FAMILIES OF GENERATING FUNCTIONS FOR THE JACOBI AND RELATED MATRIX POLYNOMIALS

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ABSTRACT. The Jacobi matrix polynomials and their orthogonality only for commutative matrices was first studied by Defez et. al. [Jacobi matrix differential equation, polynomial solutions and their properties. *Comput. Math. Appl.* 48 (2004), 789–803]. It is known that orthogonal matrix polynomials comprise an emerging field of study, with important results in both theory and applications continuing to appear in the literature. The main object of this paper is to derive various families of linear, multilateral and multilinear generating functions for the Jacobi matrix polynomials and the Gegenbauer matrix polynomials. Recurrence relations of Jacobi matrix polynomials are obtained. Some special cases of the results presented in this study are also indicated.

1. INTRODUCTION

The book by Gohberg, Lancaster and Rodman [6] is a good source for matrix polynomials considering orthogonal matrix polynomials. Recently, the classical orthogonal polynomials have been extended to the orthogonal matrix polynomials [2, 10, 11, 14]. In [4] the authors introduced and studied Jacobi matrix polynomials. Jódar and Cortés introduced and studied the hypergeometric matrix function $F(A, B; C; z)$ and the hypergeometric matrix differential equation in [9] and the explicit closed form general solution of it has been given in [12]. In [3] the authors introduced the Chebyshev matrix polynomials and gave some result with Chebyshev matrix polynomials. In [8] the authors introduced a new system of matrix polynomials, namely the Gegenbauer matrix polynomials (see also [15]).

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Throughout this paper, for a matrix $A \in \mathbb{C}^{N\times N}$ its spectrum is denoted by $\sigma(A)$. The two-norm of A, which will be denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where, for a vector $y \in \mathbb{C}^N$, $\|y\|_2 = \left(y^T y\right)^{1/2}$ is the Euclidean norm of y. I and θ will denote the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively. We say that a matrix A in $\mathbb{C}^{N\times N}$ is a positive stable if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A. If $A_0, A_1, ..., A_n$ are elements of $\mathbb{C}^{N\times N}$ and $A_n \neq \theta$, then we call

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0$$

a matrix polynomial of degree n in x. From [9], one can see

$$(P)_n = P(P+I)(P+2I)...(P+(n-1)I); \; n \geq 1; \; (P)_0 = I. \tag{1.1}$$

From (1.1), we can obtain

$$(P)_{n-k} = (-1)^k (P)_n [(I - P - nI)_k]^{-1}; \; 0 \leq k \leq n, \tag{1.2}$$

where $P + nI$ is invertible for every integer $n \geq 0$. From the relation (1.3) of [13], we see that

$$\frac{(-1)^k}{(n-k)!} I = \frac{(-nI)_k}{n!}; \; 0 \leq k \leq n. \tag{1.3}$$

The hypergeometric matrix function $F(A, B; C; z)$ has been given in the form [9]

$$F(A,B;C;z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{n!} \left[(C)_n\right]^{-1} z^n \tag{1.4}$$

for matrices A, B and C in $\mathbb{C}^{N\times N}$ such that $C + nI$ is invertible for all integer $n \geq 0$ and for $|z| < 1$. For any matrix A in $\mathbb{C}^{N\times N}$ the authors exploited the following relation due to [9]

$$(1-x)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} x^n \;, |x| < 1. \tag{1.5}$$

The Jacobi matrix polynomials have been given as in [4] so that $P_n^{(A,B)}(x)$ for parameter matrices A and B whose eigenvalues, z, all satisfy $Re(z) > -1$. For any positive integer n, the nth Jacobi matrix polynomial $P_n^{(A,B)}(x)$ is defined by

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{n!} F\left(A + B + (n+1)I, -nI; B + I; \frac{1+x}{2}\right) \times \Gamma^{-1}(B+I)\Gamma(B+(n+1)I). \tag{1.6}$$

We can write that

$$P_n^{(A,B)}(x) = \frac{1}{n!}F\left(A+B+(n+1)I, -nI; A+I; \frac{1-x}{2}\right) \times\Gamma^{-1}(A+I)\Gamma(A+(n+1)I), \tag{1.7}$$

or

(1.8)

$$P_n^{(A,B)}(x) = \frac{1}{n!}\left(\frac{x+1}{2}\right)^n F\left(-nI, -(B+nI); A+I; \frac{x-1}{x+1}\right)(A+I)_n.$$

In [3], for D and $C \in \mathbb{C}^{N\times N}$, suppose that D is positive stable, $DC = CD$, and that $C - D + kI$ and $C + kI$ are invertible for all nonnegative integers k. Then, for $|t| < 1$,

$$\text{(1.9)}\quad F(-nI, D; C; t) = (1-t)^n F\left(-nI, C-D; C; \frac{-t}{1-t}\right), n = 0, 1, 2, \ldots.$$

