage $x_2 \in V(X)$ of $f(b)$ such that $\{x_1, x_2\} \in E(X)$ and analogously for every preimage of $f(b)$. A homomorphism f is called *quasi-strong* if $\{f(a), f(b)\} \in E(Y)$ implies that there exists a preimage $x_1 \in V(X)$ of $f(a)$ which is adjacent to every preimage of $f(b)$ and analogously for preimage of $f(b)$. A homomorphism f is called *strong* if $\{f(a), f(b)\} \in E(Y)$ implies that any preimage of $f(a)$ is adjacent to any preimage of $f(b)$. A homomorphism f is called an *isomorphism* if f is bijective and f^{-1} is a homomorphism. A homomorphism (resp. isomorphism) f from X to itself is called an *endomorphism* (resp. *automorphism*) of X (see [2] and its references). The sets of all endomorphisms, half-strong endomorphisms, locally-strong endomorphisms, quasi-strong endomorphisms, strong endomorphisms and automorphisms of the graph X are denoted by $End(X)$, $hEnd(X)$, $lEnd(X)$, $qEnd(X)$, $sEnd(X)$ and $Aut(X)$, respectively. A graph X is called *unretractive* if $End(X) = Aut(X)$. Clearly, for any graph X, we always have

$$Aut(X) \subseteq sEnd(X) \subseteq qEnd(X) \subseteq lEnd(X) \subseteq hEnd(X) \subseteq End(X).$$

Let f be a endomorphism of graph X. A subgraph of X is called the *endomorphic image* of X under f, denoted by I_f, if $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(X)$. By ρ_f we denote the equivalence relation on $V(X)$ induced by f, i.e., for $a, b \in V(X)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class containing $a \in V(X)$ with respect to ρ_f. The partition π of $V(X)$ corresponding to ρ_f is called the *kernel* of f. By X/ρ_f we denote the *factor graph* of X under ρ_f, that is a graph with $V(X/\rho_f) = V(X)/\rho_f$ and $\{[a]_{\rho_f}, [b]_{\rho_f}\} \in E(X/\rho_f)$ if and only if there exist $c \in [a]_{\rho_f}$ and $d \in [b]_{\rho_f}$ such that $\{c, d\} \in E(X)$. Define $i_f : V(X/\rho_f) \longrightarrow V(I_f)$ with $i_f([x]_{\rho_f}) = f(x)$ for $x \in V(X)$. Obviously, i_f is well defined.

An element a of semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. A semigroup S is called regular if all its elements are regular. An element e in S is called *idempotent* if $e^2 = e$. We denote by $Idpt(X)$ the set of all idempotent endomorphisms of graph X. A graph X is said to be End-regular if its endomorphism monoid $End(X)$ is regular.

Let S be a semigroup. Green's relation $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{D}$ on S are defined by

$$\begin{aligned}(a,b) \in \mathcal{L} &\Leftrightarrow S^1a = S^1b,\\ (a,b) \in \mathcal{R} &\Leftrightarrow aS^1 = bS^1,\\ (a,b) \in \mathcal{J} &\Leftrightarrow S^1aS^1 = S^1bS^1\end{aligned}$$

and $\mathcal{H} = \mathcal{L} \wedge \mathcal{R}$, $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$.

The $\mathcal{L}$-class ($\mathcal{R}$-class, $\mathcal{J}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class) containing the element a will be denoted by L_a (R_a, J_a, H_a, D_a). It is known that $\mathcal{D} = \mathcal{J}$ for any finite semigroup S. Define a partial order " $\leq$ " among the $\mathcal{J}$-classes of S by: $J_a \leq J_b$ if $S^1aS^1 \subseteq S^1bS^1$. Clearly, $J_{xay} \leq J_a$ for all $a \in S$ and for all $x, y \in S^1$.

For undefined concepts in this paper we refer to [5,9]. The following results quoted from the references will be used later.

Lemma 1.1 ([4]) Let X be a graph, f and g be two regular elements of $End(X)$. Then
(1) $f\mathcal{L}g$ if and only if $\rho_f = \rho_g$;
(2) $f\mathcal{R}g$ if and only if $I_f = I_g$;
(3) $f\mathcal{D}g$ if and only if $I_f \cong I_g$.

Lemma 1.2 ([8]) $Aut(K(n,3)) \cong D_n$, the dihedral group of degree n.

Lemma 1.3 ([10]) Let X be a graph and let $f \in End(X)$. Then

(1) $f \in hEnd(X)$ if and only if I_f is an induced subgraph of X;
(2) If f is regular, then $f \in hEnd(X)$.

Lemma 1.4 ([11]) Let X be a graph and let $f \in End(X)$. Then f is regular if and only if there exist $g, h \in Idpt(X)$ such that $\rho_g = \rho_f$ and $I_h = I_f$.

Lemma 1.5 ([12]) Let X be a graph. Then the mapping i_f is an isomorphism from X/ρ_f to I_f.

2 Main results

In this section, we will characterize the endomorphism monoid of circulant complete graph $K(n, 4)$. Recall that $\{i, j\} \in E(K(n, 4))$ if and only if $4 \leq |i - j| \leq n - 4$ for any $i, j \in \{1, 2, \cdots, n\}$. By K_n^* we denote a graph obtained by deleting an edge from K_n, that is $K_n^* = K_n - e$, where e is any edge of K_n. For convenience, we view i as $i(mod\ n)$ for any positive integer i and view n and 1 as consecutive integers.

Lemma 2.1 Let $f \in End(K(n, 4))$. Then
(1) If $f(x_1) = f(x_2)$ for some $x_1, x_2 \in V(K(n, 4))$, then $|x_1 - x_2| \leq 3$ or $n - 3 \leq |x_1 - x_2| \leq n - 1$.
(2) There are no distinct $x_1, x_2, x_3, x_4, x_5 \in V(K(n, 4))$ such that $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5)$.

Proof (1) Note that $\{x_1, x_2\} \in E(K(n, 4))$ if and only if $4 \leq |x_1 - x_2| \leq n - 4$. If $f(x_1) = f(x_2)$, then $\{x_1, x_2\} \notin E(K(n, 4))$ and so $|x_1 - x_2| \leq 3$ or $n - 3 \leq |x_1 - x_2| \leq n - 1$.
(2) It follows immediately from (1).

To study the endomorphism monoid of $K(n, 4)$, we divide it into four cases, that is, $n = 4m$, $n = 4m + 1$, $n = 4m + 2$ and $n = 4m + 3$, where $m \geq 2$. We first consider the case of $n = 4m$.

Lemma 2.2 (1) $\omega(K(4m, 4)) = m$.
(2) There are exactly four subgraphs in $K(4m, 4)$ isomorphic to K_m, say, $< 1, 5, \cdots, 4m - 3 >$, $< 2, 6, \cdots, 4m - 2 >$, $< 3, 7, \cdots, 4m - 1 >$ and $< 4, 8, \cdots, 4m >$.
(3) $K(4m, 4)$ does not contain a subgraph isomorphic to K_{m+1}^*.

Proof (1) and (2) are obvious.
(3) Suppose that G is a subgraph of $K(4m, 4)$ which is isomorphic to K_{m+1}^*. Without loss of generality, we may assume that $G = K_m \cup \{i\}$. Then i is adjacent to exactly $m - 1$ vertices in K_m. By the definition of $K(4m, 4)$ and (2), there is no such vertex, hence (3) holds.

Lemma 2.3 Let $f \in End(K(4m,4))$. If $f(x_1) = f(x_2) = f(x_3) = f(x_4)$ for some distinct $x_1, x_2, x_3, x_4 \in V(K(4m,4))$, then x_1, x_2, x_3 and x_4 are four consecutive numbers in $V(K(4m,4))$. In this case, $\rho_f = \{[i, i+1, i+2, i+3], [i+4, i+5, i+6, i+7], \cdots, [i+4m-4, i+4m-3, i+4m-2, i+4m-1]\}$ for some $i \in \{1,2,3,4\}$ and $I_f \cong K_m$.

Proof Suppose $f(x_1) = f(x_2) = f(x_3) = f(x_4)$ for some $x_1, x_2, x_3, x_4 \in V(K(4m,4))$. Then $\{x_i, x_j\} \notin E$ for any $i, j = 1,2,3,4$. Hence x_1, x_2, x_3, x_4 are four consecutive numbers. Without loss of generality, we suppose that $x_1 = i$, $x_2 = i+1$, $x_3 = i+2$ and $x_4 = i+3$ for some $i \in V(K(4m,4))$.

