The minimal positive index of inertia of signed unicyclic graphs

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Abstract: The positive index of inertia of a signed graph Γ, denoted by $i_+(\Gamma)$, is the number of positive eigenvalues of the adjacency matrix $A(\Gamma)$ including multiplicities. In this paper we investigate the minimal positive index of inertia of signed unicyclic graphs of order n with fixed girth and characterize the extremal graphs with the minimal positive index. Finally, we characterize the signed unicyclic graphs with the positive indices 1 and 2.

1 Introduction

Let $G = (V, E)$ be a simple connected graph of order n with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. The adjacency matrix $A(G) = (a_{ij})_{n\times n}$ of G is defined as follows: $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. A *signed graph* Γ is a pair $(G, \ )$ where $G = (V, E)$ is a simple graph, called the *underlying graph*, and is a sign function for edges $: E \quad \{+, -\}$. Sometimes, the underlying graph can be written as $|\Gamma|$. It is evident that $V(\Gamma) = V(G) = V$, $E(G) = E$ but $E(\Gamma) = E$. The *adjacency matrix* of the signed graph Γ is $A(\Gamma) = (a_{ij})_{n\times n}$ with $a_{ij} = \ (v_i v_j) a_{ij}$, where a_{ij} is an element in the adjacency matrix of the underlying graph G. If all edges are signed to positive, $A(G, \ )$ is exactly the ordinary $A(G)$. Moreover, we write $(G, +)$ instead of $(G, \ )$. The positive index of

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inertia of a signed graph Γ, denoted by $i_+(\Gamma)$, is the number of positive eigenvalues of the adjacency matrix $A(\Gamma)$ including multiplicities.

Let $\Gamma = (G, \ )$ be a signed graph. A *matching* of Γ is a set of independent edges of the underlying graph G. The *matching number* of Γ, denoted by $m(\Gamma)$, is the size of a maximal matching of Γ. For a graph Γ with at least two vertices, a vertex $v \ \ V(\Gamma)$ is called *unsaturated* in Γ if there exists a maximum matching M in which no edge is incident with v; otherwise, v is called *saturated* in Γ. The graph Γ is called *acyclic* (resp. *unicyclic*) if its underlying graph G is acyclic (resp. unicyclic).

The study of signed graphs attracted much attention since they have some applications in the fields of social psychology and chemistry, see [5, 9, 13, 14, 15] for details. Zaslavsky [20] and Chaiken [1] obtained the Matrix-Tree Theorem for signed graphs, respectively. Hou et al [10, 11] investigated the Laplacian eigenvalues of signed graphs and got some bounds for the largest and the least eigenvalues. Fan et al [6] studied the nullity of signed unicyclic graphs and characterized the signed unicyclic graphs of order n with nullity $n - i$ for $i = 2, 3, 4, 5$. In [18, 19], the authors studied the inertia of bicyclic graphs and weighted unicyclic graphs.

Let $\mathbf{U}_n$ be the set of signed unicyclic graphs of order n and $\mathbf{U}_{n,k}$ be the set of signed unicyclic graphs of order n with fixed girth k. In this paper, we shall investigate the minimal positive index of signed unicyclic graphs of order n with fixed girth k. This paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we investigate the minimal positive index among all unicyclic graphs in $\mathbf{U}_{n,k}$. In Section 4, we characterize the extremal signed unicyclic graphs with the minimal positive index in $\mathbf{U}_{n,k}$. Finally we characterize the signed unicyclic graphs with the positive indices 1 and 2.

2 Preliminaries

Lemma 2.1. *Let M be an $n \times n$ Hermitian matrix. Let N be the Hermitian matrix obtained by bordering M as follows:*

$$N = \begin{pmatrix} M & y \\ y & a \end{pmatrix},$$

where y is a column vector and a is a real number. Then $i_+(N) - 1 \leq i_+(M) \leq i_+(N)$.

Lemma 2.2. *Let* Γ *be an induced signed subgraph of* Γ. *Then* $i_+(\Gamma\ ) \leq i_+(\Gamma)$.