In [4] if $f(z)$ and $g(z)$ are holomorphic functions in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{N\times N}$ for which $\sigma(A) \subset \Omega$, then

$$f(A)g(A) = g(A)f(A).$$

Hence, if $B \in \mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B) \subset \Omega$ so, and if $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \tag{1.10}$$

Furthermore, in [4], the reciprocal scalar Gamma function, $\Gamma^{-1}(z) = 1/\Gamma(z)$, is an entire function of the complex variable z. Thus, for any $C \in \mathbb{C}^{N\times N}$, the Riesz-Dunford functional calculus [5] shows that $\Gamma^{-1}(C)$ is well defined and is, indeed, the inverse of $\Gamma(C)$. Hence: if $C \in \mathbb{C}^{N\times N}$ is such that $C + nI$ is invertible for every integer $n \geq 0$, then

$$(C)_n = \Gamma(C+nI)\Gamma^{-1}(C). \tag{1.11}$$

The aim of this paper is to derive various families of linear, multilinear and multilateral generating functions for the Jacobi matrix polynomials and the Gegenbauer matrix polynomials. We consider the special cases of them. Some recurrence relations for Jacobi matrix polynomials are also obtained.

2. Generating functions for the Jacobi matrix polynomials

In this section, we derive families of linear generating functions for the Jacobi matrix polynomials. We have the following main theorem.

Theorem 2.1. *Assume that all eigenvalues z of the matrices A and B of the Jacobi matrix polynomials $P_n^{(A,B)}(x)$ satisfy the condition $Re(z) > -1$.*

Then we have

$$\sum_{n=0}^{\infty}(A+B+I)_n\ P_n^{(A,B)}(x)\left[(B+I)_n\right]^{-1}r^n$$

$$(2.1) = \quad (1+r)^{-(A+B+I)}F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};B+I;\frac{2r(x+1)}{(1+r)^2}\right),$$

where $|r|<1$ *and* $A,B\in\mathbb{C}^{N\times N}$.

Proof. By (1.6), we easily see that

$$\sum_{n=0}^{\infty}(A+B+I)_nP_n^{(A,B)}(x)\ \left[(B+I)_n\right]^{-1}r^n=\sum_{n=0}^{\infty}(A+B+I)_n\frac{(-1)^n}{n!}$$

$$\times\sum_{k=0}^{n}\left\{\frac{(\ (n+1)I+A+B)_k\ (-nI)_k\left[(B+I)_k\right]^{-1}}{k!}\right.$$

$$\left.\times\left(\frac{1+x}{2}\right)^k\Gamma^{-1}(B+I)\Gamma\left[B+(n+1)I\right]\left[(B+I)_n\right]^{-1}r^n\right\}.$$

Using (1.10) and (1.11) we get

$$(B+I)_n=\Gamma^{-1}(B+I)\Gamma\left[B+(n+1)I\right].$$

Also using (1.3) we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(A+B+I)_nP_n^{(A,B)}(x)\ \left[(B+I)_n\right]^{-1}r^n\\ =\ &\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(A+B+I)_n(-1)^{n+k}(\ (n+1)I+A+B)_k}{(n-k)!\ k!}\\ &\times\left[(B+I)_k\right]^{-1}\left(\frac{1+x}{2}\right)^k r^n.\end{aligned}$$

Then it follows from (1.1) that

$$(A+B+I)_n(\ (n+1)I+A+B)_k\ =(A+B+I)_{n+k},$$

we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}(A+B+I)_nP_n^{(A,B)}(x)\ \left[(B+I)_n\right]^{-1}r^n\\ =\ &\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(A+B+I)_{n+2k}\ (-1)^n}{n!k!}\left[(B+I)_k\right]^{-1}\left(\frac{1+x}{2}\right)^k r^{n+k}\end{aligned}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{(A+B+I)_{2k}\,(-1)^n r^n}{n!}\left(\frac{1+x}{2}\right)^k \right.$$
$$\left. \times\,\frac{(A+B+I+2kI)_n\, r^k}{k!}\left[(B+I)_k\right]^{-1}\right\}.$$

By (1.5), since

$$\sum_{n=0}^{\infty}\frac{(A+B+I+2kI)_n\,(-1)^n r^n}{n!} = (1+r)^{-(A+B+(2k+1)I)},$$

we get

$$\sum_{n=0}^{\infty}(A+B+I)_n P_n^{(A,B)}(x)\ \left[(B+I)_n\right]^{-1} r^n$$
$$= \sum_{k=0}^{\infty}\frac{(A+B+I)_{2k}\,(1+r)^{-(A+B+(2k+1)I)} r^k}{k!}\left[(B+I)_k\right]^{-1}\left(\frac{1+x}{2}\right)^k.$$