We claim that $f(i+4) = f(i+5) = f(i+6) = f(i+7)$. Otherwise, there exists $t \in \{4,5,6\}$ such that $f(i+t) \neq f(i+7)$. Let $A = \{i+3, i+7, \cdots, i+4m-1\}$. Then the subgraph of $K(4m,4)$ induced by A is isomorphic to K_m. Since $i+t$ is adjacent to every vertex of $A \setminus \{i+3, i+7\}$, $f(i+t)$ is adjacent to every vertex of $f(A) \setminus \{f(i+3), f(i+7)\}$. Note that $\{f(i+t), f(i+3)\} = \{f(i+t), f(i)\} \in E$. Now $f(i+t) \neq f(i+7)$ implies that $f(i+t) \notin f(A)$. Hence the subgraph of $K(4m,4)$ induced by $\{f(i+t)\} \cup f(A)$ is isomorphic to K^*_{m+1}. This contradicts Lemma 2.2 (3). A similar argument will show that $f(i+4k) = f(i+4k+1) = f(i+4k+2) = f(i+4k+3)$ for $k = 2,3,\cdots,m-1$. Therefore $\rho_f = \{[i, i+1, i+2, i+3], [i+4, i+5, i+6, i+7], \cdots, [i+4m-4, i+4m-3, i+4m-2, i+4m-1]\}$. Clearly, $I_f \cong K(4m,4)/\rho_f \cong K_m$.

Lemma 2.4 Let $f \in End(K(4m,4))$. If $|[a]_{\rho_f}| \leq 3$ for any $a \in V(K(4m,4))$ and $f(x_1) = f(x_2) = f(x_3)$ for some $x_1, x_2, x_3 \in V(K(4m,4))$, then x_1, x_2 and x_3 are three consecutive numbers in $V(K(4m,4))$.

Proof Suppose $f(x_1) = f(x_2) = f(x_3)$ for some distinct $x_1, x_2, x_3 \in V(K(4m,4))$. By Lemma 2.1, $|x_i - x_j| \leq 3$ or $n-3 \leq |x_i - x_j| \leq n-1$ for any $i, j = 1,2,3$. Without loss of generality, we may suppose $x_1 < x_2 < x_3$ and $x_1 = i$. If x_1, x_2 and x_3 are not three consecutive numbers in $V(K(4m,4))$, then $x_3 = i+3$. Since $|[a]_{\rho_f}| \leq 3$ for any $a \in V(K(4m,4))$, then there exist $t \in \{1,2\}$ such that $f(i+t) \neq f(i)$. Let $A = \{i+3, i+4+t, i+8+t \cdots, i+4m-8, i+4m-4+t\}$. It is easy to see that the subgraph of $K(4m,4)$ induced by $A \backslash \{i+3\}$ is isomorphic to K_{m-1}. Since $i+3$ is adjacent to every vertex of $A \setminus \{i+3, i+4+t\}$, $f(i+3)$ is adjacent to every vertex of $f(A) \backslash \{f(i+3), f(i+4+t)\}$. Now $\{f(i+3), f(i+4+t)\} = \{f(i), f(i+4+t)\} \in E$ implies that the subgraph of $K(4m,4)$ induced by $f(A)$ is isomorphic to K_m. Since $i+t$ is adjacent to every vertex of $A \setminus \{i+3\}$, $f(i+t)$ is adjacent to every vertex of $f(A) \setminus \{f(i+3)\}$. Now $f(i+t) \neq f(i)$ implies that the subgraph of $K(4m,4)$ induced by $f(A) \cup \{f(i+t)\}$ is isomorphic to K^*_{m+1}. A contradiction.

Lemma 2.5 Let $f \in End(K(4m,4))$. If $|[a]_{\rho_f}| \leq 3$ for any $a \in$

$V(K(4m,4))$ and $f(i) = f(i+1) = f(i+2)$ for some $i \in V(K(4m,4))$, then $\rho_f = \{[i,i+1,i+2],[i+3],[i+4,i+5,i+6],[i+7],\cdots,[i+4m-4,i+4m-3,i+4m-2],[i+4m-1]\}$ for some $i \in \{1,2,3,4\}$ and $I_f \cong \overline{C_{2m}}$.

Proof Suppose $f(i) = f(i+1) = f(i+2)$ for some $i \in V(K(4m,4))$. Then $f(i+4) = f(i+5) = f(i+6)$. Otherwise, there exist $t \in \{5,6\}$ such that $f(i+t) \neq f(i+4)$. Let $A = \{i+2, i+10, i+14, \cdots, i+4m-2\}$. Then the subgraph of $K(4m,4)$ induced by A is isomorphic to K_{m-1}. Since $i+4$ is adjacent to every vertex of $A\setminus\{i+2\}$, $f(i+4)$ is adjacent to every vertex of $f(A)\setminus\{f(i+2)\}$. Now $\{f(i+4), f(i+2)\} = \{f(i+4), f(i)\} \in E$ implies that the subgraph of $K(4m,4)$ induced by $f(A)\cup\{f(i+2)\}$ is isomorphic to K_m. Since $i+t$ is adjacent to every vertex of $A\setminus\{i+2\}$, $f(i+t)$ is adjacent to every vertex of $f(A)\setminus\{f(i+2)\}$. Now $\{f(i+t), f(i+2)\} = \{f(i+t), f(i)\} \in E$ implies that the subgraph of $K(4m,4)$ induced by $f(A) \cup \{f(i+t)\}$ is isomorphic to K_m. Now $f(i+4) \neq f(i+t)$ implies that the subgraph of $K(4m,4)$ induced by $f(A) \cup \{f(i+4), f(i+t)\}$ is isomorphic to K^*_{m+1}. A contradiction.

Since $|[a]_{\rho_f}| \leq 3$ for any $a \in V(K(4m,4))$, $[i+3]_{\rho_f} = \{i+3\}$. A similar argument will show that $[i+7]_{\rho_f} = \{i+7\}$, $f(i+8) = f(i+9) = f(i+10)$, $\cdots$, $f(i+4m-4) = f(i+4m-3) = f(i+4m-2)$, $[i+4m-1]_{\rho_f} = \{i+4m-1\}$. Therefore $\rho_f = \{[i,i+1,i+2],[i+3],[i+4,i+5,i+6],[i+7],\cdots,[i+4m-4,i+4m-3,i+4m-2],[i+4m-1]\}$. Clearly, $I_f \cong K(4m,4)/\rho_f \cong \overline{C_{2m}}$.

Lemma 2.6 Let $f \in End(K(4m,4))$. If $|[a]_{\rho_f}| \leq 2$ for any $a \in V(K(4m,4))$ and $f(x_1) = f(x_2)$ for some distinct $x_1, x_2 \in V(K(4m,4))$, then $|x_1 - x_2| = 1$ or $|x_1 - x_2| = n-1$.

Proof Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in V(K(4m,4))$. Then $|x_1 - x_2| \leq 3$ or $n-3 \leq |x_1 - x_2| \leq n-1$. Without loss of generality, we can suppose $x_1 < x_2$ and $x_1 = i$. If $|x_1 - x_2| \neq 1$ and $|x_1 - x_2| \neq n-1$, then $x_2 = i+2$ or $x_2 = i+3$. Let $A = \{x_2, i+5, i+9 \cdots, i+4m-7, i+4m-3\}$. It is easy to see that the subgraph of $K(4m,4)$ induced by $A\setminus\{x_2\}$ is isomorphic to K_{m-1}. Since x_2 is adjacent to every vertex of $A \setminus \{x_2, i+5\}$, $f(x_2)$ is adjacent to every vertex of $f(A)\setminus\{f(x_2), f(i+5)\}$. Now $\{f(x_2), f(i+5)\} = \{f(i), f(i+5)\} \in E$ implies that the subgraph of $K(4m,4)$ induced by $f(A)$ is isomorphic to K_m. Since $i+1$ is adjacent to every vertex of $A \setminus \{x_2\}$, $f(i+1)$ is adjacent to every vertex of $f(A) \setminus \{f(x_2)\}$. Since $|[a]_{\rho_f}| \leq 2$ for any $a \in V(K(4m,4))$, $f(i+1) \neq f(x_2)$, and so $f(i+1) \notin f(A)$. Thus the subgraph of $K(4m,4)$ induced by $f(A)\cup\{f(i+1)\}$ is isomorphic to K^*_{m+1}. A contradiction. Therefore $|x_1 - x_2| = 1$ or $|x_1 - x_2| = n-1$.

Lemma 2.7 Let $f \in End(K(4m,4))$. If $|[a]_{\rho_f}| \leq 2$ for any $a \in V(K(4m,4))$ and $f(i) = f(i+1)$ for some $i \in V(K(4m,4))$, then

(1) $\rho_f = \{[i,i+1],[i+2,i+3],[i+4,i+5],[i+6,i+7],\cdots,[i+4m-$

$4, i+4m-3], [i+4m-2, i+4m-1]\}$ and $I_f \cong \overline{C_{2m}}$, or

(2) $\rho_f = \{[i, i+1], [i+2], [i+3], [i+4, i+5], [i+6], [i+7], \cdots, [i+4m-4, i+4m-3], [i+4m-2], [i+4m-1]\}$ and $I_f \cong K(3m, 3)$.