The sign of a cycle C of Γ is the product of the signs of all edges, denoted by $sgn(C) = \Pi_{e\ \ C}\ \ (e)$. A signed graph is said to be *balanced* if the signs of all cycles are positive, or equivalently, all cycles have even number of negative edges; otherwise, it is called *unbalanced*.

Lemma 2.3. *[10] Let* Γ *be a signed graph. Then* Γ *is balanced if and only if* $\Gamma = (G,\ \ )\ \ \ (G, +)$.

Lemma 2.4.
1. *[16] If* C_n *is balanced, the eigenvalues of* C_n *are* $2cos\frac{2\pi i}{n}$ $(i = 0, 1, \cdots, n-1)$.
2. *[6] If* C_n *is unbalanced, the eigenvalues of* C_n *are* $2cos\frac{(2i-1)\pi}{n}$ $(i = 1, 2, \cdots, n)$.
3. *[16] The eigenvalues of signed path* P_n *are* $2cos\frac{\pi i}{n+1}$ $(i = 1, \cdots, n)$.

From Lemmas 2.3 and 2.4, it follows that

Lemma 2.5. *Let* C_n, P_n *be the signed cycle, signed path of order* n, *respectively. Then*

1. *If* C_n *is balanced, then* $i_+(C_n) = \begin{cases} \frac{n}{2} - 1, & n \quad 0\ (mod\ 4), \\ \frac{n+1}{2}, & n \quad 1\ (mod\ 4), \\ \frac{n}{2}, & n \quad 2\ (mod\ 4), \\ \frac{n-1}{2}, & n \quad 3\ (mod\ 4). \end{cases}$

2. *If* C_n *is unbalanced, then* $i_+(C_n) = \begin{cases} \frac{n}{2}, & n \quad 0\ (mod\ 4), \\ \frac{n-1}{2}, & n \quad 1\ (mod\ 4), \\ \frac{n}{2} - 1, & n \quad 2\ (mod\ 4), \\ \frac{n+1}{2}, & n \quad 3\ (mod\ 4). \end{cases}$

3. $i_+(P_n) = \begin{cases} \frac{n}{2}, & n \quad 0\ (mod\ 2), \\ \frac{n-1}{2}, & n \quad 1\ (mod\ 2). \end{cases}$

The following result can be immediately deduced from Theorem 8.14 in [2] and is implicit in Theorem 3 in [12].

Lemma 2.6. *Let* T *be a signed tree. Then* $i_+(T) = i_-(T) = m(T)$.

The following result follows from Theorem 1.1(b) in [7].

Lemma 2.7. *Let* Γ *be a signed graph containing a pendant vertex* v *with unique neighbor* u. *Then* $i_+(\Gamma) = i_+(\Gamma - u - v) + 1$.

3 The minimal positive index of graphs in $\mathbf{U}_{n,k}$

Lemma 3.1. *Let Γ_0 be a signed graph of order $n - p$ such that $u \quad V(\Gamma_0)$ and S_p be a signed star with non-central vertices $\{v_2, v_2, \cdots, v_p\}$. Assume that Γ_1 is a signed graph obtained from Γ_0 and S_p by inserting an edge between u and the center v_1 of S_p. Let $\Gamma_2 = \Gamma_1 - \{v_1v_2, v_1v_3, \cdots, v_1v_p\} + \{uv_2, uv_3, \cdots, uv_p\}$ be a new signed graph such that $\quad (uv_i) = \quad (v_1v_i)$ $(i = 2, 3, \cdots, p)$. Then $i_+(\Gamma_1) \geq i_+(\Gamma_2)$.*

Proof. From Lemma 2.7, we have

$$i_+(\Gamma_1) = i_+(\Gamma_1 - v_1 - v_2) + 1 = i_+(\Gamma_0) + 1;$$
$$i_+(\Gamma_2) = i_+(\Gamma_2 - v_2 - u) + 1 = i_+(\Gamma_0 - u) + 1.$$

From Lemma 2.2, we have $i_+(\Gamma_0) \geq i_+(\Gamma_0 - u)$. This implies the result. □