From (1.1) we may write that

$$(A+B+I)_{2k} = 2^{2k}\left(\frac{A+B+I}{2}\right)_k\left(\frac{A+B+2I}{2}\right)_k,$$

which implies

$$\sum_{n=0}^{\infty}(A+B+I)_n P_n^{(A,B)}(x)\ \left[(B+I)_n\right]^{-1} r^n$$
$$= (1+r)^{-(A+B+I)}\sum_{k=0}^{\infty}\frac{2^{2k}\left(\frac{A+B+I}{2}\right)_k\left(\frac{A+B+2I}{2}\right)_k\ r^k}{k!(1+r)^{2k}}$$
$$\times\left[(B+I)_k\right]^{-1}\left(\frac{1+x}{2}\right)^k$$
$$= (1+r)^{-(A+B+I)}\ F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};B+I;\ \frac{2r(x+1)}{(1+r)^2}\right).$$

The proof is completed. □

In a similar manner as in the proof of Theorem 2.1, (1.7) and (1.8) yield the following results, respectively.

Theorem 2.2. *Assume that all eigenvalues z of the matrices A and B of the Jacobi matrix polynomials $P_n^{(A,B)}(x)$ satisfy the condition $Re(z) > -1$.*

Then we have

$$\sum_{n=0}^{\infty}(A+B+I)_n P_n^{(A,B)}(x)\ \left[(A+I)_n\right]^{-1} r^n$$

$$(2.2)\quad = (1-r)^{-(A+B+I)}\ F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};A+I;\frac{2r(x-1)}{(1-r)^2}\right),$$

where $|r|<1$ *and* $A,B \in \mathbb{C}^{N\times N}$.

Theorem 2.3. *Let* $A,B,C,D \in \mathbb{C}^{N\times N}$. *We get*

$$\sum_{n=0}^{\infty} P_n^{(A,B)}(x)t^n = F_4\left(I+B,I+A;I+A,I+B;\frac{(x-1)\,t}{2};\frac{(x+1)\,t}{2}\right),$$

where $AB=BA$ *and* $F_4\,(A,B;C,D;x,y)$ *is the matrix version of the Appell's function of two variables which is defined by*

$$F_4\,(A,B;C,D;x,y)=\sum_{n,k=0}^{\infty}(A)_{n+k}\,(B)_{n+k}\,(D)_n^{-1}\,(C)_k^{-1}\,\frac{x^k y^n}{k!n!},$$

where $C+nI$ *and* $D+nI$ *are invertible for every integer* $n\geq 0$.

Theorem 2.4. *Let* $A,B,C,D \in \mathbb{C}^{N\times N}$. *We have*

$$\sum_{n=0}^{\infty}(C)_n(D)_n(I+B)_n^{-1}P_n^{(A,B)}(x)(I+A)_n^{-1}t^n$$

$$= F_4\left(C,D;I+A,I+B;\frac{(x-1)\,t}{2};\frac{(x+1)\,t}{2}\right)$$

where $A+nI$ *and* $B+nI$ *are invertible for every integer* $n\geq 0$.

3. Multilinear and multilateral generating functions for the Jacobi matrix polynomials

In this section, we derive several families of bilinear and bilateral generating functions for the Jacobi matrix polynomials generated by (2.1) and given explicitly by (1.6).

We first state our result as the following

Theorem 3.1. *Corresponding to a non-vanishing function* $\Omega_\mu(y_1,\ldots,y_s)$ *of* s *complex variables* $y_1,\ldots,y_s$ $(s\in\mathbb{N})$ *and of complex order* μ, *let*

$$(3.1)\qquad \Lambda_{\mu,\nu}(y_1,\ldots,y_s;z):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\ldots,y_s)z^k$$

$$(a_k\neq 0,\ \ \mu,\nu\in\mathbb{C})$$

and
(3.2)

$$\Theta_{n,p,\mu,\nu}(x;y_1,\dots,y_s;\zeta) := \sum_{k=0}^{[n/p]} a_k(A+B+I)_{n-pk}\ P_{n-pk}^{(A,B)}(x) \times\ [(B+I)_{n-pk}]^{-1}\,\Omega_{\mu+\nu k}(y_1,\dots,y_s)\zeta^k$$

where $A, B \in \mathbb{C}^{N\times N}$, $n,p \in \mathbb{N}$ *and (as usual)* $[\lambda]$ *represents the greatest integer in* $\lambda \in \mathbb{R}$. *Then we have*

$$(3.3)\quad \sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(x;y_1,\dots,y_s;\frac{\eta}{t^p}\right)t^n = (1+t)^{-(A+B+I)}F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};B+I;\frac{2t(x+1)}{(1+t)^2}\right) \times\Lambda_{\mu,\nu}(y_1,\dots,y_s;\eta)$$