Proof Suppose $f(i) = f(i+1)$ for some $i \in V(K(4m, 4))$. Then $f(i+4) = f(i+5)$. Otherwise, Let $A = \{i+1, i+9, i+13, \cdots, i+4m-7, i+4m-3\}$. It is easy to see that the subgraph of $K(4m, 4)$ induced by A is isomorphic to K_{m-1}. Since $i+4$ is adjacent to every vertex of $A \setminus \{i+1\}$, $f(i+4)$ is adjacent to every vertex of $f(A) \setminus \{f(i+1)\}$. Now $\{f(i+4), f(i+1)\} = \{f(i+4), f(i)\} \in E$ implies that $f(A) \cup \{f(i+4)\}$ is isomorphic to K_m. Since $i+5$ is adjacent to every vertex of A, $f(i+5)$ is adjacent to every vertex of $f(A)$. Now $f(i+4) \neq f(i+5)$ implies that the subgraph of $K(4m, 4)$ induced by $f(A) \cup \{f(i+4), f(i+5)\}$ is isomorphic to K^*_{m+1}. A contradiction. A similar argument will show that $f(i+4k) = f(i+4k+1)$ for $k = 2, \cdots, m-1$.

Since $|[a]_{\rho_f}| \leq 2$ for any $a \in V(K(4m, 4))$, by Lemma 2.6, we have $f(i+2) = f(i+3)$ or $[i+2]_{\rho_f} = \{i+2\}$. In the former case, a similar argument of the first paragraph will show that $f(i+4k+2) = f(i+4k+3)$ for any $k \in \{0, 1, \cdots, m-1\}$. Therefore $\rho_f = \{[i, i+1], [i+2, i+3], [i+4, i+5], [i+6, i+7], \cdots, [i+4m-4, i+4m-3], [i+4m-2, i+4m-1]\}$. Clearly, $I_f \cong K(4m, 4)/\rho_f \cong \overline{C_{2m}}$. In the later case, we have $[i+3]_{\rho_f} = \{i+3\}$. If $f(i+4t+2) = f(i+4t+3)$ for some $k \in \{1, 2, \cdots, m-1\}$. A similar argument of the first paragraph will show that $f(i+4k+2) = f(i+4k+3)$ for any $k = 0, 1, \cdots, m-1$. A contradiction. Thus $f(i+4k+2) \neq f(i+4k+3)$ for any $k \in \{0, 1, \cdots, m-1\}$. Therefore $\rho_f = \{[i, i+1], [i+2], [i+3], [i+4, i+5], [i+6], [i+7], \cdots, [i+4m-4, i+4m-3], [i+4m-2], [i+4m-1]\}$. Clearly, $I_f \cong K(4m, 4)/\rho_f \cong K(3m, 3)$.

Theorem 2.8 Let $f \in End(K(4m, 4))$. Then

(1) $f \in Aut(K(4m, 4))$, or

(2) $\rho_f = \{[i, i+1, i+2, i+3], [i+4, i+5, i+6, i+7], \cdots, [i+4m-4, i+4m-3, i+4m-2, i+4m-1]\}$ for some $i \in \{1, 2, 3, 4\}$ and $I_f \cong K_m$, or

(3) $\rho_f = \{[i, i+1, i+2], [i+3], [i+4, i+5, i+6], [i+7], \cdots, [i+4m-4, i+4m-3, i+4m-2], [i+4m-1]\}$ for some $i \in \{1, 2, 3, 4\}$ and $I_f \cong \overline{C_{2m}}$, or

(4) $\rho_f = \{[i, i+1], [i+2, i+3], [i+4, i+5], [i+6, i+7], \cdots, [i+4m-4, i+4m-3], [i+4m-2, i+4m-1]\}$ for some $i \in \{1, 2\}$ and $I_f \cong \overline{C_{2m}}$, or

(5) $\rho_f = \{[i, i+1], [i+2], [i+3], [i+4, i+5], [i+6], [i+7], \cdots, [i+4m-4, i+4m-3], [i+4m-2], [i+4m-1]\}$ for some $i \in \{1, 2, 3, 4\}$ and $I_f \cong K(3m, 3)$.

Proof It follows directly from Lemmas 2.3, 2.5 and 2.7.

Theorem 2.9 $End(K(4m, 4))$ $(m \geq 2)$ is regular.

Proof To prove that $End(K(4m,4))$ is regular, let $f \in End(K(4m,4))$. We only need to show that there exist two idempotent endomorphisms g and h such that $\rho_g = \rho_f$ and $I_h = I_f$.

Define a mapping g from $V(K(4m,4))$ to itself by $g(x) = i$, where i is the least number in $[x]_{\rho_f}$. Then it is easy to check $g \in End(K(4m,4))$, $\rho_f = \rho_g$ and $g^2 = g$.

If $I_f \cong K_m$, then we can define a mapping h from $V(K(4m,4))$ to itself by

$$h(x) = \begin{cases} x, & \text{if } x \in V(I_f), \\ x-1, & \text{if } x \notin V(I_f) \textit{ and } x-1 \in V(I_f), \\ x-2, & \text{if } x-i \notin V(I_f)(i=0,1) \textit{ and } x-2 \in V(I_f), \\ x-3, & \text{if } x-i \notin V(I_f)(i=0,1,2) \textit{ and } x-3 \in V(I_f). \end{cases}$$

Then $h \in End(K(4m,4))$ and $I_h = I_f$ and $h^2 = h$.

If $I_f \cong \overline{C_{2m}}$, then we can define a mapping h from $V(K(4m,4))$ to itself by

$$h(x) = \begin{cases} x, & \text{if } x \in V(I_f), \\ x+1, & \text{if } x \notin V(I_f) \textit{ and } x+1 \in V(I_f), \\ x+2, & \text{if } x+i \notin V(I_f)(i=0,1) \textit{ and } x+2 \in V(I_f). \end{cases}$$

Then $h \in End(K(4m,4))$, $I_h = I_f$ and $h^2 = h$.

If $I_f \cong K(3m,3)$, then we can define a mapping h from $V(K(4m,4))$ to itself by

$$h(x) = \begin{cases} x, & \text{if } x \in V(I_f), \\ x+1, & \text{if } x \notin V(I_f). \end{cases}$$

Then $h \in End(K(4m,4))$, $I_h = I_f$ and $h^2 = h$. We complete the proof.

Theorem 2.10 *$End(K(4m,4)) = qEnd(K(4m,4))$, $sEnd(K(4m,4)) = Aut(K(4m,4))$.*

Proof Let $f \in End(K(4m,4))$ and $a, b \in V(I_f)$. In the following, we divide it into four cases to discuss:

(1) Assume $\rho_f = \{[i,i+1,i+2,i+3],[i+4,i+5,i+6,i+7],\cdots,[i+4m-4,i+4m-3,i+4m-2,i+4m-1]\}$ for some $i \in \{1,2,3,4\}$. Then $f^{-1}(a) = \{s,s+1,s+2,s+3\}$ and $f^{-1}(b) = \{t,t+1,t+2,t+3\}$ for some $s,t \in V(K(4m,4))$. Clearly, there exists $x \in f^{-1}(a)$ such that x is adjacent to every vertex of $f^{-1}(b)$ and there exists $y \in f^{-1}(b)$ such that y is adjacent to every vertex of $f^{-1}(a)$. Hence $f \in qEnd(K(4m,4))$.

(2) Assume $\rho_f = \{[i,i+1,i+2],[i+3],[i+4,i+5,i+6],[i+7],\cdots,[i+4m-4,i+4m-3,i+4m-2],[i+4m-1]\}$ for some $i \in \{1,2,3,4\}$. If $[x]_{\rho_f} = \{j,j+1,j+2\}$, $[y]_{\rho_f} = \{j+3\}$ or $[y]_{\rho_f} = \{j-1\}$ for some $j \in V(K(4m,4))$, then $\{m,n\} \notin E$ for any $m \in [x]_{\rho_f}$ and $n \in [y]_{\rho_f}$.