Lemma 3.2. *Let Γ_0 be a signed graph of order $n - l - t$ and $u, v \quad V(\Gamma_0)$. Let S_{l+1}, S_{t+1} be two signed stars of order $l + 1$, $t + 1$, respectively. Assume that Γ_1 is a signed graph obtained from Γ_0, S_{l+1} and S_{t+1} by identifying u with the center of S_{l+1}, v with the center of S_{t+1}, respectively. Γ_2 is a signed graph obtained from Γ_0, S_{l+1} and S_{t+1} by identifying u with the centers of S_{l+1} and S_{t+1}. Then $i_+(\Gamma_1) \geq i_+(\Gamma_2)$.*

Proof. Let u_1, v_1 be two pendant vertices of S_{l+1}, S_{t+1}, respectively. From Lemma 2.7, we have

$$\begin{aligned} i_+(\Gamma_1) &= i_+(\Gamma_1 - u_1 - u) + 1 = i_+(\Gamma_1 - u_1 - u - v_1 - v) + 2 = i_+(\Gamma_0 - u - v) + 2; \\ i_+(\Gamma_2) &= i_+(\Gamma_2 - u_1 - u) + 1 = i_+(\Gamma_0 - u) + 1. \end{aligned}$$

From Lemma 2.1, $i_+(\Gamma_0 - u - v) \geq i_+(\Gamma_0 - u) - 1$ which implies $i_+(\Gamma_1) \geq i_+(\Gamma_2)$. □

Let $H_{n,k}$ be an ordinary unicyclic graph obtained from a cycle C_k by adding $n - k$ pendant vertices to a vertex of C_k.

Theorem 3.3. *Let $\Gamma \quad \mathbf{U}_{n,k}$ be a signed unicyclic graph of order n with girth k $(3 \leq k \leq n - 2)$. Then $i_+(\Gamma) \geq \frac{k}{2}$. This bound is sharp.*

Proof. Let C_k be the unique signed cycle in Γ. Assume that T_i $(1 \leq i \leq t)$ is all disjoint signed trees rooted at vertices on C_k. From Lemma 3.1, the positive index does not increase if all T_i's become signed stars whose centers are on C_k. From

Lemma 3.2, the positive index does not increase if all the above signed stars are attached at the same vertex on C_k. Hence some signed unicyclic graph Γ with the underlying graph $H_{n,k}$ attains the minimal positive index in $\mathbf{U}_{n,k}$. From Lemmas 2.7 and 2.5, we have

$$i_+(\Gamma) \quad = \quad i_+(P_{k-1}) + 1 = \begin{cases} \frac{k+1}{2}, & \text{if } k \text{ is odd,} \\ \frac{k}{2}, & \text{if } k \text{ is even.} \end{cases}$$

This completes the proof. □

Corollary 3.4. *Let $\Gamma \quad \mathbf{U}_n$ be a signed unicyclic graph of order n with pendant vertices. Then $i_+(\Gamma) \geq 2$. This bound is sharp.*

4 The extremal signed unicyclic graphs

The authors in [6, 8] investigated a class of (signed) graphs with pendant trees and expressed their nullities in terms of their subgraphs. In the following we adopt the notations in [6]. Let Γ_1 be a signed graph containing a vertex u and Γ_2 be a signed graph of order n disjoint from Γ_1. For $1 \leq k \leq n$, the k-joining graph of Γ_1 and Γ_2 with respect to u, denoted by $\Gamma_1(u) \quad {}^k\,\Gamma_2$, is obtained from $\Gamma_1 \quad \Gamma_2$ by joining u and some k vertices of Γ_2 with signed edges.

Let Γ be a signed unicyclic graph and C_k be the unique signed cycle of Γ. For each vertex $v \quad V(C_k)$, let $\Gamma\{v\}$ be the tree rooted at v and containing v. Clearly, $\Gamma\{v\}$ is an induced subgraph of Γ. The unicyclic graph Γ is said to be *Type I* if there exists a vertex v on the cycle such that v is saturated in $\Gamma\{v\}$; otherwise, Γ is said to be *Type II*.