provided that each member of (3.3) *exists.*

Proof. For convenience, let S denote the first member of the assertion (3.3) of Theorem 3.1. Then, upon substituting for the polynomials

$$\Theta_{n,p,\mu,\nu}\left(x;y_1,\dots,y_s;\frac{\eta}{t^p}\right)$$

from the definition (3.2) into the left-hand side of (3.3), we obtain

$$(3.4)\qquad S = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k(A+B+I)_{n-pk}\ P_{n-pk}^{(A,B)}(x) \times [(B+I)_{n-pk}]^{-1}\,\Omega_{\mu+\nu k}(y_1,\dots,y_s)\eta^k t^{n-pk}.$$

Upon inverting the order of summation in (3.4), if we replace n by $n+pk$, we can write

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k\,(A+B+I)_n\ P_n^{(A,B)}(x)\,[(B+I)_n]^{-1}\,\Omega_{\mu+\nu k}(y_1,\dots,y_s)\eta^k t^n \\ &= \sum_{n=0}^{\infty}(A+B+I)_n\ P_n^{(A,B)}(x)\,[(B+I)_n]^{-1}\,t^n \sum_{k=0}^{\infty} a_k\Omega_{\mu+\nu k}(y_1,\dots,y_s)\eta^k \\ &= (1+t)^{-(A+B+I)}F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};B+I;\frac{2t(x+1)}{(1+t)^2}\right) \\ &\quad\times\Lambda_{\mu,\nu}(y_1,\dots,y_s;\eta), \end{aligned}$$

which completes the proof of Theorem 3.1. □

By expressing the multivariable function

$$\Omega_{\mu+\nu k}(y_1,\dots,y_s)\ (k\in\mathbb{N}_0,\ s\in\mathbb{N})$$

in terms of simpler function of one and more variables, we can give further applications of Theorem 3.1. For example, if we set

$$s = 1 \text{ and } \Omega_{\mu+\nu k}(y) = L^{(C,\lambda)}_{\mu+\nu k}(y)$$

in Theorem 3.1, where the nth Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ is defined by [7]

$$L_n^{(A,\lambda)}(x) = \sum_{n=0}^{n} \frac{(-1)^k \lambda^k}{k!\,(n-k)!} (A+I)_n \left[(A+I)_k\right]^{-1} x^k,$$

where A is a matrix in $\mathbb{C}^{N\times N}$, $A + nI$ is invertible for every integer $n \geq 0$ and λ is a complex number with $\mathrm{Re}(\lambda) > 0$ and generated by

$$\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x) t^n = (1-t)^{-(A+I)} \exp\left(\frac{-\lambda x t}{1-t}\right), \tag{3.5}$$

$$|t| < 1, \ -\infty < x < \infty,$$

then we obtain the following result which provides a class of bilateral generating functions for the Jacobi matrix polynomials and the Laguerre matrix polynomials polynomials.

Corollary 3.2. *If* $\Lambda_{\mu,\nu}(y;z) := \sum\limits_{k=0}^{\infty} a_k L^{(C,\lambda)}_{\mu+\nu k}(y) z^k$ *where* $(a_k \neq 0, \ \mu,\nu \in \mathbb{C})$; *and*

$$\begin{aligned}\Theta_{n,p,\mu,\nu}(x;y;\zeta) \ \ : \ &= \sum_{k=0}^{[n/p]} a_k (A+B+I)_{n-pk}\ P^{(A,B)}_{n-pk}(x) \\ &\times \left[(B+I)_{n-pk}\right]^{-1} L^{(C,\lambda)}_{\mu+\nu k}(y)\zeta^k\end{aligned}$$

where $n, p \in \mathbb{N}$. *Then we have*

(3.6)

$$\begin{aligned}\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(x;y;\frac{\eta}{t^p}\right) t^n = \ & (1+t)^{-(A+B+I)} \\ & \times F\left(\tfrac{A+B+I}{2}, \tfrac{A+B+2I}{2}; B+I; \tfrac{2t(x+1)}{(1+t)^2}\right) \Lambda_{\mu,\nu}(y;\eta\end{aligned}$$

provided that each member of (3.6) *exists.*

Remark 3.1. *Using the generating relation* (3.5) *for the Laguerre matrix polynomials and taking* $a_k = 1$, $\mu = 0$, $\nu = 1$, *we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}(A+B+I)_{n-pk}\; P_{n-pk}^{(A,B)}(x)\;\left[(B+I)_{n-pk}\right]^{-1} L_k^{(C,\lambda)}(y)\eta^k t^{n-pk}$$
$$= (1+t)^{-(A+B+I)}F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};B+I;\frac{2t(x+1)}{(1+t)^2}\right)$$
$$\times (1-\eta)^{-(C+I)}\exp\left(\frac{-\lambda y\eta}{1-\eta}\right),$$

where $|\eta| < 1$, $-\infty < y < \infty$.