Since f is regular, $f \in hEnd(K(4m,4))$. Hence $\{f(x), f(y)\} \notin E$. Let $a, b \in V(I_f)$ and $\{a,b\} \in E(K(4m,4)$. If $f^{-1}(a) = \{s, s+1, s+2\}$ and $f^{-1}(b) = \{t, t+1, t+2\}$ for some $s, t \in V(K(4m,4))$, then there exists $x \in f^{-1}(a)$ such that x is adjacent to every vertex of $f^{-1}(b)$ and there exists $y \in f^{-1}(b)$ such that y is adjacent to every vertex of $f^{-1}(a)$; If $f^{-1}(a) = \{s\}$ and $f^{-1}(b) = \{t\}$ for some $s, t \in V(K(4m,4))$, then $\{s,t\} \in E$; If $f^{-1}(a) = \{s, s+1, s+2\}$ and $f^{-1}(b) = \{t\}$ for some $s, t \in V(K(4m,4))$, by discuss above, $t \neq s+3$ and $t \neq s-1$. So $\{s,t\} \in E$, $\{s+1, t\} \in E$ and $\{s+2, t\} \in E$. Hence $f \in qEnd(K(4m,4))$.

(3) Assume $\rho_f = \{[i, i+1], [i+2, i+3], [i+4, i+5] \cdots, [i+4m-2, i+4m-1]\}$ for some $i \in \{1,2\}$. If $[x]_{\rho_f} = \{t, t+1\}$ and $[y]_{\rho_f} = \{t+2, t+3\}$ for some $t \in V(K(4m,4))$, then $\{m,n\} \notin E$ for any $m \in [x]_{\rho_f}$ and $n \in [y]_{\rho_f}$. Since f is regular, $f \in hEnd(K(4m,4))$. Hence $\{f(x), f(y)\} \notin E$. Let $a, b \in V(I_f)$ and $\{a,b\} \in E(K(4m,4)$. Then $f^{-1}(a) = \{s, s+1\}$ and $f^{-1}(b) = \{t, t+1\}$ for some $s, t \in V(K(4m,4))$. By discuss above, $t \neq s+2$ and $s \neq t+2$. So there exists $x \in f^{-1}(a)$ such that x is adjacent to every vertex of $f^{-1}(b)$ and there exists $y \in f^{-1}(b)$ such that y is adjacent to every vertex of $f^{-1}(a)$. Hence $f \in qEnd(K(4m,4))$.

(4) Assume $\rho_f = \{[i, i+1], [i+2], [i+3], [i+4, i+5], [i+6], [i+7], \cdots, [i+4m-4, i+4m-3], [i+4m-2], [i+4m-1]\}$ for some $i \in \{1,2,3,4\}$. If $[x]_{\rho_f} = \{j, j+1\}$ and $[y]_{\rho_f} = \{j+t\}$ $(t = -2, -1, 2, 3)$ for some $j \in V(K(4m,4))$, then $\{m,n\} \notin E$ for any $m \in [x]_{\rho_f}$ and $n \in [y]_{\rho_f}$. Since f is regular, $f \in hEnd(K(4m,4))$. Hence $\{f(x), f(y)\} \notin E$. Let $a, b \in V(I_f)$ and $\{a,b\} \in E(K(4m,4)$. If $f^{-1}(a) = \{s, s+1\}$ and $f^{-1}(b) = \{t, t+1\}$ for some $s, t \in V(K(4m,4))$, then there exists $x \in f^{-1}(a)$ such that x is adjacent to every vertex of $f^{-1}(b)$ and there exists $y \in f^{-1}(b)$ such that y is adjacent to every vertex of $f^{-1}(a)$; If $f^{-1}(a) = \{s\}$ and $f^{-1}(b) = \{t\}$ for some $s, t \in V(K(4m,4))$, then $\{s,t\} \in E$ since f is half-strong; If $f^{-1}(a) = \{s, s+1\}$ and $f^{-1}(b) = \{t\}$ for some $s, t \in V(K(4m,4))$, by discuss above, $t \neq s-2, s-1, s+2, s+3$. So $\{s,t\} \in E$ and $\{s+1, t\} \in E$. Hence $f \in qEnd(K(4m,4))$.

Let $f \in sEnd(K(4m,4))$. If $f(x_1) = f(x_2)$ for some $x_1, x_2 \in V(K(4m,4))$, then $N(x_1) = N(x_2)$. Note that there are no two vertices in $K(4m,4)$ having the same adjacent set. Hence $f(x_1) \neq f(x_2)$ for any $x_1, x_2 \in V(K(4m,4))$ and so $f \in Aut(K(4m,4))$.

The next theorem characterizes the automorphism group of $K(n,4)$. Define

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 3 & 4 & \cdots & n-1 & n & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 1 & n & n-1 & \cdots & 4 & 3 & 2 \end{pmatrix}.$$

Then $A, B \in Aut(K(4m,4))$, $A^n = (1)$, $B^2 = (1)$, $BA = A^{-1}B$ and $A^k \neq (1)$ for any $0 < k < n$ (where (1) is the identity mapping of V).

Theorem 2.11 $Aut(K(n,4)) \cong D_n$, the dihedral group of degree n.

Proof Let $f \in Aut(K(n,4))$ with $f(1) = j$ for some $j \in V(K(n,4))$. Since $n-2, n-1, n, 2, 3, 4$ are the only six vertices in $V(K(n,4))$ that are not adjacent to 1, $f(n-2), f(n-1), f(n), f(2), f(3), f(4)$ are the only six vertices in $V(K(n,4))$ that are not adjacent to j. Thus $f(n-2), f(n-1), f(n), f(2), f(3), f(4) \in \{j-3, j-2, j-1, j+1, j+2, j+3\}$. Note that $\{n-2, t\} \in E$ for $t = 2,3,4$ and $\{4, n-k\} \in E$ for $k = 0,1,2$. Then $f(n-2)$ is adjacent to $f(t)$ for $t = 2,3,4$ and $f(4)$ is adjacent to $f(n-k)$ for $t = 0,1,2$. Since only $j-3$ and $j+3$ in $\{j-3, j-2, j-1, j+1, j+2, j+3\}$ are adjacent to three vertices in $\{j-3, j-2, j-1, j+1, j+2, j+3\}$, $f(n-2), f(4) \in \{j-3, j+3\}$. If $f(n-2) = j-3$, then $f(4) = j+3$. It follows that $f(n-1), f(n), f(2), f(3) \in \{j-2, j-1, j+1, j+2\}$. Note that $\{n-1, 3\} \in E$. Then $\{f(n-1), f(3)\} \in E$. Since only $j-2$ and $j+2$ in $\{j-2, j-1, j+1, j+2\}$ are adjacent, $f(n-1), f(3) \in \{j-2, j+2\}$. It is easy to see that $\{n-1, 4\} \in E$. Then $\{f(n-1), f(4)\} = \{f(n-1), j+3\} \in E$. So $f(n-1) = j-2$ and $f(3) = j+2$. Now it follows that $f(n), f(2) \in \{j-1, j+1\}$. Note that $\{n, 4\} \in E$. Then $\{f(n), f(4)\} = \{f(n), j+3\} \in E$. So $f(n) = j-1$ and $f(2) = j+1$. Since $f(2) = j+1$, $f(\{n-1, n, 1, 3, 4, 5\}) = \{j-2, j-1, j, j+2, j+3, j+4\}$. Hence $f(5) = j+4$. A similar argument will show that $f(t) = j+t-1$ for any $t \in \{1, 2, \cdots, n\}$. Therefore $f = A^j$. If $f(n-2) = j+3$, then $f(4) = j-3$. Similarly, we can show that $f(n-1) = j+2$, $f(n) = j+1$, $f(2) = j-1$ and $f(3) = j-2$. Since $f(2) = j-1$, $f(\{n-1, n, 1, 3, 4, 5\}) = \{j-4, j-3, j-2, j, j+1, j+2\}$. Hence $f(5) = j-4$. A similar argument will show that $f(t) = j-t+1$ for any $t \in \{1, 2, \cdots, n\}$. Therefore $f = BA^j$. Consequently, $Aut(K(n,4)) \cong D_n$.

Theorem 2.12 $\mid End(K(4m,4)) \mid = 16m! + 248m$.