By modifying the proofs of Theorems 3.1, 3.3 and 4.1 in [8] and Theorems 2.9, 2.10 and 3.1 in [6], one has

Lemma 4.1. *Let T be a signed tree with matching number $m(T)$. Assume that Γ is a signed graph of order n, then for each integer k ($1 \leq k \leq n$):*

1. If u is a saturated vertex in T, then we have

$$i_+(T(u) \quad {}^k\,\Gamma) = i_+(T) + i_+(\Gamma) = m(T) + i_+(\Gamma).$$

2. If u is an unsaturated vertex in T, then we have

$$i_+(T(u) \quad {}^k\,\Gamma) = i_+(T - u) + i_+(\Gamma + u) = m(T) + i_+(\Gamma + u),$$

where $\Gamma + u$ is the subgraph of $T(u)$ $^{k}\Gamma$ induced by the vertices of Γ and u.

From Lemma 4.1, it follows that

Lemma 4.2. *Let Γ $\mathbf{U}_{n,k}$ be a signed unicyclic graph and C_k be the unique signed cycle of Γ. Then the following statements hold:*

1. *If Γ is of Type I and v $V(C_k)$ is saturated in $\Gamma\{v\}$, then $i_+(\Gamma) = i_+(\Gamma\{v\}) + i_+(\Gamma - \Gamma\{v\})$.*

2. *If Γ is of Type II, then $i_+(\Gamma) = i_+(\Gamma - C_k) + i_+(C_k)$.*

Let G be an ordinary graph obtained from a cycle C_k and a star S_{n-k} by inserting an edge between a vertex on C_k and the center of S_{n-k}.

Theorem 4.3. *Let Γ $\mathbf{U}_{n,k}$ be a signed unicyclic graph with the minimal positive index $\frac{k}{2}$. Let C_k be the unique signed cycle with vertex set $\{v_0, v_1, \cdots, v_{k-1}\}$ in Γ. Then*

1. *If $k = n$, Γ is one of the following graphs: balanced cycle C_n and n 0 (mod 4), or n 3 (mod 4); unbalanced cycle C_n and n 1 (mod 4), or n 2 (mod 4); if $k = n - 1$, Γ is a signed graph with $H_{n,n-1}$ as the underlying graph.*

2. *If Γ is of Type I and $3 \leq k \leq n - 2$. Assume that v_0 $V(C_k)$ is saturated in $\Gamma\{v_0\}$. Then $\Gamma - C_k$ is a set of isolated vertices and all pendant vertices are adjacent to some vertices with even index on C_k.*

3. *If Γ is of Type II and $3 \leq k \leq n - 2$. Then $|\Gamma| \cong G$, C_k is balanced and k 0 (mod 4), or k 3 (mod 4); $|\Gamma| \cong G$, C_k is unbalanced and k 1 (mod 4), or k 2 (mod 4)*

Proof. From Lemmas 2.5 and 2.7, it is easy to verify that the results hold if $k = n$, $n - 1$. In the following we consider the case $3 \leq k \leq n - 2$.

Assume that Γ is of *Type I*. Let C_k be the unique signed cycle in Γ. We divide two steps to characterize the signed unicyclic graphs with the minimal positive index in this case.

Step 1. Verifying that $\Gamma\{v_0\}$ is a star and $m(\Gamma - \Gamma\{v_0\}) = \frac{k-1}{2}$.

Since v_0 $V(C_k)$ is saturated in $\Gamma\{v_0\}$. From Lemmas 2.6 and 4.2, we have

$$i_+(\Gamma) = i_+(\Gamma\{v_0\}) + i_+(\Gamma - \Gamma\{v_0\}) = m(\Gamma\{v_0\}) + m(\Gamma - \Gamma\{v_0\}).$$

If k is even, then $m(\Gamma\{v_0\}) + m(\Gamma - \Gamma\{v_0\}) = \frac{k}{2}$. Note that $m(\Gamma\{v_0\}) \geq 1$ and $m(\Gamma - \Gamma\{v_0\}) \geq \frac{k-2}{2}$. Therefore it follows that $m(\Gamma\{v_0\}) = 1$ and $m(\Gamma - \Gamma\{v_0\}) = \frac{k-2}{2}$.