Choosing $s = 1$ and $\Omega_{\mu+\nu k}(y) = P_{\mu+\nu k}^{(A,B)}(y)$, $(\mu,\nu \in \mathbb{N}_0)$, in Theorem 3.1 we obtain the following class of bilinear generating function for the Jacobi matrix polynomials.

Corollary 3.3. *If* $\Lambda_{\mu,\nu}(y;z) := \sum_{k=0}^{\infty} a_k P_{\mu+\nu k}^{(A,B)}(y)z^k$ *where* ($a_k \neq 0$, $\mu,\nu \in \mathbb{C}$); *and*

$$\Theta_{n,p,\mu,\nu}(x;y;\zeta) \;:\; = \sum_{k=0}^{[n/p]} a_k(A+B+I)_{n-pk}\; P_{n-pk}^{(A,B)}(x)$$
$$\times\left[(B+I)_{n-pk}\right]^{-1} P_{\mu+\nu k}^{(A,B)}(y)\zeta^k$$

where $n,p \in \mathbb{N}$. *Then we have*

(3.7)
$$\sum_{n=0}^{\infty}\Theta_{n,p,\mu,\nu}\left(x;y;\frac{\eta}{t^p}\right)t^n = (1+t)^{-(A+B+I)}$$
$$\times F\left(\tfrac{A+B+I}{2},\tfrac{A+B+2I}{2};B+I;\tfrac{2t(x+1)}{(1+t)^2}\right)\Lambda_{\mu,\nu}(y;\eta)$$

provided that each member of (3.7) *exists.*

Remark 3.2. *Using Theorem 2.3 and taking* $a_k = 1$, $\mu = 0$, $\nu = 1$, *we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}(A+B+I)_{n-pk}\; P_{n-pk}^{(A,B)}(x)\;\left[(B+I)_{n-pk}\right]^{-1} P_k^{(A,B)}(y)\eta^k t^{n-pk}$$
$$= (1+t)^{-(A+B+I)}F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};B+I;\frac{2t(x+1)}{(1+t)^2}\right)$$
$$\times F_4\left(I+B,I+A;I+A,I+B;\frac{(y-1)\,\eta}{2};\frac{(y+1)\,\eta}{2}\right).$$

Furthermore, for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(y_1,\dots,y_s)$, $(s \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorems 3.1 can be applied in order to derive various families of multilinear and multilateral generating functions for the Jacobi matrix polynomials.

4. Gegenbauer matrix polynomials via Jacobi matrix polynomials

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$. Then, the Gegenbauer matrix polynomials are defined by [8]

$$(1-2xr+r^2)^{-A}=\sum_{n=0}^{\infty}C_n^A(x)r^n. \tag{4.1}$$

Observe that (4.1) yields the following *explicit* representation

$$C_n^A(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k\,(A)_{n-k}}{k!\,(n-2k)!}\,(2x)^{n-2k}.$$

By taking $A=B$ in Theorem 2.1 we get

$$\sum_{n=0}^{\infty}(2A+I)_n\,P_n^{(A,A)}(x)\,[(A+I)_n]^{-1}\,r^n = (1+r)^{-(2A+I)} \times F\left(\frac{2A+I}{2};-;\frac{2r(x+1)}{(1+r)^2}\right).$$

Using (1.5) we have

(4.2)

$$\sum_{n=0}^{\infty}(2A+I)_n\,P_n^{(A,A)}(x)\,[(A+I)_n]^{-1}\,r^n = (1+r)^{-(2A+I)}\left(1-\frac{2r(x+1)}{(1+r)^2}\right)^{-(\cdot} = (1-2xr+r^2)^{-(A+I/2)}.$$

Comparing (4.1) and (4.2), we can write

$$C_n^{A+I/2}(x)=(2A+I)_n\,P_n^{(A,A)}(x)\,[(A+I)_n]^{-1}. \tag{4.3}$$

Now by (4.1), we can write the following result for the Gegenbauer matrix polynomials via Theorem 3.1 without its proof:

Theorem 4.1. *Corresponding to a non-vanishing function $\Omega_\mu(y_1,\dots,y_s)$ of s complex variables $y_1,\dots,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1,\dots,y_s;z):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\dots,y_s)z^k \tag{4.4}$$

$$(a_k\neq 0,\ \ \mu,\nu\in\mathbb{C})$$

and

$$\Theta_{n,p,\mu,\nu}(x;y_1,...,y_s;\zeta) := \sum_{k=0}^{[n/p]} a_k C_{n-pk}^{A}(x)\Omega_{\mu+\nu k}(y_1,...,y_s)\zeta^k,$$

where $A, B \in \mathbb{C}^{N\times N}$, $n, p \in \mathbb{N}$ *and (as usual)* $[\lambda]$ *represents the greatest integer in* $\lambda \in \mathbb{R}$. *Then we have*

$$(4.5)\quad \sum_{n=0}^{\infty}\Theta_{n,p,\mu,\nu}\left(x;y_1,...,y_s;\frac{\eta}{t^p}\right)t^n = (1-2xr+r^2)^{-A}\Lambda_{\mu,\nu}(y_1,...,y_s;\eta)$$

provided that each member of (4.5) *exists.*

The generating relation (4.1) yields the following addition formula for the Gegenbauer matrix polynomials

$$(4.6)\qquad C_n^{A+B}(x) = \sum_{k=0}^{n} C_{n-k}^{A}(x)C_k^{B}(x),$$

where $AB = BA$ and $A, B \in \mathbb{C}^{N\times N}$.

Precisely the same manner as described Theorem 3.1 and using (4.6), we can prove the following result.

Theorem 4.2. *For a non-vanishing function* $\Omega_\mu(y_1,...,y_s)$ *of complex variables* $y_1,...,y_s$ $(s \in \mathbb{N})$ *and for* $p \in \mathbb{N}$; $\mu, \nu \in \mathbb{C}$; *let*

$$\Lambda_{\mu,\nu,\alpha,\beta}^{n,p}(x;y_1,...,y_s;z) := \sum_{k=0}^{[n/p]} a_k C_{n-pk}^{A+B}(x)\Omega_{\mu+\nu k}(y_1,...,y_s)z^k,$$

$$(a_k \neq 0;\quad n,k \in \mathbb{N}_0;\ \mathbb{N}_0 = \mathbb{N}\cup\{0\}).$$

Then we have

(4.7)

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l\, C_{n-k}^{A}(x)C_{k-pl}^{B}(x)\Omega_{\mu+\nu l}(y_1,...,y_s)z^l = \Lambda_{\mu,\nu,\alpha,\beta}^{n,p}(x;y_1,...,y_s;z)$$

provided that each member of (4.7) *exists.*

If we set

$$s = 2 \text{ and } \Omega_{\mu+\nu k}(y,z) = H_{\mu+\nu k}(y,z,B)$$

in Theorem 4.1, where the two-variable Hermite matrix polynomials $H_n(x,y,A)$ is defined by means of the generating function [1]

$$(4.8)\quad \exp\left(xt\sqrt{2A} - yt^2 I\right) = \sum_{n=0}^{\infty}\frac{1}{n!}H_n(x,y,A)t^n;\ |t| < \infty,\ A \in \mathbb{C}^{N\times N},$$

then we obtain the following result which provides a class of bilateral generating functions for the two-variable Hermite matrix polynomials and the Gegenbauer matrix polynomials defined by (4.1).

Corollary 4.3. *If* $\Lambda_{\mu,\psi}(y,z;r) := \sum_{k=0}^{\infty} a_k H_{\mu+\nu k}(y,z,B) r^k$ *where* ($a_k \neq 0$, $\mu,\nu \in \mathbb{C}$); *and*

$$\Theta_{n,p,\mu,\psi}(x_1,\dots,x_r;y,z;\zeta) := \sum_{k=0}^{[n/p]} a_k C^A_{n-pk}(x) H_{\mu+\nu k}(y,z,B)\zeta^k,$$

where $n,p \in \mathbb{N}$ *and* $A,B \in \mathbb{C}^{N\times N}$. *Then we have*

$$(4.9) \quad \sum_{n=0}^{\infty} \Theta_{n,p,\mu,\psi}\left(x_1,\dots,x_r;y,z;\frac{\eta}{t^p}\right) t^n = (1-2xr+r^2)^{-A} \Lambda_{\mu,\psi}(y,z;\eta)$$

provided that each member of (4.9) *exists.*

Remark 4.1. *Using the generating relation* (4.8) *for the two-variable Hermite matrix polynomials and taking* $a_k = 1$, $\mu = 0$, $\nu = 1$, *we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} C^A_{n-pk}(x) H_k(y,z,B)\eta^k t^{n-pk} = (1-2xr+r^2)^{-A} \times \exp\left(y\eta\sqrt{2B} - z\eta^2 I\right),$$

where $|\eta| < \infty$.

Choosing $s = 1$ and $\Omega_{\mu+\nu k}(y_1,\dots,y_r) = C^D_{\mu+\nu k}(y)$, $(\mu,\nu \in \mathbb{N}_0)$, in Theorem 4.2 we obtain the following class of bilinear generating function for the Gegenbauer matrix polynomials.