Proof Let $f \in End(K(4m,4)) \setminus Aut(K(4m,4))$. If $\rho_f = \{[i, i+1, i+2, i+3], [i+4, i+5, i+6, i+7], \cdots, [i+4m-4, i+4m-3, i+4m-2, i+4m-1]\}$, then $I_f \cong K_m$. Now it is easy to see that there are four ρ_f when $i = 1,2,3,4$ and there are four subgraphs in $K(4m,4)$ isomorphic to K_m. Take one ρ_f and one I_f of them. Then the number of mappings from $K(4m,4)/\rho_f$ to I_f is equal to the number of the automorphisms of graph I_f. It is $m!$ when $I_f \cong K_m$. Hence there are $16m!$ different endomorphisms in this case. If $\rho_f = \{[i, i+1, i+2], [i+3], [i+4, i+5, i+6], [i+7], \cdots, [i+4m-4, i+4m-3, i+4m-2], [i+4m-1]\}$, then $I_f \cong \overline{C_{2m}}$. Now it is easy to see that there are four ρ_f when $i = 1,2,3,4$ and there are only six subgraph in $K(4m,4)$ isomorphic to $\overline{C_{2m}}$. Take one ρ_f and one I_f of them. Then there are $4m$ cases to map $K(4m,4)/\rho_f$ to $\overline{C_{2m}}$. Hence there are $96m$

different endomorphisms in this case. If $\rho_f = \{[i, i+1], [i+2, i+3], [i+4, i+5], [i+6, i+7], \cdots, [i+4m-4, i+4m-3], [i+4m-2, i+4m-1]\}$, then $I_f \cong \overline{C_{2m}}$. Now it is easy to see that there are two ρ_f when $i = 1, 2$ and there are only six subgraph in $K(4m, 4)$ isomorphic to $\overline{C_{2m}}$. Take one ρ_f and one I_f of them. Then there are $4m$ cases to map $K(4m, 4)/\rho_f$ to $\overline{C_{2m}}$. Hence there are $48m$ different endomorphisms in this case. If $\rho_f = \{[i, i+1], [i+2], [i+3], [i+4, i+5], [i+6], [i+7], \cdots, [i+4m-4, i+4m-3], [i+4m-2], [i+4m-1]\}$, then $I_f \cong K(3m, 3)$. Now it is easy to see there are four ρ_f when $i = 1, 2, 3, 4$ and there are only four subgraphs in $K(4m, 4)$ isomorphic to $K(3m, 3)$. Take one ρ_f and one I_f of them. By Lemma 1.2, there are $6m$ cases to map $K(4m, 4)/\rho_f$ to $K(3m, 3)$. Hence there are $96m$ different endomorphisms in this case . By Theorem 2.11, $| Aut(K(4m, 4)) | = 8m$. Therefore $| End(K(4m, 4)) | = 16m! + 248m$.

Next we consider the case of $n = 4m + 1$. It is easy to see that $K(9, 4)$ is C_9. Since any odd cycle is unretractive, in the following, let $m \geq 3$.

Lemma 2.13 (1) $\omega(K(4m+1, 4)) = m$.

(2) $K(4m+1, 4)$ does not contain a subgraph isomorphic to $\overline{C_{2m+1}}$.

(3) For any clique K_m of size m, there are only two vertices $x_1, x_2 \in V(K(4m+1, 4)) \setminus K_m$ that are adjacent to $m-1$ vertices of K_m. They are non-adjacent.

(4) For any clique K_m of size m, there are no two vertices $x_1, x_2 \in V(K(4m+1, 4)) \setminus K_m$ that are adjacent to the same $m-1$ vertices of K_m.

Proof (1) and (2) are obvious.

(3), (4). It is easy to see that the clique K_m of size m has form $< i, i+5, i+9, \cdots, i+4m-3 >$ for some $i \in V(K(4m+1, 4))$. Thus only $i+1, i+4 \in V(K(4m+1, 4)) \setminus K_m$ that are adjacent to $m-1$ vertices of K_m. Clearly, $\{i+1, i+4\} \notin E$. In particular, $i+1$ is adjacent to every vertex of $K_m \setminus \{i\}$ and $i+4$ is adjacent to every vertex of $K_m \setminus \{i+5\}$.

Lemma 2.14 Let $f \in End(K(4m+1, 4))$. Then $f(i) \neq f(i+3)$ for any $i \in V(K(4m+1, 4))$.

Proof Suppose $f(i) = f(i+3)$ for some $i \in V(K(4m+1, 4))$. Let $A = \{i+4, i+8, \cdots, i+4m\}$. It is easy to see that the subgraph of $K(4m+1, 4)$ induced by A is isomorphic to K_m. Since $i+3$ is adjacent to every vertex of $A \setminus \{i+4\}$, $f(i+3)$ is adjacent to every vertex of $f(A) \setminus \{f(i+4)\}$. Now $\{f(i+3), f(i+4)\} = \{f(i), f(i+4)\} \in E$ implies that the subgraph of $K(4m+1, 4)$ induced by $f(A) \cup \{f(i+3)\}$ is isomorphic to K_{m+1}. A contradiction.

Lemma 2.15 Let $f \in End(K(4m+1, 4))$. Then $f(i) \neq f(i+2)$ for any $i \in V(K(4m+1, 4))$.

Proof Suppose $f(i) = f(i+2)$ for some $i \in V(K(4m+1,4))$. In the following, we will show that $f(i+4) = f(i+6)$.

Suppose $f(i+4) \neq f(i+6)$. Let $A = \{i+2, i+6, \cdots, i+4m-2\}$. Then the subgraph of $K(4m+1,4)$ induced by A is isomorphic to K_m. Since $i+4$ is adjacent to every vertex of $A \setminus \{i+2, i+6\}$, $f(i+4)$ is adjacent to every vertex of $f(A) \setminus \{f(i+2), f(i+6)\}$. Now $\{f(i+4), f(i+2)\} = \{f(i+4), f(i)\} \in E$ implies that the subgraph of $K(4m+1,4)$ induced by $f(A) \cup \{f(i+4)\}$ is isomorphic to K^*_{m+1}.

We claim that $f(i+4m-1) = f(i+4m-2)$. Otherwise, since $i+4m-1$ is adjacent to every vertex of $A \setminus \{i+4m-2\}$, $f(i+4m-1)$ is adjacent to every vertex of $f(A) \setminus \{f(i+4m-2)\}$. Now $f(i+4m-1) \neq f(i+4m-2)$ implies that the subgraph of $K(4m+1,4)$ induced by $f(A) \cup \{f(i+4m-1)\}$ is isomorphic to K^*_{m+1}. Note that $\{f(i+4), f(i+4m-1)\} \in E$. This is a contradiction to Lemma 2.13(3). Now $f(i+4m-5) = f(i+4m-6)$ for any $m \geq 4$. Otherwise, since $i+4m-5$ is adjacent to every vertex of $A \setminus \{i+4m-2, i+4m-6\}$, $f(i+4m-5)$ is adjacent to every vertex of $f(A) \setminus \{f(i+4m-2), f(i+4m-6)\}$. Now $\{f(i+4m-5), f(i+4m-2)\} = \{f(i+4m-5), f(i+4m-1)\} \in E$ implies that the subgraph of $K(4m+1,4)$ induced by $f(A) \cup \{f(i+4m-5)\}$ is isomorphic to K^*_{m+1}. Note that $\{f(i+4), f(i+4m-5)\} \in E$. This is a contradiction to Lemma 2.13(3). A similar argument will show that $f(i+4k+2) = f(i+4k+3)$ for any $k = 2, 3, \cdots, m-3$. Now $f(i+6) = f(i+7)$. Otherwise, since $i+7$ is adjacent to every vertex of $A \setminus \{i+6, i+10\}$, $f(i+7)$ is adjacent to every vertex of $f(A) \setminus \{f(i+6), f(i+10)\}$. Now $\{f(i+7), f(i+10)\} = \{f(i+7), f(i+11)\} \in E$ implies that $f(i+7)$ is adjacent to every vertex of $f(A) \setminus \{f(i+6)\}$. Note that $f(i+4)$ is adjacent to every vertex of $f(A) \setminus \{f(i+6)\}$. This is a contradiction to Lemma 2.13(4).

We claim that $f(i+5) = f(i+6)$ or $f(i+5) = f(i+4)$. Otherwise, since $i+5$ is adjacent to every vertex of $A \setminus \{i+2, i+6\}$, $f(i+5)$ is adjacent to every vertex of $f(A) \setminus \{f(i+2), f(i+6)\}$. Now $\{f(i+5), f(i+2)\} = \{f(i+5), f(i)\} \in E$ implies that $f(i+5)$ is adjacent to every vertex of $f(A) \setminus \{i+6\}$. This is a contradiction to Lemma 2.13(4).

If $f(i+5) = f(i+6)$, then $f(i+9) = f(i+10)$. Otherwise, since $i+9$ is adjacent to every vertex of $A \setminus \{i+6, i+10\}$, $f(i+9)$ is adjacent to every vertex of $f(A) \setminus \{f(i+6), f(i+10)\}$. Now $\{f(i+9), f(i+6)\} = \{f(i+9), f(i+5)\} \in E$ implies that the subgraph of $K(4m+1,4)$ induced by $f(A) \cup \{f(i+9)\}$ is isomorphic to K^*_{m+1}. Note that $\{f(i+9), f(i+4)\} \in E$. This contradicts Lemma 2.13(3). A similar argument will show that $f(i+4k+1) = f(i+4k+2)$ for any $k = 3, \cdots, m$. If $f(i+3) = f(i+4)$, then $\{f(i+4), f(i+6)\} = \{f(i+3), f(i+7)\} \in E$. Since $f(i+4)$ is adjacent to every vertex of $f(A) \setminus \{f(i+6)\}$, the subgraph of $K(4m+1,4)$ induced by $f(A) \cup \{f(i+4)\}$ is isomorphic to K_{m+1}. A contradiction. Therefore $\rho_f = [i, i+1, i+2], [i+3], [i+4], [i+5, i+6, i+7], [i+8], [i+$

$9, i+10, i+11], \cdots, [i+4m-3, i+4m-2, i+4m-1], [i+4m]$. Clearly, $I_f \cong K(4m+1,4)/\rho_f \cong \overline{C_{2m+1}}$. This contradicts Lemma 2.13(2).