If k is odd, then

$$m(\Gamma\{v_0\}) + m(\Gamma - \Gamma\{v_0\}) = \frac{k+1}{2}.$$

Note that $m(\Gamma\{v_0\}) \geq 1$ and $m(\Gamma - \Gamma\{v_0\}) \geq \frac{k-1}{2}$. So it follows that $m(\Gamma\{v_0\}) = 1$ and $m(\Gamma - \Gamma\{v_0\}) = \frac{k-1}{2}$.

Step 2. Characterizing the extremal signed unicyclic graphs

Let $\Gamma = \Gamma - \Gamma\{v_0\}$ and $P = v_1 v_2 \cdots v_{k-1}$ be a path in Γ .

Claim 1. $\Gamma - P$ is a set of isolated vertices.

Assume that $\Gamma - P$ contains P_2 as an induced subgraph. Then

$$m(\Gamma) \geq m(P_2) + m(P) = 1 + \left\lfloor \frac{k-1}{2} \right\rfloor \geq 1 + \frac{k-2}{2} = \frac{k}{2}.$$

This contradicts to the fact obtained in Step 1. So this claim holds.

Claim 2. All pendant vertices in Γ must be adjacent to the vertices with even index on C_k.

Assume that there exists some pendant vertices adjacent to a vertex with odd index on C_k. This yields that $m(\Gamma - \Gamma\{v_0\}) = \frac{k-1}{2} + 1$, which is a contradiction.

Assume that Γ is of *Type II*. From Lemmas 2.6, 4.2, we have

$$i_+(\Gamma) = i_+(C_k) + i_+(\Gamma - C_k) = i_+(C_k) + m(\Gamma - C_k).$$

If C_k is balanced, then we have

$$m(\Gamma - C_k) = \begin{cases} 1, & k \quad 0 \pmod 4, \\ 0, & k \quad 1 \pmod 4, \\ 0, & k \quad 2 \pmod 4, \\ 1, & k \quad 3 \pmod 4. \end{cases}$$

If C_k is unbalanced, then we have

$$m(\Gamma - C_k) = \begin{cases} 0, & k \quad 0 \pmod 4, \\ 1, & k \quad 1 \pmod 4, \\ 1, & k \quad 2 \pmod 4, \\ 0, & k \quad 3 \pmod 4. \end{cases}$$

If $m(\Gamma - C_k) = 0$, then any vertex not on C_k is a pendant vertex attached at C_k in Γ. This contradicts to the fact that Γ is *Type II*. So $\Gamma - C_k$ is a star. This implies the result. □

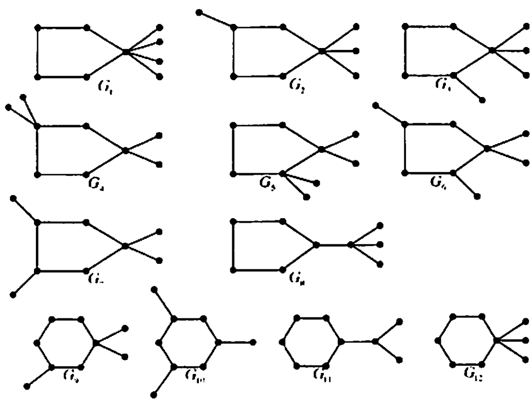

Figure 1: *Twelve ordinary unicyclic graphs*

Example 4.4. *Let G_i $(i = 1, 2, \cdots, 12)$ be twelve ordinary unicyclic graphs (see Fig. 1).*

In $\mathbf{U}_{9,5}$, Γ has the minimal positive index 3 if and only if Γ is one of the following signed unicyclic graphs: the signed unicyclic graphs with G_i $(i = 1, \cdots, 7)$ as the underlying graph, the unbalanced unicyclic graph with G_8 as the underlying graph.