Corollary 4.4. *If*

$$\Lambda^{n,p}_{\mu,\nu}(x;y;z) := \sum_{k=0}^{[n/p]} a_k C^{A+B}_{n-pk}(x) C^D_{\mu+\nu k}(y) z^k,$$

where $a_k \neq 0$, $n,p \in \mathbb{N}$, $\mu,\nu \in \mathbb{N}_0$, A, B, $D \in \mathbb{C}^{N\times N}$. *Then we have*

$$(4.10) \qquad \sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l\, C^A_{n-k}(x) C^B_{k-pl}(x) C^D_{\mu+\nu l}(y) z^l = \Lambda^{n,p}_{\mu,\nu}(x;y;z)$$

provided that each member of (4.10) *exists.*

5. Recurrence relations for Jacobi matrix polynomials

Assume that $\Psi(u)$ has the formal power-series expansion

$$(5.1) \qquad \Psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n \quad , \quad \gamma_0 \neq 0\ , \ \gamma_n \in \mathbb{C}^{N\times N}.$$

Define the matrix polynomials $f_n(x)$ by

$$(1-t)^{-C}\Psi\left(\frac{-4xt}{(1-t)^2}\right)=\sum_{n=0}^{\infty} f_n(x)t^n \quad , \quad C\in\mathbb{C}^{N\times N}. \tag{5.2}$$

Theorem 5.1. *The polynomials* $f_n(x)$ *defined by* (5.1) *and* (5.2) *have the following properties:*

$$f_n(x)=\frac{(C)_n}{n!}\sum_{k=0}^{n}(-nI)_k(C+nI)_k\left[\left(\frac{C}{2}\right)_k\left(\frac{C}{2}+\frac{I}{2}\right)_k\right]^{-1}\gamma_k x^k, \tag{5.3}$$

$$xf_n'(x)-nf_n(x)=-(C+(n-1)I)f_{n-1}(x)-xf_{n-1}'(x) \quad , \quad n\geq 1, \tag{5.4}$$

$$xf_n'(x)-nf_n(x)=-C\sum_{k=0}^{n-1} f_k(x)-2x\sum_{k=0}^{n-1} f_k'(x) \quad , \quad n\geq 1, \tag{5.5}$$

$$xf_n'(x)-nf_n(x)=\sum_{k=0}^{n-1}(-1)^{n-k}(C+2kI)f_k(x) \quad , \quad n\geq 1, \tag{5.6}$$

where $C\in\mathbb{C}^{N\times N}$ and $C+nI$ is invertible for every integer $n\geq 0$.

Proof. To obtain (5.3), consider

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n(x)t^n &= \sum_{k=0}^{\infty}(1-t)^{-C+2kI}(-4)^k\gamma_k t^k x^k \\
&= \sum_{n,k=0}^{\infty}\frac{(C+2kI)_n}{n!}\left[(C)_{2k}\right]\left[(C)_{2k}\right]^{-1}(-4)^k\gamma_k t^{n+k}x^k \\
&= \sum_{n,k=0}^{\infty}\frac{(C)_{n+2k}}{n!}\left[(C)_{2k}\right]^{-1}(-4)^k\gamma_k t^{n+k}x^k \\
&= \sum_{n,k=0}^{\infty}(-1)^k\frac{(C)_{n+2k}}{n!}\left[\left(\frac{C}{2}\right)_k\left(\frac{C}{2}+\frac{I}{2}\right)_k\right]^{-1}\gamma_k t^{n+k}x^k \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{(C)_{n+k}}{(n-k)!}\left[\left(\frac{C}{2}\right)_k\left(\frac{C}{2}+\frac{I}{2}\right)_k\right]^{-1}\gamma_k t^n x^k \\
&= \sum_{n=0}^{\infty}\frac{(C)_n}{n!}\sum_{k=0}^{n}(-nI)_k(C+nI)_k \\
&\quad \times\left[\left(\frac{C}{2}\right)_k\left(\frac{C}{2}+\frac{I}{2}\right)_k\right]^{-1}\gamma_k x^k t^n,
\end{aligned}$$

whence (5.3) immediately.

In order to derive (5.4),(5.5) and (5.6), put

$$G(x,t) = (1-t)^{-C}\Psi\left(\frac{-4xt}{(1-t)^2}\right).$$

Then

$$\frac{\partial G}{\partial x} = -4t(1-t)^{-C-2I}\Psi',$$

$$\frac{\partial G}{\partial t} = C(1-t)^{-C-I}\Psi - 4x(1+t)(1-t)^{-C-3I}\Psi'.$$

Therefore G satisfies the partial differential equation

$$x(1+t)\frac{\partial G}{\partial x} - t(1-t)\frac{\partial G}{\partial t} = -CtG. \tag{5.7}$$