If $f(i+5) = f(i+4)$, then $f(i+1) = f(i+2)$. Otherwise, since $i+1$ is adjacent to every vertex of $A \setminus \{i+2\}$, $f(i+1)$ is adjacent to every vertex of $f(A) \setminus \{f(i+2)\}$. Thus $f(i+1)$ and $f(i+4)$ are adjacent to $m-1$ vertices of $f(A)$. Note that $\{f(i+1), f(i+4)\} = \{f(i+1), f(i+5)\} \in E$. This contradicts Lemma 2.13(3). We claim that $f(i+4k) \neq f(i+4k-1)$ for any $k = 2, 3, \cdots, m$. Otherwise, suppose $f(i+4t) = f(i+4t-1)$ for some $t \in \{2, 3, \cdots, m\}$ and $f(i+4r) \neq f(i+4r-1)$ for any $r < t$. Since $f(i+4t-1) = f(i+4t-2)$, $f(i+4t) = f(i+4t-1) = f(i+4t-2)$. By our hypothesis $f(i+4t-4) \neq f(i+4t-5)$. Note that $f(i+4t-5) = f(i+4t-6)$, so $f(i+4t-4) \neq f(i+4t-6)$. Since $i+4t-4$ is adjacent to every vertex of $A \setminus \{i+4t-2, i+4t-6\}$, $f(i+4t-4)$ is adjacent to every vertex of $f(A) \setminus \{f(i+4t-2), f(i+4t-6)\}$. Now $\{f(i+4t-4), f(i+4t-2)\} = \{f(i+4t-4), f(i+4t)\} \in E$ implies that the subgraph of $K(4m+1,4)$ induced by $f(A) \cup \{f(i+4t-4)\}$ is isomorphic to K^*_{m+1}. Note that $\{f(i+4t-4), f(i+4)\} \in E$ for any $t \geq 2$. This contradicts Lemma 2.13(3). Note that $f(i+4)$ is adjacent to every vertex of $f(A) \setminus \{f(i+6)\}$. If $f(i+3) = f(i+4)$, then $\{f(i+4), f(i+6)\} = \{f(i+3), f(i+7)\} \in E$. Thus the subgraph of $K(4m+1,4)$ induced by $f(A) \cup \{f(i+4)\}$ is isomorphic to K_{m+1}. A contradiction. Therefore $f(i+3) \neq f(i+4)$. Let $B = \{i+2, i+4, i+6, \cdots, i+4m-2, i+4m, i+3\}$. By the discussion above, $f(x_1) \neq f(x_2)$ for any $x_1, x_2 \in B$ and $x_1 \neq x_2$. Then $f(B)$ contains a subgraph isomorphic to $\overline{C_{2m+1}}$. This is a contradiction to Lemma 2.13(2). This contradiction yields that $f(i+4) = f(i+6)$.

A similar argument will show that $f(i+8) = f(i+10)$, $\cdots$, $f(i+4m-4) = f(i+4m-2)$, $f(i+4m) = f(i+1)$, $f(i+3) = f(i+5)$, $\cdots$, $f(i+4m-1) = f(i)$. Thus $f(i+4m-1) = f(i) = f(i+2)$. This is a contradiction since $\{i+4m-1, i+2\} \in E$. The proof is complete.

Lemma 2.16 Let $f \in End(K(4m+1,4))$. Then $f(i) \neq f(i+1)$ for any $i \in V(K(4m+1,4))$.

Proof Suppose $f(i) = f(i+1)$ for some $i \in V(K(4m+1,4))$. In the following, we show that $f(i+4) = f(i+5)$.

Suppose $f(i+4) \neq f(i+5)$. Let $A = \{i+1, i+9, i+13 \cdots, i+4m-3\}$. It is easy to see that the subgraph of $K(4m+1,4)$ induced by A is isomorphic to K_{m-1}. Since $i+4$ is adjacent to every vertices of $A \setminus \{i+1\}$, $f(i+4)$ is adjacent to every vertex of $f(A) \setminus \{f(i+1)\}$. Now $\{f(i+4), f(i+1)\} = \{f(i+4), f(i)\} \in E$ implies that the subgraph of $K(4m+1,4)$ induced by $f(A) \cup \{f(i+4)\}$ is isomorphic to K_m. Since $i+5$ is adjacent to every vertices of A, $f(i+5)$ is adjacent to every vertex of $f(A)$. Note that $f(i+4) \neq f(i+5)$. Hence the subgraph of $K(4m+1,4)$ induced by

$f(A) \cup \{f(i+4), f(i+5)\}$ is isomorphic to K^*_{m+1}.

We claim that $f(i+4m-2) = f(i+4m-3)$. Otherwise, let $B = A \cup \{i+4\}$. Then the subgraph of $K(4m+1,4)$ induced by $f(B)$ is isomorphic to K_m. Since $i+4m-2$ is adjacent to every vertices of $B \setminus \{i+4m-3\}$, $f(i+4m-2)$ is adjacent to every vertex of $f(B) \setminus \{f(i+4m-3)\}$. Since $f(i+4m-2) \neq f(i+4m-3)$, the subgraph of $K(4m+1,4)$ induced by $f(B) \cup \{f(i+4m-2)\}$ is isomorphic to K^*_{m+1}. Note that $\{i+5, i+4m-2\} \in E$ for any $m \geq 3$. This contradicts Lemma 2.13(3). Now $f(i+4m-6) = f(i+4m-7)$ for any $m \geq 4$. Otherwise, since $i+4m-6$ is adjacent to every vertices of $B \setminus \{i+4m-3, i+4m-7\}$, $f(i+4m-6)$ is adjacent to every vertex of $f(B) \setminus \{f(i+4m-3), f(i+4m-7)\}$. Now $\{f(i+4m-6), f(i+4m-3)\} = \{f(i+4m-6), f(i+4m-2)\} \in E$ and $f(i+4m-6) \neq f(i+4m-7)$ implies that the subgraph of $K(4m+1,4)$ induced by $f(B) \cup \{f(i+4m-6)\}$ is isomorphic to K^*_{m+1}. Note that $\{f(i+4m-6), f(i+5)\} \in E$ for any $m \geq 4$. This contradicts Lemma 2.13(3). A similar argument will show that $f(i+4k-2) = f(i+4k-3)$ for $k = 3, 4, \cdots, m-1$.

By Lemma 2.15, $f(i+6) \neq f(i+4)$. Since $i+6$ is adjacent to every vertices of $B \setminus \{i+4, i+9\}$, $f(i+6)$ is adjacent to every vertex of $f(B) \setminus \{f(i+4), f(i+9)\}$. Note that $\{i+6, i+9\} = \{i+6, i+10\} \in E$. If $f(i+5) \neq f(i+6)$, then $f(i+5)$ and $f(i+6)$ are adjacent to the same $m-1$ vertices of clique $f(B)$. This contradicts Lemma 2.13(4). Therefore $f(i+5) = f(i+6)$.