In $\mathbf{U}_{9,6}$, Γ has the minimal positive index 3 if and only if Γ is one of the following signed unicyclic graphs: the signed unicyclic graphs with G_i $(i = 9, 10, 11)$ as the underlying graph, the unbalanced unicyclic graph with G_{12} as the underlying graph.

Next we define four ordinary unicyclic graphs as follows (see Fig. 2): $G_1(n_1, n_2)$ $(n_1, n_2 \geq 0, n_1 + n_2 = n - 3)$ is obtained from C_3 by attaching n_1, n_2 pendant vertices at two different vertices on C_3; $G_2(n_1, n_2)$ $(n_1, n_2 \geq 0, n_1 + n_2 = n - 4)$ is obtained from C_4 by attaching n_1, n_2 pendant vertices at two nonadjacent vertices on C_4; $G_3(n_1)$ $(n_1 = n - 3)$ is obtained from C_3 by inserting an edge between a vertex on C_3 and the center of S_{n_1}; $G_4(n_2)$ $(n_2 = n - 4)$ is obtained from C_4 by inserting an edge between a vertex on C_4 and the center of S_{n_2}.

From Theorem 4.3, we have

Theorem 4.5. *Let Γ $\mathbf{U}_n$ be a signed unicyclic graph. Then we have*

1. *If $i_+(\Gamma) = 1$, then Γ is balanced C_3, or balanced C_4.*

2. *If $i_+(\Gamma) = 2$, then Γ is one of the following signed unicyclic graphs: the signed unicyclic graphs with $G_i(n_1, n_2)$ $(i = 1, 2)$ as the underlying graph; the balanced signed unicyclic graphs with $G_3(n_1)$, or $G_4(n_2)$ as the underlying graph.*

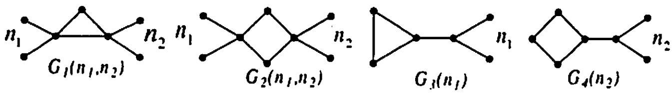

Figure 2: *Four ordinary unicyclic graphs*

From Theorem 3.3, we obtain the following result which can be deduced from Theorem 6 in [4].

Corollary 4.6. *Let G be an ordinary unicyclic graph of order n with pendant vertices and fixed girth k $(3 \leq k \leq n-2)$. Then $i_+(G) \geq \frac{k}{2}$. This bound is sharp.*

Corollary 4.7. *Let G be an ordinary unicyclic graph with the minimal positive index $\frac{k}{2}$ $(3 \leq k \leq n-2)$. Let C_k be the unique cycle with vertex set $\{v_0, v_1, \cdots, v_{k-1}\}$ in G. Then*

1. *G is of Type I. Assume that v_0 $V(C_k)$ is saturated in $G\{v_0\}$. Then $G\{v_0\}$ is a star and all pendant vertices are adjacent to some vertices with even index on C_k.*

2. *G is of Type II. Then $G \cong G$ and k 0 (mod 4), or k 3 (mod 4).*

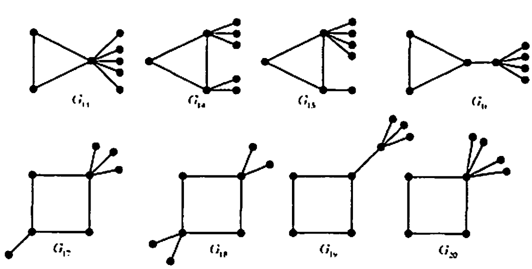

Figure 3: *Eight ordinary unicyclic graphs in* $\mathbf{U}_8$

Example 4.8. *All unicyclic graphs of order 8 with two positive eigenvalues are G_is $(i = 13, \cdots, 20)$ (see Fig. 3).*

Remark. Example 4.8 corresponds to the result obtained by Cvetković et al in [3]. $G_i s$ ($i = 13, \cdots, 20$) are the graphs corresponding to the graphs numbered by 89, 86, 88, 80, 43, 40, 30, 45 in [3].

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