Equation (5.7) can be put in the forms

$$x\frac{\partial G}{\partial x} - t\frac{\partial G}{\partial t} = -CtG - t^2\frac{\partial G}{\partial t} - xt\frac{\partial G}{\partial x}, \tag{5.8}$$

$$x\frac{\partial G}{\partial x} - t\frac{\partial G}{\partial t} = -\frac{Ct}{1-t}G - \frac{2xt}{1-t}\frac{\partial G}{\partial x}, \tag{5.9}$$

$$x\frac{\partial G}{\partial x} - t\frac{\partial G}{\partial t} = -\frac{Ct}{1+t}G - \frac{2t^2}{1+t}\frac{\partial G}{\partial x}. \tag{5.10}$$

Since

$$G(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n,$$

(5.8) yields that

$$\begin{aligned}\sum_{n=0}^{\infty}\left[xf_n'(x) - nf_n(x)\right]t^n &= -C\sum_{n=0}^{\infty} f_n(x)t^{n+1} - \sum_{n=0}^{\infty} nf_n(x)t^{n+1} \\ &\quad -\sum_{n=0}^{\infty} xf_n'(x)t^{n+1} \\ &= \sum_{n=1}^{\infty}\left[-(C+(n-1)I)f_{n-1}(x) - xf_{n-1}'(x)\right]t^n\end{aligned}$$

which leads to (5.4).

Equation (5.9) implies that

$$\begin{aligned}\sum_{n=0}^{\infty}\left[xf_n'(x)-nf_n(x)\right]t^n &= -C\left(\sum_{n=0}^{\infty}t^{n+1}\right)\left(\sum_{k=0}^{\infty}f_k(x)t^k\right)\\ &\quad -2x\left(\sum_{n=0}^{\infty}t^{n+1}\right)\left(\sum_{k=0}^{\infty}f_k'(x)t^k\right)\\ &= -C\sum_{n=0}^{\infty}\sum_{k=0}^{n}f_k(x)t^{n+1}-2x\sum_{n=0}^{\infty}\sum_{k=0}^{n}f_k'(x)t^{n+1}\\ &= -C\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}f_k(x)t^{n}-2x\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}f_k'(x)t^{n}\end{aligned}$$

which leads to (5.5).

From (5.10) we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\left[xf_n'(x)-nf_n(x)\right]t^n &= -C\left(\sum_{n=0}^{\infty}(-1)^nt^{n+1}\right)\left(\sum_{k=0}^{\infty}f_k(x)t^k\right)\\ &\quad -2\left(\sum_{n=0}^{\infty}(-1)^nt^{n+1}\right)\left(\sum_{k=0}^{\infty}kf_k(x)t^k\right)\\ &= -\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n-k}(C+2kI)f_k(x)t^{n+1}\\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^{n-k}(C+2kI)f_k(x)t^{n},\end{aligned}$$

which gives (5.6). □

If we choose

$$C=A+B+I \quad ; \quad \gamma_n=\frac{(I+A+B)_{2n}}{2^{2n}n!}(I+A)_n^{-1}$$

in Theorem 5.1, we see that the matrix polynomials f_n is

$$f_n(v)=(I+A+B)_nP_n^{(A,B)}(1-2v)(I+A)_n^{-1}.$$

Hence, Theorem 5.1 gives following results, when put in therms of x rather than v.

Corollary 5.2. *The Jacobi matrix polynomials have the following recurrence relations:*

$$\begin{aligned}&(x-1)\left[(A+B+nI)\frac{dP_n^{(A,B)}(x)}{dx}+\frac{dP_{n-1}^{(A,B)}(x)}{dx}(A+nI)\right]\\ &\quad=(A+B+nI)\left[nP_n^{(A,B)}(x)-P_{n-1}^{(A,B)}(x)(A+nI)\right],\end{aligned}$$

$$(x-1)\frac{dP_n^{(A,B)}(x)}{dx} - nP_n^{(A,B)}(x) = -(A+B+I)_n^{-1}\sum_{k=0}^{n-1}\{(A+B+I)_k$$

$$\times\left[(A+B+I)P_k^{(A,B)}(x) + 2(x-1)\frac{dP_k^{(A,B)}(x)}{dx}\right](I+A)_k^{-1}(A+I)_n\Big\},$$

$$(x-1)\frac{dP_n^{(A,B)}(x)}{dx} - nP_n^{(A,B)}(x) = (A+B+I)_n^{-1}\sum_{k=0}^{n-1}\{(-1)^{n-k}$$

$$\times (A+B+2kI+I)(A+B+I)_k P_k^{(A,B)}(x)(I+A)_k^{-1}(A+I)_n\Big\},$$

where all eigenvalues z of the matrices A and B of the Jacobi matrix polynomials $P_n^{(A,B)}(x)$ satisfy the condition $Re(z) > -1$ and $A+B+nI$ is invertible for every integer $n \geq 0$.

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