Let $C = A \cup \{i+5\}$. It is easy to see that the subgraph of $K(4m+1,4)$ induced by C is isomorphic to K_m. Then $f(C)$ is also isomorphic to K_m. Since $f(i) = f(i+1)$ and $f(i+5) = f(i+6)$, by Lemma 2.15, $f(i+2) \neq f(i+1)$ and $f(i+4) \neq f(i+5)$. Hence $f(i+2), f(i+4) \notin f(C)$. Since $i+2$ is adjacent to every vertices of $C \setminus \{i+1, i+5\}$, $f(i+2)$ is adjacent to every vertex of $f(C) \setminus \{f(i+1), f(i+5)\}$. Note that $\{i+1, i+5\} = \{i+1, i+6\} \in E$. Then $f(i+2)$ is adjacent to every vertex of $f(C) \setminus \{f(i+1)\}$. A similar argument will show that $f(i+4)$ is adjacent to every vertex of $f(C) \setminus \{f(i+5)\}$. Thus $f(C) \subseteq N(f(i+2)) \cup N(f(i+4))$. By Lemma 2.13(3), $\{f(i+2), f(i+4)\} \notin E$. Without loss of generality, suppose $f(i+2) < f(i+4)$ and $f(i+2) = j$. Then $f(i+4) \in \{j+1, j+2, j+3\}$. If $f(i+4) = j+1$, then $N(j) \cup N(j+1) = \{j+4, j+5, \cdots, j+4m-2\}$. But the vertices in $\{j+4, j+5, \cdots, j+4m-2\}$ can not induced a subgraph isomorphic to K_m. A contradiction. If $f(i+4) = j+2$, then $N(j) \cup N(j+2) = \{j+4, j+5, \cdots, j+4m-1\}$. But the vertices in $\{j+4, j+5, \cdots, j+4m-1\}$ can not induced a subgraph isomorphic to K_m. A contradiction. Therefore $f(i+4) = j+3$. Now $N(j) \cup N(j+3) = \{j+4, j+5, \cdots, j+4m\}$. Since only $\{j+4, j+8, \cdots, j+4m\} \subseteq \{j+4, j+5, \cdots, j+4m\}$ can induced a subgraph isomorphic to K_m, $f(C) = \{j+4, j+8, \cdots, j+4m\}$. Note that only $j+4 \in f(C)$ is not adjacent to $j+3$ and only $j+4m \in f(C)$ is not adjacent to j, then $f(i+5) = j+4$ and $f(i+1) = j+4m$. Since $i+3$ is adjacent to every vertex of $C \setminus \{i+1, i+5\}$, $f(i+3) \in \{j+1, j+2\}$. Without loss of generality, we may suppose that $f(i+3) = j+1$. Since $f(i+4k-2) = f(i+4k-3)$ for $k = 3, 4, \cdots, m-1$, by Lemma 2.15, $f(i+4k-3) \neq f(i+4k-4)$ for $k = 3, 4, \cdots, m-1$. Since $i+8$ is adjacent to every vertices of $C \setminus \{i+5, i+9\}$, $f(i+8)$ is adjacent to every vertex of $f(C) \setminus \{f(i+5), f(i+9)\}$. Note that $\{i+4, i+8\} \in E$, then $f(i+8) = j+7$ and $f(i+9) = j+8$. A similar argument will show that $f(i+4t-3) = j+4t-4$ and $f(i+4t-4) = j+4t-5$ for $t = 4, \cdots, m$. Since $i+4m$ is adjacent to every vertex of $C \setminus \{i+1, i+4m-3\}$, $f(i+4m)$ is adjacent to every vertex of $f(C) \setminus \{f(i+1), f(i+4m-3)\}$. Thus $f(i+4m) \in \{j+4m-3, j+4m-4, j+4m-5\}$. Now $\{i+4m, i+4m-4\} \in E$ implies that $f(i+4m) = j+4m-1$. Note that $\{i+3, i+4m\} \in E$, but now $\{f(i+3), f(i+4m)\} = \{j+1, j+4m-1\} \notin E$. This contradiction yields that $f(i+4) = f(i+5)$.

A similar argument will show that $f(i+8) = f(i+9)$, $f(i+12) = f(i+13)$, $\cdots$, $f(i+4m) = f(i+4m+1)$. Thus $f(i) = f(i+1) = f(i+4m)$. Note that $i+1 = (i+4m)+2$. This contradicts Lemma 2.15. The proof is complete.

Theorem 2.17 $K(4m+1, 4)$ is unretractive.

Proof It follows directly from Lemmas 2.1, 2.14, 2.15 and 2.16.

Lemma 2.18 (1) $\omega(K(4m+2, 4)) = m$.

(2) There are only two subgraphs in $K(4m+2, 4)$ isomorphic to $\overline{C_{2m+1}}$, say, $G_1 =< 1, 3, \cdots, 4m+1 >$ and $G_2 =< 2, 4, \cdots, 4m+2 >$.

(3) For any $t \in V(K(4m+2, 4)) \setminus G_i$ $(i = 1, 2)$, t is adjacent to $2m-3$ vertices of G_i.

(4) $\omega(K(4m+3, 4)) = m$.

Proof (1), (2) and (4) are obvious.

(3) It is easy to see that t is adjacent to exactly $G_i \setminus \{t+1, t-1, t+3, t-3\}$.

Lemma 2.19 Let $f \in End(K(4m+2, 4)$. If $f(x_1) = f(x_2)$ for some $x_1, x_2 \in V(K(4m+2, 4))$, then $|x_1 - x_2| = 1$ or $|x_1 - x_2| = n-1$.

Proof Let $f \in End(K(4m+2, 4))$ and $x_1, x_2 \in V(K(4m+2, 4))$ be such that $f(x_1) = f(x_2)$. Without loss of generality, we suppose $x_1 < x_2$ and $x_1 = i$. If $|x_1 - x_2| \neq 1$ and $|x_1 - x_2| \neq n-1$, then $x_2 = i+2$ or $x_2 = i+3$. Let $A = \{i, i+4, \cdots, i+4m-4\}$. Then the subgraph of $K(4m+2, 4)$ induced by A is isomorphic to K_m. Since $i+4m$ is adjacent to every vertex of $A \setminus \{i\}$, $f(i+4m)$ is adjacent to every vertex of $f(A) \setminus \{f(i)\}$. Now if $x_2 = i+2$, then $\{f(i+4m), f(i)\} = \{f(i+4m), f(i+2)\} \in E$; if $x_2 = i+3$, then $\{f(i+4m), f(i)\} = \{f(i+4m), f(i+3)\} \in E$. Thus the subgraph of $K(4m+2,4)$ induced by $f(A) \cup \{f(i+4m)\}$ is isomorphic to K_{m+1}. A contradiction. Therefore $|x_1 - x_2| = 1$ or $|x_1 - x_2| = n-1$.

Lemma 2.20 Let $f \in End(K(4m+2,4))$. If $f(i) = f(i+1)$ for some $i \in V(K(4m+2,4))$, then $\rho_f = \{[i,i+1],[i+2,i+3],[i+4,i+5],[i+6,i+7],\cdots,[i+4m-2,i+4m-1],[i+4m,i+4m+1]\}$ and $I_f \cong \overline{C_{2m+1}}$.

Proof Suppose $f(i) = f(i+1)$ for some $i \in V(K(4m+2,4))$. Then $f(i+4) = f(i+5)$.

Suppose for contradiction that $f(i+4) \neq f(i+5)$. First we claim that $f(i+3) \neq f(i+4)$. Otherwise, let $A = \{i+3, i+7, \cdots, i+4m-1\}$. Then the subgraph of $K(4m+2,4)$ induced by A is isomorphic to K_m. Since $i+1$ is adjacent to every vertex of $A \setminus \{i+3\}$, $f(i+1)$ is adjacent to every vertex of $f(A) \setminus \{f(i+3)\}$. Now $\{f(i+1), f(i+3)\} = \{f(i), f(i+4)\} \in E$ implies that the subgraph of $K(4m+2,4)$ induced by $f(A) \cup \{f(i+1)\}$ is isomorphic to K_{m+1}. A contradiction. Let $B = \{i+1, i+3, \cdots, i+4m-1, i+4m+1\}$. It is easy to see that the subgraph of $K(4m+2,4)$ induced by B is isomorphic to $\overline{C_{2m+1}}$. By Lemma 2.18(2), there are only two subgraphs in $K(4m+2,4)$ isomorphic to $\overline{C_{2m+1}}$. Furthermore, they are induced subgraphs of $K(4m+2,4)$. By Lemma 2.19, $f(x_1) \neq f(x_2)$ for any $x_1, x_2 \in B$. Hence the subgraph of $K(4m+2,4)$ induced by $f(B)$ is isomorphic to $\overline{C_{2m+1}}$. Since $f(i+4) \neq f(i+5)$ and $f(i+4) \neq f(i+3)$, by Lemma 2.19, $f(i+4) \notin f(B)$. Since $i+4$ is adjacent to every vertex of $B \setminus \{i+1, i+3, i+5, i+7\}$, $f(i+4)$ is adjacent to every vertex of $f(B) \setminus \{f(i+1), f(i+3), f(i+5), f(i+7)\}$. Now $\{f(i+4), f(i+1)\} = \{f(i+4), f(i)\} \in E$ implies that $f(i+4)$ is adjacent to $2m-2$ vertices of $f(B)$. This is a contradiction to Lemma 2.18(3). Therefore $f(i+4) = f(i+5)$.

A similar argument will show that $f(i+8) = f(i+9)$, $\cdots$, $f(i+4m) = f(i+4m+1)$, $f(i+2) = f(i+3)$, $\cdots$, $f(i+4m-2) = f(i+4m-1)$. Therefore $\rho_f = \{[i,i+1],[i+2,i+3],[i+4,i+5],[i+6,i+7],\cdots,[i+4m-2,i+4m-1],[i+4m,i+4m+1]\}$. Clearly, $I_f \cong K(4m+2,4)/\rho_f \cong \overline{C_{2m+1}}$.

Theorem 2.21 Let $f \in End(K(4m+2,4))$. Then

(1) $f \in Aut(K(4m+4,4))$, or

(2) $\rho_f = \{[i,i+1],[i+2,i+3],[i+4,i+5],[i+6,i+7],\cdots,[i+4m-2,i+4m-1],[i+4m,i+4m+1]\}$ for some $i \in \{1,2\}$ and $I_f \cong \overline{C_{2m+1}}$.

Proof It follows directly from Lemmas 2.1, 2.19 and 2.20.

Theorem 2.22 $End(K(4m+2,4))$ $(m \geq 2)$ is regular.

Proof To prove $End(K(4m+2,4))$ is regular, let $f \in End(K(4m+2,4)) \setminus Aut(K(4m+2,4))$. We only need to show that there exist two idempotent endomorphisms g and h such that $\rho_g = \rho_f$ and $I_h = I_f$.

Define a mapping g from $V(K(4m+2,4))$ to itself by $g(x) = i$, where i

is the odd number in $[x]_{\rho_f}$. Then it is easy to check $g \in End(K(4m+2,4))$, $\rho_f = \rho_g$ and $g^2 = g$.

Now $I_f \cong \overline{C_{2m+1}}$. By Lemma 2.18, the subgraphs of $K(4m+2,4)$ isomorphic to $\overline{C_{2m+1}}$ is G_1 or G_2.

Define a mapping h from $V(K(4m+2,4))$ to itself by

$$h(x) = \begin{cases} x & \text{if } x \in V(I_f), \\ x-1 & \text{if } x \notin V(I_f). \end{cases}$$

Then $h \in End(K(4m+2,4))$, $I_h = I_f$ and $h^2 = h$.

By Lemma 1.1 and Theorem 2.21, there are only two $\mathcal{D}$-classes in $M = End(K(4m+2,4))$, say,

$$J_1 = \{f | I_f \cong \overline{C_{2m+1}}\} \quad \text{and} \quad J_2 = \{f | I_f \cong K(4m+2,4)\}.$$

It is clear that $J_2 = Aut(K(4m+2,4))$.

For $i = 1, 2$, let

$$\rho_i = \{[i, i+1], [i+2, i+3], \cdots, [i+4m-2, i+4m-1], [i+4m, i+4m+1]\}.$$

Then by Lemma 1.1 and Theorem 2.21, there are two $\mathcal{R}$-classes and two $\mathcal{L}$-classes in J_1, ρ_i determines an $\mathcal{L}$-class L_{1i} and G_j determines an $\mathcal{R}$-class R_{1j} in J_1. Let

$$h_{i1} = \begin{pmatrix} i & i+1 & i+2 & i+3 & \cdots & i+4m & i+4m+1 \\ i & i & i+2 & i+2 & \cdots & i+4m & i+4m \end{pmatrix};$$

$$h_{i2} = \begin{pmatrix} i & i+1 & i+2 & i+3 & \cdots & i+4m & i+4m+1 \\ i+1 & i+1 & i+3 & i+3 & \cdots & i+4m+1 & i+4m+1 \end{pmatrix}.$$

Denote $e_{11} = h_{11}$, $e_{12} = h_{12}$, $e_{21} = h_{22}$, $e_{22} = h_{21}$. Then it is easy to verify that e_{ij} $(i, j = 1, 2)$ is the idempotent endomorphism in $L_{1i} \cap R_{1j}$. Therefore every $\mathcal{H}$-class in J_1 is a maximal subgroup of $End(K(4m+2,4))$.

For $i, j = 1, 2$, $f \in L_{1i} \cap R_{1j}$, let $\overline{f}$ be the automorphism of G_i such that $\overline{f}(s) = f(s)$. Then $f = \overline{f} e_{ij}$ and $\{\overline{f} | f \in L_{1i} \cap R_{1j}\} = Aut(G_i) \cong D_{2m+1}$, the dihedral group of degree $2m+1$. It follows that $L_{1i} \cap R_{1j} = D_{2m+1} e_{ij}$. The $\mathcal{D}$-structure of $End(K(4m+2,4))$ is shown as in Figure 1.

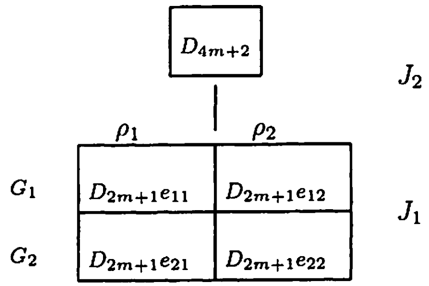

Figure 1 Structure of End($K(4m+2,4)$)

Recall that if every $\mathcal{H}$-class in a semigroup S is a group, then S is a complete regular semigroup. We have the following theorem.

Theorem 2.23 *End*$(K(4m+2,4))$ is completely regular.

Recall that if a and b are $\mathcal{D}$-equivalent elements in a semigroup S, then $|H_a| = |H_b|$. We have the following theorem.

Theorem 2.24 $|$ *End*$(K(4m+2,4))$ $| = 24m+12$.

Proof Clearly, there are two $\mathcal{D}-$classes J_1 and J_2 in *End*$(K(4m+2,4))$. It follows that $|J_1| = 4|D_{2m+1}| = 16m+8$. By Lemma 2.12, $|J_2| = 8m+4$. Therefore $|$ *End*$(K(4m+2,4))$ $| = |J_1| + |J_2| = 24m+12$.

Theorem 2.25 *End*$(K(4m+2,4)) = q$*End*$(K(4m+2,4))$, s*End*$(K(4m+2,4)) =$ *Aut*$(K(4m+2,4))$.

Proof Let $f \in$ *End*$(K(4m+2,4)) \setminus$ *Aut*$(K(4m+2,4))$. Then $\rho_f = \{[i,i+1],[i+2,i+3],[i+4,i+5]\cdots,[i+4m,i+4m+1]\}$ for some $i \in \{1,2\}$. If $[x]_{\rho_f} = \{t,t+1\}$ and $[y]_{\rho_f} = \{t+2,t+3\}$ for some $t \in V(K(4m+2,4))$, then $\{s,k\} \notin E$ for any $s \in [x]_{\rho_f}$ and $k \in [y]_{\rho_f}$. Since f is regular, $f \in h$*End*$(K(4m+2,4))$. Hence $\{f(x),f(y)\} \notin E$. Let $a,b \in V(I_f)$ and $\{a,b\} \in E(K(4m+2,4))$. Then $f^{-1}(a) = \{s,s+1\}$ and $f^{-1}(b) = \{t,t+1\}$ for some $s,t \in V(K(4m+2,4))$. By discuss above, $t \neq s+2$ and $s \neq t+2$. So there exists $x \in f^{-1}(a)$ such that x is adjacent to every vertex of $f^{-1}(b)$ and there exists $y \in f^{-1}(b)$ such that y is adjacent to every vertex of $f^{-1}(a)$. Hence $f \in q$*End*$(K(4m+2,4))$.

Since there are no two vertices in $K(4m+2,4)$ having the same adjacent set, s*End*$(K(4m+2,4)) =$ *Aut*$(K(4m+2,4))$.

Theorem 2.26 $K(4m+3,4)$ is unretractive.

Proof Let $f \in$ *End*$(K(4m+3,4))$. If there exist $x_1,x_2 \in V(K(4m+3,4))$ such that $f(x_1) = f(x_2)$, by Lemma 2.1, $|x_1 - x_2| \leq 3$ or $n-3 \leq |x_1 - x_2| \leq n-1$. Without loss of generality, we suppose $x_1 < x_2$ and

$x_1 = i$. Then $x_2 = i+1$, or $x_2 = i+2$, or $x_2 = i+3$. Let $A = \{i, i+4, \cdots, i+4m-4\}$. It is easy to see that the subgraph of $K(4m+3,4)$ induced by A is isomorphic to K_m. Since $i+4m$ is adjacent to every vertex of $A \setminus \{i\}$, $f(i+4m)$ is adjacent to every vertex of $f(A) \setminus \{f(i)\}$. Now if $x_2 = i+1$, then $\{f(i+4m), f(i)\} = \{f(i+4m), f(i+1)\} \in E$; if $x_2 = i+2$, then $\{f(i+4m), f(i)\} = \{f(i+4m), f(i+2)\} \in E$; if $x_2 = i+3$, then $\{f(i+4m), f(i)\} = \{f(i+4m), f(i+3)\} \in E$. Thus the subgraph of $K(4m+3,4)$ induced by $f(A) \cup \{f(i+4m)\}$ is isomorphic to K_{m+1}. A contradiction. Hence f is a bijective from $V(K(4m+3,4))$ to itself. Therefore $K(4m+3,4)$ is unretractive.

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